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SPECIAL INVOLUTIONS AND BULKY PARABOLIC SUBGROUPS IN FINITE COXETER GROUPS

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Abstract

The conjugacy classes of so-called special involutions parameterize the constituents of the action of a finite Coxeter group on the cohomology of the complement of its complexified hyperplane arrangement. In this note we give a short intrinsic characterisation of special involutions in terms of so-called bulky parabolic subgroups.

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1. Introduction

A finite Coxeter group W with root system Φ spanning a Euclidean vector space V acts on the space V as a finite reflection group. Any involution in W decomposes the space V into the direct sum of a 1-eigenspace and a (-1)-eigenspace. With respect to such a decomposition, one might ask whether the projection of some root $\alpha \in \Phi$ onto an eigenspace is proportional to a root contained in the eigenspace or not. In general, such a projection need not be proportional to a root. An involution is called special if every root yields at least one projection which is proportional to a root. Special involutions have been introduced by Felder and Veselov [3] to describe the cohomology of the complement of the complexified hyperplane arrangement of W.

In this note, we give a short intrinsic characterisation of special involutions in terms of a property of parabolic subgroups. It is well known that the normalizer $N_W(P)$ of a parabolic subgroup P of W splits over P. Here we call the parabolic subgroup P bulky, if the semidirect product $N_W(P) = P \rtimes N$ is in fact a direct product $P \times N$. We

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show that an involution is special if and only if its corresponding parabolic subgroup is bulky; see Theorem 2.3.

In [3] Felder and Veselov considered the standard and twisted actions of a finite Coxeter group W on the cohomology $H^*(\mathcal{M}_W)$ of the complement of the complexified hyperplane arrangement $\mathcal{M}_W$ of W. The twisted action is obtained by combining the standard action with complex conjugation; we refer the reader to [3] for precise statements. In a case by case argument, Felder and Veselov obtain a formula for all finite Coxeter groups W for the standard action

$$H^*(\mathcal{M}_W) \cong \sum_{\sigma \in X_W} (2 \cdot 1_{\langle\sigma\rangle}^W - \varrho)$$

as $\mathbb{C}W$-modules, where X_W is a set of representatives of the W-conjugacy classes of the special involutions in W, ϱ is the regular representation of W, and $1_{\langle\sigma\rangle}^W$ is the $\mathbb{C}W$-module induced from the trivial $\mathbb{C}\langle\sigma\rangle$-module. In case W is crystallographic, this formula can be deduced from earlier work of Lehrer [8, 9] and Fleischmann–Janiszczak [4, 5]. The new aspect of [3] in the theory is a uniform geometric description of the sets X_W of W-conjugacy classes of special involutions used in the formula above.

Felder and Veselov give a similar formula for the twisted action where the summation is taken over the set of even elements from X_W.

2. Notation and preliminaries

Throughout, W denotes a finite Coxeter group, generated by a set of simple reflections $S \subseteq W$; see [1] or [6] for a general introduction into the theory of Coxeter groups. For $J \subseteq S$, let W_J be the parabolic subgroup of W generated by J and denote by w_J the unique word in W_J of maximal length (with respect to S). Let $T = S^W$ be the set of all reflections of W. Let Φ be a root system with Coxeter group W and Φ_J the root subsystem of Φ corresponding to W_J. Set $V := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$. Then V affords the usual reflection representation of W. For each involution $\sigma \in W$ we have a direct sum decomposition $V = V_1 \oplus V_{-1}$, where V_1 and V_{-1} are the 1 and (-1)-eigenspaces of V of σ, respectively. For $\epsilon = \pm 1$ let $\Phi_\epsilon := \Phi \cap V_\epsilon$. Note that for $\sigma = w_J$ we have $\Phi_{-1} = \Phi_J$. Following [3], we say that an involution σ in W is *special*, if for any root $\alpha \in \Phi$ at least one of its projections onto V_ϵ is proportional to a root in Φ_ϵ. Clearly, this definition does not depend on the choice of root system for W.

The conjugacy classes of involutions in W have been classified by Richardson [10, Theorem A] and Springer [11] in terms of the parabolic subgroups of W whose longest element is central. More precisely, each involution is conjugate to a longest element w_J which is central in W_J for some $J \subseteq S$.

The normalisers of parabolic subgroups of finite Coxeter groups have been described by Howlett [7] and Brink and Howlett [2]. Accordingly, the normaliser

$N_W(W_J)$ of W_J in W is a semi-direct product of the form $W_J \rtimes N_J$, where N_J is itself a semi-direct product of a Coxeter group of known type and a group M_J, [7, Corollary 7]. It turns out, however, that in the case when w_J is central in W_J the group M_J is trivial.

PROPOSITION 2.1. *M_J acts faithfully as inner graph automorphisms on W_J. In particular, if w_J is central in W_J, then $M_J = \{1\}$.*

PROOF. According to the tables in [2], if W is irreducible and J is such that $|S \setminus J| = 2$, then a non-trivial generator of the group M_J exists only when either W is of type E_7 and W_J is of type $A_4 \times A_1$, or W is of type D_{2n} and W_J is of type A_{2n-2} or $A_{2k} \times A_{2l}$ with $k \neq l$ and $k+l = n-1$. Let us say that W_J is an *M-parabolic* subgroup of W in such a case. An easy check shows that if W_J is an M-parabolic subgroup of W, then M_J induces the same non-trivial graph automorphism on W_J as conjugation by w_J. In general, it follows from [2] that M_J is trivial unless a conjugate L of J lies in a subset $K \subseteq S$ such that $|K \setminus L| = 2$ and $W_K = W_N \times W_{K'}$ and $W_L = W_N \times W_{L'}$ for suitable subsets $K', L', N \subseteq S$ and $W_{L'}$ is an M-parabolic subgroup of $W_{K'}$. Now M_L induces a non-trivial inner automorphism on W_L and so does M_J on W_J.

By [7, Corollary 9], M_J intersects the centraliser of J in N_J trivially, and hence acts faithfully on W_J. □

The centraliser of the involution w_J and the normaliser of the parabolic subgroup W_J of W coincide; see [3, Proposition 7]. We give a new proof of this property.

PROPOSITION 2.2. *For each $J \subseteq S$ the element w_J is central in W_J if and only if $C_W(w_J) = N_W(W_J)$.*

PROOF. Suppose w_J is central in W_J. Then $W_J \subseteq C_W(w_J) \cap N_W(W_J)$. To show that $C_W(w_J) = N_W(W_J)$ it thus suffices to consider the set $D_J = \{x \in W : l(sx) > l(x) \text{ and } l(xs) > l(x) \text{ for all } s \in J\}$ of distinguished double coset representatives of W_J in W.

We have $l(w^x) = l(w)$ for all $w \in W_J$, $x \in N_J = \{x \in D_J : J^x = J\}$. In particular, $w_J^x = w_J$ for $x \in N_J$. Hence $N_W(W_J) \subseteq C_W(w_J)$.

Conversely, let $x \in C_W(w_J) \cap D_J$. Then $w_J \in W_J \cap W_J^x = W_{J \cap J^x}$; see, for example, [6, (2.1.12)]. It follows that $J = J^x$ whence $C_W(w_J) \subseteq N_W(W_J)$. □

We call the parabolic subgroup W_J *bulky* (in W) if $N_W(W_J) = W_J \times N_J$, that is, if N_J acts trivially on W_J. The main result of this note is the following theorem.

THEOREM 2.3. *Let $J \subseteq S$ be such that w_J is central in W_J. Then the involution w_J is special if and only if W_J is bulky in W.*

In our arguments we do make use of the classification of the irreducible Coxeter groups and the structure of the root systems of Weyl groups. We also use the notation and labelling of the Dynkin diagram of W as in [1, Planches I–IX].

3. Special involutions and bulky parabolic subgroups

We maintain the notation from the previous sections.

LEMMA 3.1. *If* $\dim V_1 = 1$ *and* $\Phi_1 \neq \varnothing$ *or if* $\dim V_{-1} = 1$, *then* w_J *is special. In particular,* $\pm\sigma$ *is special for every reflection* $\sigma \in T$.

PROOF. The projection of any root onto a one-dimensional space generated by a root α is clearly proportional to α. □

REMARK 3.2. The element w_J is central in W_J if and only if W_J has no components of type A_n with $n \geq 2$, of type D_{2n+1} with $n \geq 2$, of type E_6, or of type $I_2(2m+1)$, $m \geq 2$; see [10, 1.12].

PROOF OF THEOREM 2.3. We may assume that W is irreducible. By [2, Theorem B] and our Proposition 2.1 the group N_J is generated by certain conjugates of elements of the form $w_L w_K$, where $L \subseteq K \subseteq S$ such that L is a conjugate of J, $|K \setminus L| = 1$ and $L^{w_K} = L$. If $s^{w_L w_K} = s$ for all $s \in L$, then $w_L w_K$ centralises W_L and so its conjugate centralises W_J. Obviously, $s^{w_L w_K} = s^{w_K}$ for all $s \in L$, since w_L is central in W_L.

Now suppose that W_J is not bulky in W, that is, N_J does not centralise W_J. Then there exists a conjugate L of J and a subset $K \subseteq S$ such that $L \subseteq K$ with $|K \setminus L| = 1$ and w_K induces a non-trivial graph automorphism on W_L. It follows that $W_K = W_N \times W_{K'}$ for suitable $N, K' \subseteq S$ where the type of $W_{K'}$ is one of those listed in Remark 3.2. Since w_L is central in W_L, it follows that $W_L = W_N \times W_{L'}$ where $W_{L'}$ is a product of components of types not listed in Remark 3.2. Inspection of the maximal parabolic subgroups of $W_{K'}$ shows that $W_{L'}$ is of type D_{2n} and $W_{K'}$ is of type D_{2n+1}, $n \geq 1$; this includes the case where A_1^2 embeds into A_3 for $n = 1$.

Without loss of generality we may assume $N = \varnothing$, $K' = K = S$ and $L' = L = J$. So let

$$\Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq 2n+1\}$$

be a root system of type D_{2n+1} and consider the simple root $\alpha = \varepsilon_1 - \varepsilon_2$. It is easy to check that $\{\varepsilon_2, \ldots, \varepsilon_{2n+1}\}$ is a basis of V_{-1} and that V_1 is the $\mathbb{R}$-span of ε_1. Obviously V_1 contains no root, hence $\Phi_1 = \Phi \cap V_1 = \varnothing$. Consequently, no projection of a root in Φ on V_1 is proportional to a root in Φ_1. Next we show that the projection of α

onto V_{-1} is not proportional to any root in $\Phi_{-1} = \Phi \cap V_{-1}$. Recall that $\Phi_{-1} = \Phi_J$, a root system of type D_{2n} consisting of the roots $\pm\varepsilon_i \pm \varepsilon_j$, with $2 \le i < j \le 2n+1$. The projection of $v \in V$ onto V_{-1} is given by $\frac{1}{2}(v - w_J(v))$. Hence α projects onto $\frac{1}{2}(\alpha - w_J(\alpha)) = -\varepsilon_2$ and therefore is not proportional to any root in Φ_{-1}. Thus w_J is not special, as required.

For the converse, suppose that W_J is bulky in W, that is, N_J acts trivially on W_J. If $J = \varnothing$ or $J = S$, then clearly w_J is a special involution. So let us assume that $J \neq \varnothing, S$. We consider the different types of irreducible Coxeter groups in turn.

If W is of type A_n $(n \ge 1)$, then, by Remark 3.2, W_J is necessarily a direct product of components of type A_1. But if there is more than one such component, N_J permutes them non-trivially. Hence W_J is of type A_1 and the claim follows by Lemma 3.1.

If W is of type C_n $(n \ge 2)$, then, by Remark 3.2, W_J is a direct product of a component of type C_m, $0 \le m < n$ and further components of type A_1. As before there cannot be more than one component of type A_1. Hence W_J is of type C_m or of type $C_m \times A_1$ for some $m < n$. In any case, W_J has a component of type C_m.

Let

$$\Phi = \{\pm 2\varepsilon_i : 1 \le i \le n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\}$$

be the root system of type C_n. Consider the maximal rank subsystem Φ' of type $C_m \times C_{n-m}$ consisting of the long roots $\{\pm 2\varepsilon_i : 1 \le i \le n\}$ and the short roots $\{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le m$ or $m+1 \le i < j \le n\}$. Let U_1 be the subspace of V spanned by $\varepsilon_1, \dots, \varepsilon_m$ and U_2 the subspace spanned by $\varepsilon_{m+1}, \dots, \varepsilon_n$. Then $U_2 \cap \Phi$ is a root system of type C_{n-m}. All the long roots $\pm 2\varepsilon_i$ of Φ are contained in Φ'. A short root $\pm\varepsilon_i \pm \varepsilon_j$ is either contained in Φ' or both its projections on U_1 and U_2 are proportional to a root in Φ'. By construction, the (-1)-eigenspace V_{-1} of w_J contains U_1. Hence every root that lies in U_1 or is proportional to a root in U_1 is also proportional to a root in V_{-1}. It remains to consider the roots in U_2. Without loss of generality we can now assume that $m = 0$. Then W_J is of type A_1 and the claim follows by Lemma 3.1.

If W is of type D_{2n+1} $(n \ge 2)$, then, by Remark 3.2, W_J is a direct product of an optional component of type D_{2m}, $1 \le m \le n$ and further components of type A_1. As before there cannot be more than one component of type A_1. And N_J acts non-trivially on a component of type D_{2m}. Hence W_J is of type A_1 and the claim follows by Lemma 3.1.

If W is of type D_{2n} $(n \ge 2)$, then, by Remark 3.2, W_J is a direct product of an optional component of type D_{2m}, $1 \le m < n$ and further components of type A_1. As before there cannot be more than one component of type A_1. The non-trivial action of the parabolic subgroup of type D_{2m+1} on a component of type D_{2m} then restricts the type of W_J to either A_1 or $D_{2(n-1)} \times A_1$. In the latter case, $V_1 \cap \Phi$ is a root system of type A_1 and so in both cases the claim follows by Lemma 3.1.

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If W is of type $I_2(m)$ ($m \geq 5$), then W_J is of type A_1 and the claim follows by Lemma 3.1.

Finally, if W has type E_6, E_7, E_8, F_4, H_3, or H_4, then the claim is established by inspection. □

REMARK 3.3. Felder and Veselov prove one implication of Theorem 2.3, namely that W_J is bulky if w_J is special in a case by case analysis [3, Proposition 10].

REMARK 3.4. Bulky parabolic subgroups can be easily classified. It turns out that if W has a central longest element, then w_J is central in W_J whenever W_J is bulky. Otherwise, W has bulky parabolic subgroups W_J which are not associated with a conjugacy class of involutions in W.

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