5

The Springer Correspondence

Sam Gunningham

Overview

The goal of these lectures is to introduce the audience to some of the key concepts and tools in the field of *geometric representation theory*, using the *Springer correspondence* as a motivating example.

- In the first lecture, we will go over the background necessary to state the Springer correspondence for an arbitrary semisimple Lie algebra.
- In the second lecture, we will study the notion of convolution in Borel–Moore homology and see how to apply it to the Springer correspondence.
- In the third lecture we will reframe these ideas in the language of perverse sheaves and intersection homology.

These notes are not intended as a detailed reference with complete proofs. Rather, they are designed to give a somewhat informal overview of the subject broadly aimed at new(ish) PhD students.

Acknowledgements I would like to thank Gwyn Bellamy for his generous help with the preparation of these lectures, and an anonymous referee for their useful comments. Any errors are, of course, my own.

Further Reading

Textbooks

- A good place to start is the textbook *Representation Theory and Complex Geometry* by Chriss and Ginzburg [4]. Chapters 2 and 3 form the basis for these lectures.
- There are many good textbooks on algebraic groups and Lie theory, e.g. Springer's book [18] is an appropriate choice.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

- For background on derived categories and perverse sheaves, there is the book of Dimca [6], and the (somewhat more technical) classic by Kashiwara and Schapira [12].
- For those looking for some further reading on geometric representation theory, the book *D-modules, Perverse Sheaves, and Representation Theory* [11] by Hotta, Takeuchi and Tanisaki and Takeuchi has good background on D-modules and perverse sheaves and a nice introduction to Kazhdan–Lusztig theory.

Other online resources There are plenty of other lecture notes, theses and the like available online. For example:

- Lecture notes by Zhiwei Yun on Springer theory and orbital integrals (see Lecture I):
  http://math.mit.edu/~zyun/ZhiweiYunPCMIv2.pdf
- Senior thesis of Dustin Clausen on the Springer correspondence:
  www.math.harvard.edu/media/clausen.pdf
- Survey of Julia Sauter on Springer theory (in a more general sense):
  https://arxiv.org/abs/1307.0973
- A great set of notes on perverse sheaves (including representation theoretic applications) by Konni Rietsch:
  https://arxiv.org/abs/math/0307349

Original papers Of course, there are also the original papers in which the subject was first developed. We give a partial list here: [17], [19], [13], [3], [16], [10], [14], [7], [15] (the introduction to this last paper of Shoji contains a nice overview of the history of the subject).

5.1 The Statement of the Springer Correspondence

The goal for this lecture We will start by stating the Springer correspondence in type A (i.e. for the symmetric group). Then we will review some of the necessary background from Lie theory to state the Springer correspondence in arbitrary type.

5.1.1 The Springer Correspondence in Type A

Motivation

Let n be a positive integer, and consider the following two sets:

- The set $\mathrm{Irrep}(S_n)$ of isomorphism classes of irreducible (complex) representations of the symmetric group S_n.
- The set Nilp_n of conjugacy classes of $n \times n$ nilpotent matrices.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

It is not too difficult to see that both these sets have cardinality equal to the set $\mathrm{Part}(n)$ of partitions of n. For example, we know that, in general, the set of irreducible representations of a finite group is in bijection with the set of conjugacy classes, and the conjugacy class of an element in the symmetric group is determined by its cycle type – a partition of n. On the other hand, conjugacy classes of nilpotent matrices are classified by their Jordan type – also a partition of n.

It is natural to ask if we can make this bijection explicit. That is, given a nilpotent conjugacy class can one construct a representation of the symmetric group?

In these lectures, we will discuss a *geometric* approach to this problem, first identified by Tonny Springer in the 1970s [17]. In this theory, the representation of the symmetric group will live in the cohomology of a certain algebraic variety (known as a Springer fibre) associated to a nilpotent matrix. (Recall that a square matrix A is said to be nilpotent if $A^N = 0$ for $N \gg 0$.)

Springer Fibres

To define the Springer fibre, let us recall that a (full) *flag* in the vector space $\mathbb{C}^n$ is defined to be a sequence of linear subspaces

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n$$

with $\dim(V_i) = i$. The set of all flags in $\mathbb{C}^n$ is denoted $\mathcal{F}\ell(n)$. This naturally sits inside a product of Grassmannians as a closed subspace, cut out by polynomial equations. In fact, it has the structure of a smooth projective algebraic variety (and thus a compact Kähler manifold).

Example 5.1 When $n = 2$, we observe that a flag is nothing more than a line in $\mathbb{C}^2$. Thus $\mathcal{F}\ell(2)$ is just the projective line $\mathbb{P}^1$.

The *Springer fibre* $\mathcal{F}\ell(n)^A$ associated to an $n \times n$ matrix A is the subspace of $\mathcal{F}\ell(n)$ consisting of flags $V_\bullet$ such that $A(V_i) \subseteq V_i$ for all $i = 0, \ldots, n$. We will see that the most interesting Springer fibres are those where A is nilpotent.

Example 5.2 If $A = 0$, the Springer fibre $\mathcal{F}\ell(n)^0$ is the entire flag variety $\mathcal{F}\ell(n)$.

Example 5.3 Suppose A is the Jordan normal form $n \times n$ matrix with a single Jordan block. Then $\mathcal{F}\ell(n)^A$ consists of a single point, namely the coordinate flag

$$\langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \langle e_1, e_2, e_3 \rangle \subseteq \cdots \subseteq \langle e_1, e_2, \ldots, e_n \rangle.$$

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

Springer fibers are typically singular and have multiple irreducible components, however, they are known to always be equidimensional – that is, every irreducible component has the same dimension $d(A)$. Specifically, if $\lambda = (\lambda_1 \leqslant \dots \lambda_r)$ is the partition corresponding to the lengths of the Jordan blocks of A, consider the dual partition $\mu = (\mu_1 \leqslant \cdots \leqslant \mu_s)$. Then the dimension of the Springer fiber is $\frac{1}{2}\sum_i \mu_i(\mu_i - 1)$ [16] II.5.5.

Example 5.4 Here is a slightly more involved example. Consider the case $n = 3$,

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Any flag preserved by A lies in one of the two following families:

$$\begin{array}{ccccccc} \langle e_1 \rangle & \subseteq & \langle e_1, \lambda e_2 + \mu e_3 \rangle & \subseteq & \langle e_1, e_2, e_3 \rangle, & \lambda, \mu \in \mathbb{C}, \\ \langle \lambda e_1 + \mu e_2 \rangle & \subseteq & \langle e_1, e_2 \rangle & \subseteq & \langle e_1, e_2, e_3 \rangle, & \lambda, \mu \in \mathbb{C}. \end{array}$$

Each of these families corresponds to a copy of $\mathbb{P}^1$ in the flag variety, and the two $\mathbb{P}^1$'s intersect at a single point $\big($corresponding to the flag $\langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \langle e_1, e_2, e_3 \rangle\big)$, see Figure 5.1.

The Springer Correspondence

Consider the top non-zero cohomology $H^{2d(A)}(\mathcal{F}\ell(n)^A)$. Cohomology here means singular cohomology of the underlying topological space in the classical topology with rational coefficients. This is a $\mathbb{Q}$-vector space whose dimension is equal to the number of irreducible components in the Springer fibre.

Theorem 5.5 (see e.g. [4], Theorem 3.6.2) *Let n be a positive integer.*

1 *For every nilpotent $n \times n$ matrix A, the vector space $H^{2d(A)}(\mathcal{F}\ell(n)^A)$ carries a natural S_n action, affording an irreducible representation of S_n.*

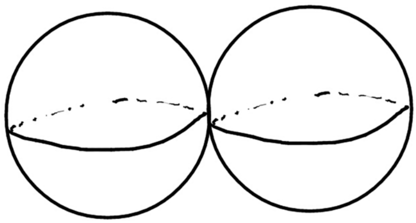


Figure 5.1 A Springer fibre in $\mathcal{F}\ell(3)$.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

2 *Each irreducible representation of S_n is isomorphic to $H^{2d(A)}(\mathcal{F}\ell(n)^A)$ for some nilpotent $n \times n$-matrix A. Moreover, the matrix A is uniquely determined up to conjugation.*

In particular, the theorem establishes a bijection between isomorphism classes of irreducible representations, and conjugacy classes of nilpotent matrices as desired.

Remark It is important to note that the action of S_n on the cohomology is not in general induced from an algebraic action on the Springer fibre itself. This is partly what makes the subject so interesting!

Example 5.6 In Example 5.4 we have that $H^2(\mathcal{F}\ell(3)^A)$ carries the unique two-dimensional irreducible representation of S_3. Try to convince yourself that this action cannot arise from automorphisms of $\mathcal{F}\ell(3)^A$.

The Springer representations have been constructed and interpreted in various contexts using convolution algebras, perverse sheaves, D-modules and vanishing cycles. As such, Springer theory provides a fantastic gateway to many of the key concepts and tools in contemporary geometric representation theory. The ideas we will see in these lectures appear all over the subject: in the theory of quiver varieties, cohomological Hall algebras, representations of finite groups of Lie type, Kazhdan-Lusztig theory and Coulomb branches, to name a few such areas.

5.1.2 The Lie Theoretic Set-up

In fact, Springer theory takes place in the wider context of *semi simple Lie algebras* (or algebraic groups) and their associated *Weyl groups*. The Springer correspondence in general exhibits an explicit bijection between the set of irreducible representations of the Weyl group and (a certain refinement of) the set of nilpotent orbits in the Lie algebra. The above example with symmetric group representations and nilpotent matrices corresponds to the special case in which the Lie algebra is $\mathfrak{sl}_n$ and the Weyl group S_n. In what follows we will give a brief outline of this set-up.

Semisimple Lie Algebras

A Lie algebra (over the complex numbers) is simple if it has no proper Lie ideals, and semisimple if it is a direct product of simple Lie algebras. It is quite remarkable that such a short and abstract definition leads to such a deep and intricate theory, as we will now describe.

The classical approach There are a number of ways of approaching the subject of semisimple Lie algebras. In the *classical* approach, we consider symmetries of vector spaces, possibly equipped with bilinear forms. This leads to the following list of examples:

- The *special linear group* SL_n consists of $n \times n$ matrices with determinant 1. Its Lie algebra $\mathfrak{sl}_n$ consists of matrices with trace 0. This is simple for $n \geqslant 2$.
- The *orthogonal group* O_n consists of $n \times n$ orthogonal matrices – those preserving the standard inner product on $\mathbb{C}^n$. Its Lie algebra $\mathfrak{so}_n$ consists of $n \times n$ skew-symmetric matrices. This is simple for $n \geqslant 5$. Moreover, $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$ and $\mathfrak{so}_3 \cong \mathfrak{sl}_2$.
- The *symplectic group* Sp_{2n} consists of $2n \times 2n$ symplectic matrices – those preserving the standard symplectic form on $\mathbb{C}^{2n}$:

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

  Its Lie algebra $\mathfrak{sp}_{2n}$ consists of $2n \times 2n$-matrices A such that $\Omega A + A\Omega = 0$. This is simple for all $n \geqslant 1$.

The root-theoretic approach In this route, we start by choosing a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (a maximal abelian subalgebra). The restriction of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ gives a decomposition in to 1-dimensional root spaces $\mathfrak{g}_\alpha$ (together with the fixed space $\mathfrak{h}$ itself):

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

The set $\Phi = \Phi(\mathfrak{g}, \mathfrak{h}) \subseteq \mathfrak{h}^*$ is called the set of roots of $\mathfrak{g}$. The real span E of the roots carries an inner product coming from the Killing form of $\mathfrak{g}$. It turns out that all the information about the semisimple Lie algebra $\mathfrak{g}$ can be encoded in terms of the Euclidean space E together with the set of roots Φ (this data is called the *root system* associated to $\mathfrak{g}$).

If one further specifies a choice of *positive roots* $\Phi_+ \subseteq \Phi$ then we obtain a triangular decomposition:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n},$$

where $\mathfrak{n}$ (respectively, $\mathfrak{n}_-$) is spanned by the root spaces of positive (respectively, negative) roots. Given the choice of positive roots, one may define the set $\Delta \subseteq \Phi_+$ of *simple roots* which form a basis of $\mathfrak{h}^*$.

Example 5.7 In the case $\mathfrak{g} = \mathfrak{sl}_n$, we can take $\mathfrak{h}$ to be the diagonal matrices. The set of roots $\alpha_{(i,j)}$ is indexed by pairs $(i, j) \in \{1, \ldots, n\}^2$, $i \neq j$. The root space $\mathfrak{g}_{\alpha_{i,j}}$ consists of matrices whose only possible non-zero entry is in the

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

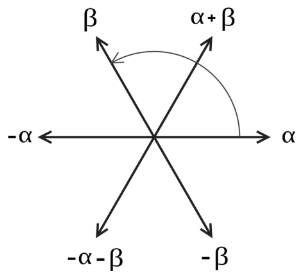


Figure 5.2 The root system of $\mathfrak{sl}_3$ (type A_2). The positive roots with respect to the simple roots $\{\alpha,\beta\}$ are $\{\alpha,\beta,\alpha+\beta\}$. (Image: Wikipedia commons https://commons.wikimedia.org/wiki/)

(i,j)-position. A standard choice for the set of positive roots is to take $\alpha_{(i,j)}$ with $i<j$. The simple roots are $\alpha_{(i,i+1)}$ for $i=1,\dots,n-1$. With this choice, $\mathfrak{n}$ (respectively $\mathfrak{n}_-$) becomes the set of strictly upper (respectively, strictly lower) triangular matrices.

Borel subalgebras and the canonical Cartan In the above presentation, we needed to pick a Cartan subalgebra $\mathfrak{h}\subseteq\mathfrak{g}$ to get started. Further choosing a subset of positive roots, we obtained a triangular decomposition $\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}$. The subspace $\mathfrak{b}:=\mathfrak{h}\oplus\mathfrak{n}$ is an example of a *Borel subalgebra*: a maximal solvable Lie subalgebra. In fact, any Borel subalgebra is G-conjugate to this one. There is another approach to this subject, where instead of choosing a Cartan and a set of positive roots, we rather consider all possible Borel subalgebras at once.

More precisely, given a Borel subalgebra $\mathfrak{b}\subseteq\mathfrak{g}$, we consider its nilpotent radical, the ideal $\mathfrak{n}(\mathfrak{b})=[\mathfrak{b},\mathfrak{b}]$ and the corresponding quotient $\mathfrak{H}(\mathfrak{b})=\mathfrak{b}/\mathfrak{n}(\mathfrak{b})$. Thus we get a short exact sequence:

$$0\to\mathfrak{n}(\mathfrak{b})\to\mathfrak{b}\to\mathfrak{H}(\mathfrak{b})\to 0. \tag{5.1.1}$$

This sequence is non-canonically split: choosing a splitting makes $\mathfrak{H}(\mathfrak{b})$ into a Cartan subalgebra of $\mathfrak{g}$ with a choice of positive roots determined by $\mathfrak{b}$.

On the other hand, it turns out that $\mathfrak{H}(\mathfrak{b})$ is actually independent of the choice of $\mathfrak{b}$ in the strongest sense. Namely, suppose $\mathfrak{b}'$ is another Borel subalgebra. Then we can choose $g\in G$ such that $Ad(g)(\mathfrak{b})=\mathfrak{b}'$ (as all Borels are conjugate), which defines an isomorphism:

$$Ad(g):\mathfrak{H}(\mathfrak{b})\cong\mathfrak{H}(\mathfrak{b}').$$

Crucially, this isomorphism is independent of the choice of g (this follows from the fact that B acts trivially on $\mathfrak{H}(\mathfrak{b})$).

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

We refer to $\mathfrak{H}$ ($= \mathfrak{H}(\mathfrak{b})$ for any Borel $\mathfrak{b}$) as the *canonical Cartan*. Moreover, choosing any splitting of $\mathfrak{H} = \mathfrak{H}(\mathfrak{b})$ into $\mathfrak{b}$ defines a root system in $\mathfrak{H}$ (together with a distinguished choice of positive roots); this root system is independent of choices. Thus we can talk about the Cartan and roots of $\mathfrak{g}$ without having to make any choices.

The Weyl Group

Now suppose $\mathfrak{g}$ is a semisimple Lie algebra. Let us also fix a connected linear algebraic group G with $\mathfrak{g} = \mathrm{Lie}(G)$. Thus G acts on $\mathfrak{g}$ via the adjoint representation (for matrix groups, this is simply the conjugation action).

The *Weyl group* is a certain finite group associated to $\mathfrak{g}$. It plays a central role in our story. There are also a number of different ways to approach its definition.

As a reflection group Recall that the root system on the canonical Cartan $\mathfrak{H}$ associated to $\mathfrak{g}$ determines a Euclidean vector space E together with a distinguished set of root hyperplanes. One definition of the Weyl group $\mathbb{W}$ is as the group generated by the reflections in the root hyperplanes. The reflections corresponding to simple roots give a set of generators for $\mathbb{W}$, giving $\mathbb{W}$ the structure of a Coxeter group. This leads to a presentation of $\mathbb{W}$ as follows:

$$\left\langle s_\alpha, \alpha \in \Delta \mid (s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \right\rangle,$$

where $m(\alpha,\beta)$ is a certain number in the set $\{1,2,3,4,5\}$ which records the angle $\angle$ formed by α and β according to the following table:

$$m(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 2 & \text{if } \alpha \perp \beta, \\ 3 & \text{if } \angle(\alpha,\beta) = 120^\circ, \\ 4 & \text{if } \angle(\alpha,\beta) = 135^\circ, \\ 5 & \text{if } \angle(\alpha,\beta) = 150^\circ. \end{cases}$$

It is a remarkable feature of root systems that these are the only possible angles that can occur. This is related to the *crystallographic restriction theorem* (see e.g. [1][Theorem 6.5.12]).

In particular, with these choices, every element $w \in \mathbb{W}$ has a well-defined notion of *length* $\ell(w)$ corresponding to the minimal number of terms appearing in any expression of w as a product of simple reflections (such an expression is called a *reduced word*). There is also a partial order $\leqslant$ on $\mathbb{W}$ characterized by the property that $v \leqslant w$ if and only if there is a reduced word expression for v that sits inside one for w.

Example 5.8 The Weyl group of $\mathfrak{sl}_n$ with respect to the Cartan of diagonal matrices is naturally identified with the symmetric group S_n, acting on $\mathfrak{h}$ by permuting the entries. The root-reflections $s_{\alpha_{(i,j)}}$ correspond to the transpositions $(i \quad j)$. Given the standard choice of positive roots we get the following presentation of S_n:

$$\left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{c} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad i = 1, \ldots n-2 \\ s_i s_j = s_j s_i, \quad i, j = 1, \ldots n-1, |i-j| \geqslant 2 \end{array} \right\rangle,$$

where $s_i = s_{\alpha_{i,i+1}}$ corresponds to the transposition $(i \quad i+1)$.

Example 5.9 The Weyl group of $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$ may both be identified with the *hyperoctahedral group* $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$, realized as the symmetries of an n-dimensional hypercube. The Weyl group of $\mathfrak{so}_{2n}$ is isomorphic to a certain index two subgroup of the hyperoctahedral group, realized as the symmetries of a demihypercube.

Via the normalizer of a Cartan Suppose now we fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, and let $H \subseteq G$ denote the centralizer of $\mathfrak{h}$ – this is a maximal torus of G with $\mathrm{Lie}(H) = \mathfrak{h}$. We define $W(\mathfrak{g}, \mathfrak{h})$ to be $N_G(\mathfrak{h})/H$. As H acts trivially on $\mathfrak{h}$, the action of $N_G(\mathfrak{h})$ naturally descends to an action of $W(\mathfrak{g}, \mathfrak{h})$ on $\mathfrak{h}$. If we further choose a Borel $\mathfrak{b}$ containing $\mathfrak{h}$ (thus giving an identification $\mathfrak{H} \cong \mathfrak{h}$), we obtain an isomorphism $W(\mathfrak{g}, \mathfrak{h}) \cong \mathbb{W}$.

The Flag variety Let $\mathcal{F}\ell = \mathcal{F}\ell(\mathfrak{g})$ denote the set of Borel subalgebras $\mathfrak{b} \subseteq \mathfrak{g}$. As any two Borels are G-conjugate, $\mathcal{F}\ell$ is naturally a homogeneous variety for G. The normalizer in G of a given Borel subalgebra $\mathfrak{b}$ is a so-called Borel subgroup $B \subseteq G$ with $\mathrm{Lie}(B) = \mathfrak{b}$. Thus, for any such choice of a basepoint in $\mathcal{F}\ell$, we get an isomorphism:

$$\mathcal{F}\ell \cong G/B.$$

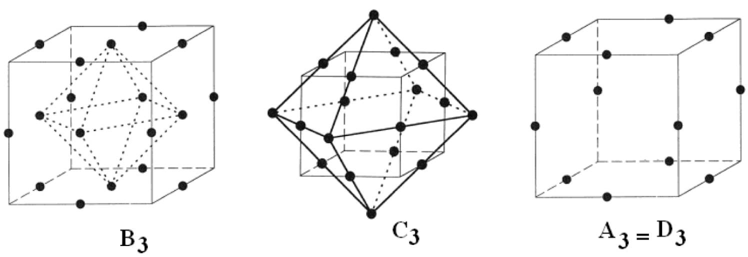


Figure 5.3 Root systems of type B_3, C_3, D_3. (Image: Wikipedia commons https://commons.wikimedia.org/wiki/ File:Root_vectors_b3_c3-d3.png)

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

In fact, $\mathcal{F}\ell$ carries the structure of a projective complex algebraic variety (in particular, a compact Kähler complex manifold) of (complex) dimension $m = \dim \mathfrak{n}$.

The Bruhat decomposition A fundamental result in this subject is that the orbits of the diagonal G-action on $\mathcal{F}\ell \times \mathcal{F}\ell$ are in bijection with the canonical Weyl group $\mathbb{W}$. To understand why this is, note that if we pick two Borels $\mathfrak{b}_1, \mathfrak{b}_2$, one can choose a Cartan subalgebra in their intersection. Then $\mathfrak{b}_1$ and $\mathfrak{b}_2$ correspond to two choices of a set of positive roots and are thus related by an element of $W(\mathfrak{g},\mathfrak{h})$ (since $W(\mathfrak{g},\mathfrak{h})$ acts simply transitively on the set of such choices). Given a pair of flags $\mathfrak{b}_1, \mathfrak{b}_2$, we say that they are in relative position $w \in \mathbb{W}$ if they lie in the G-orbit corresponding to w.

There are a number of equivalent expressions of this idea, known as the Bruhat decomposition. For example if we fix a Borel subalgebra $\mathfrak{b}$ with normalizer B, then we can identify G-orbits in $\mathcal{F}\ell \times \mathcal{F}\ell$ with B orbits in $\mathcal{F}\ell \cong G/B$ (or equivalently B-double cosets in G). If we further fix a Cartan $\mathfrak{h}$ with corresponding maximal torus H, then we obtain a locally closed decomposition:

$$G = \bigsqcup_{w \in W(\mathfrak{g},\mathfrak{h})} B\dot{w}B,$$

or equivalently

$$G/B = \bigsqcup_{w \in W(\mathfrak{g},\mathfrak{h})} B\dot{w}B/B,$$

where $\dot{w}$ denotes any lift of w to $N_G(H)$. The subsets $B\dot{w}B/B$ are called Bruhat cells – they are affine spaces of dimension $\ell(w)$. These Bruhat cells define a basis for the homology of $\mathcal{F}\ell$.

Example 5.10 Given a pair of flags $U_\bullet, V_\bullet$ in $\mathbb{C}^n$, the numbers:

$$n_{ij} = \dim\left(\frac{U_i \cap V_j}{U_{i-1} \cap V_j + U_i \cap V_{j-1}}\right)$$

define a permutation matrix and thus correspond to an element w of S_n. We say that $U_\bullet, V_\bullet$ are in relative position w.

The Characteristic Polynomial Map

An element $x \in \mathfrak{g}$ is called semisimple if it is contained in some Cartan subalgebra $\mathfrak{h}$. It follows that the set $\mathfrak{c}$ of semisimple conjugacy classes in $\mathfrak{g}$ are in bijection with W-orbits $\mathfrak{c} = \mathfrak{h}/W$ for any given Cartan $\mathfrak{h}$ (or better, in bijection with the canonical $\mathfrak{H}/\mathbb{W}$). It turns out that $\mathfrak{c}$ carries the natural structure of an

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

affine space: it is isomorphic to $\mathbb{C}^r$ where $r := \dim \mathfrak{h}$ is the *rank* of $\mathfrak{g}$. There is a natural G-invariant map:

$$\chi : \mathfrak{g} \to \mathfrak{c},$$

which is defined by taking an element $x \in \mathfrak{g}$ to the unique semisimple conjugacy class in the closure of $G \cdot x$. This is called the *characteristic polynomial map*.

In the language of algebraic geometry, χ is equal to the composite of the quotient map $\mathfrak{g} \to \mathfrak{g}//G := \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G)$ with the Chevalley isomorphism $\mathfrak{g}//G \cong \mathfrak{h}/W$, induced by the inclusion of a Cartan $\mathfrak{h} \hookrightarrow \mathfrak{g}$. In other words, the coordinate ring of $\mathfrak{c}$ is identified with the ring of G-invariant polynomial functions on $\mathfrak{g}$ (or alternatively, the ring of $\mathbb{W}$-invariant functions on $\mathfrak{H}$). The fact that $\mathfrak{c}$ is an affine space corresponds to the statement that $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$ is a polynomial ring, i.e. is isomorphic to $\mathbb{C}[a_1,\ldots,a_n]$ for some elements $a_1,\ldots,a_n$ (the analogues of the elementary symmetric functions). The degrees (minus 1) of the generators a_i are called the *exponents* of $\mathfrak{g}$.

Example 5.11 The characteristic polynomial map for $\mathfrak{sl}_n$ takes a matrix A to the collection of coefficients of its characteristic polynomial $p_A(t)$ (ignoring the coefficient of t^{n-1}, which is zero by definition) (thought of as an element of the affine space $\mathbb{C}^{n-1}$). Thus, the fibres of χ consist of matrices with a fixed characteristic polynomial (or equivalently, a fixed (multi)set of eigenvalues, counted with multiplicity). In each such fibre, the G-orbits are parameterized by the possible minimal polynomials; if the minimal polynomial has distinct roots, the element is semisimple (i.e. diagonalizable); if the minimal polynomial is equal to the characteristic polynomial, the element is regular (i.e. has maximal size Jordan blocks).

Each fibre of χ is a finite union of G-orbits in $\mathfrak{g}$, and each contains a unique closed orbit (consisting of semisimple elements) and a unique open orbit (consisting of so-called *regular* elements). In particular, the central fibre

$$\mathcal{N} = \chi^{-1}(0)$$

is called the *nilpotent cone* of $\mathfrak{g}$ and its elements are called nilpotent.

Remark The multiplicative group $\mathbb{C}^\times$ naturally acts on $\mathfrak{g}$ (as it does on any vector space). There is a corresponding action on $\mathfrak{c}$ (with certain weights) making χ equivariant and with fixed point $0 \in \mathfrak{c}$. It follows that $\mathcal{N}$ is a *cone*: it carries an action of $\mathbb{C}^\times$ with a unique fixed point $0 \in \mathfrak{g}$.

At the other extreme, the generic fibres of χ consist of a single G-orbit which is both regular and semisimple. Such elements are naturally called regular semisimple. The open subset of regular semisimple elements in $\mathfrak{g}$ is denoted

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

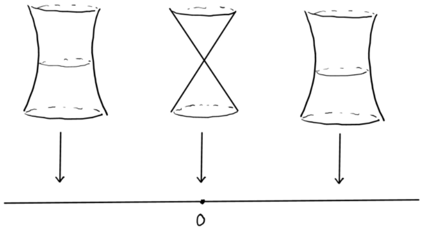


Figure 5.4 A cartoon of the characteristic polynomial map for $\mathfrak{sl}_2$.

$\mathfrak{g}^{rs}$. If $\mathfrak{h}$ is a Cartan subalgebra, the intersection $\mathfrak{g}^{rs} \cap \mathfrak{h}$ is denoted $\mathfrak{h}^{\mathrm{reg}}$; it coincides with the subset of $\mathfrak{h}$ where W acts freely, or equivalently with the complement of the root hyperplanes in $\mathfrak{h}$.

Example 5.12 Regular semisimple elements of $\mathfrak{sl}_n$ are precisely those with distinct eigenvalues. The nilpotent elements are nilpotent matrices in the usual sense (which are characterized by the property that all their eigenvalues are zero, or equivalently, their characteristic polynomial is equal to t^n).

Example 5.13 For $\mathfrak{g} = \mathfrak{sl}_2$ we can be more explicit. The map χ may be identified with

$$\mathfrak{sl}_2 \to \mathbb{C},$$
$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto -\det(A) = a^2 - bc.$$

There are two possibilities: if $d = -\det(A) \neq 0$, then the eigenvalues are distinct and A is regular semisimple. In this case $\chi^{-1}(d)$ is a smooth quadric consisting of a single G-orbit. On the other hand $\chi^{-1}(0)$ is a singular conic which is a union of two orbits: the zero orbit $\{0\}$ and the regular nilpotent orbit.

The Killing–Cartan–Dynkin Classification and Exceptional Types

Using the axiomatics of root systems, the simple Lie algebras were classified by Killing and Cartan at the end of the 19th century, and later refined by Dynkin. According to Wikipedia, “the classification is widely considered one of the most elegant results in mathematics” – I would be inclined to agree! The classification consists of four infinite families which correspond to the classical Lie algebras as follows:

- $\mathfrak{sl}_{n+1}$ is type A_n;
- $\mathfrak{so}_{2n+1}$ is type B_n;

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

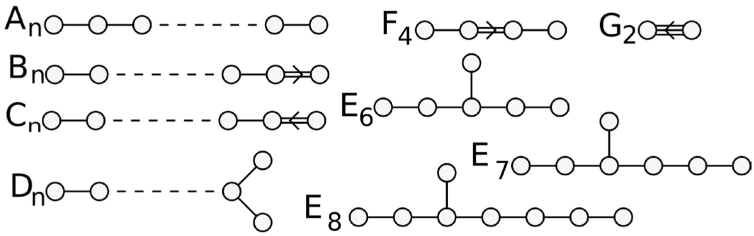


Figure 5.5 The finite Dynkin diagrams. (Image: Wikipedia commons https://en.wikipedia.org/wiki/File: Finite_Dynkin_diagrams.svg)

- $\mathfrak{sp}_{2n}$ is type C_n;
- $\mathfrak{so}_{2n}$ is type D_n.

It turns out there are precisely five more "exceptional" Lie algebras which are denoted by E_6, E_7, E_8, F_4, G_2 (the index always refers to the rank). With a bit of work, one can fit the exceptional groups into the classical paradigm using the octonions – see e.g. [2].

We can encode the isomorphism type of $\mathfrak{g}$ in a certain graph called the *Dynkin diagram*. The nodes of the Dynkin diagram correspond to the simple roots, and the number of edges between two nodes is determined by their angle (it is equal to $m(\alpha,\beta)-2$ from the above table). In the case of multiple edges (types B,C,F,G) the two roots have different length, in which case one also draws an arrow going from the long root to the short root.

Example 5.14 The Weyl group of type G_2 is a dihedral group of order 12, acting naturally on the 2-dimensional Cartan $\mathfrak{H}$ as symmetries of a hexagon.

Example 5.15 The Weyl group of type E_8 has order 696729600. It has the unique finite simple group of order 174182400 as a composition factor.

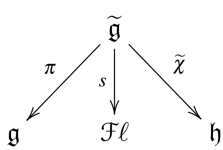
5.1.3 The Springer Correspondence in General Type

The Grothendieck–Springer Simultaneous Resolution

Define the *Grothendieck–Springer space* as follows:

$$\widetilde{\mathfrak{g}} = \{(x,\mathfrak{b}) \in \mathfrak{g} \times \mathcal{F}\ell \mid x \in \mathfrak{b}\}.$$

There are natural maps as indicated below.



The map s remembers the flag $\mathfrak{b}$ and forgets the element x. This realizes $\widetilde{\mathfrak{g}}$ as a kind of tautological vector bundle over $\mathcal{F}\ell$ (the fibre over the point corresponding a Borel subalgebra $\mathfrak{b}$ is $\mathfrak{b}$ itself).

Remark If we fix a preferred Borel subgroup $B \subseteq G$ with corresponding subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$, then we can write:

$$\widetilde{\mathfrak{g}} = G \times^B \mathfrak{b}.$$

In other words, $\widetilde{\mathfrak{g}}$ is the associated adjoint vector bundle to the B-torsor $G \to G/B \cong \mathcal{F}\ell$.

On the other hand, we can forget the flag and remember x to define the map π. The fibres $\mathcal{F}\ell^x := \pi^{-1}(x)$ are called *Springer fibres*. Explicitly, we have:

$$\mathcal{F}\ell^x = \{\mathfrak{b} \in \mathcal{F}\ell \mid x \in \mathfrak{b}\} \subseteq \mathcal{F}\ell.$$

In other words $\mathcal{F}\ell^x$ is the collection of Borel subalgebras which contain x. More on this later.

Finally, the map $\widetilde{\chi}$ is defined as follows. Given a Borel $\mathfrak{b} \subseteq \mathfrak{g}$, recall that $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ is identified with the canonical Cartan $\mathfrak{H}$. The map $\widetilde{\chi}$ is defined by taking $(x,\mathfrak{b}) \in \widetilde{\mathfrak{g}}$ to $x \operatorname{mod}[\mathfrak{b},\mathfrak{b}] \in \mathfrak{H}$.

Remark Fixing a preferred Borel B again, and writing $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$ for the nilpotent radical, we see that there is a short exact sequence of associated vector bundles:

$$G \times^B \mathfrak{n} \to G \times^B \mathfrak{b} \to G \times^B \mathfrak{H}.$$

Note that B acts trivially on $\mathfrak{H}$, so the right-most term is canonically trivial (this is one way to think about the well-definedness of the canonical Cartan):

$$G \times^B \mathfrak{H} \cong G/B \times \mathfrak{H}.$$

The resulting morphism $\widetilde{\mathfrak{g}} \to G/B \times \mathfrak{H}$ is precisely $(s, \widetilde{\chi})$.

We think of $\widetilde{\chi}$ as a lift of the characteristic polynomial map χ from $\mathfrak{c} = \mathfrak{H}/\mathbb{W}$ to $\mathfrak{H}$, as indicated by the following diagram.

$$\begin{array}{ccc} \widetilde{\mathfrak{g}} & \xrightarrow{\widetilde{\chi}} & \mathfrak{H} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathfrak{g} & \xrightarrow[\chi]{} & \mathfrak{c} \end{array} \qquad (5.1.2)$$

Note that $\widetilde{\mathfrak{g}}$ carries an action of G making the maps π and s equivariant, and $\widetilde{\chi}$ invariant.

We have the following key property:

Proposition 5.16 *The map $\widetilde{\chi}$ is a smooth morphism. In particular, the fibres $\widetilde{\chi}^{-1}(t)$ are all smooth.*

The reason this fact is cool is that the original map χ is not smooth – one of the fibres is the nilpotent cone which is generally singular. The diagram (5.1.2) above is referred to as the Grothendieck–Springer simultaneous resolution, because it simultaneously resolves the singularities of (the fibres of) the morphism χ.

Regular semisimple Springer fibres If x is regular semisimple, it is not too hard to show that it is contained in exactly $|\mathbb{W}|$-many Borel subgroups (namely, those Borels containing $\mathfrak{h} = C_{\mathfrak{g}}(x)$). In fact, the collection of such Borels is naturally a torsor for $\mathbb{W}$.

We have the following relative version of this fact:

Proposition 5.17 *There is a free and properly discontinuous action of $\mathbb{W}$ on the locus $\widetilde{\mathfrak{g}}^{rs}$ such that the map*

$$\pi^{rs} : \widetilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$$

is identified with the quotient. In other words π^{rs} is a $\mathbb{W}$-Galois covering.

The Springer Resolution

We have seen that the Springer fibres of regular semisimple elements are boring: just discrete sets. At the other end of the spectrum we have the nilpotent cone.

Consider the space

$$\widetilde{\mathcal{N}} := \widetilde{\chi}^{-1}(0) \subseteq \widetilde{\mathfrak{g}}.$$

Note that for an element $(x, \mathfrak{b})$ we have $\widetilde{\chi}(x, \mathfrak{b}) = 0$ if and only if $x \in \mathcal{N}$, or equivalently $x \in \mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$. In particular, $\widetilde{\mathcal{N}}$ is a vector bundle over $\mathcal{F}\ell$ whose fibre over $\mathfrak{b}$ is $\mathfrak{n}(\mathfrak{b})$.

Restricting π to $\widetilde{\mathcal{N}}$ we get a map

$$\rho : \widetilde{\mathcal{N}} \to \mathcal{N}$$

called the *Springer resolution*. The following proposition establishes the basic properties of the Springer resolution:

Proposition 5.18 *The map ρ is a resolution of singularities. That is:*

1. *The variety $\widetilde{\mathcal{N}}$ is smooth (i.e. non-singular).*
2. *The map ρ is proper (i.e. has compact fibers).*
3. *There is a dense open subset $U \subseteq \mathcal{N}$ such that $\rho|_{\rho^{-1}(U)}$ is an isomorphism.*

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

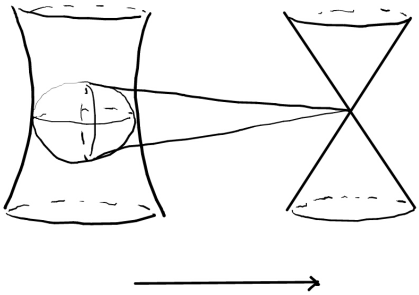


Figure 5.6 A cartoon of the Springer resolution for $\mathfrak{sl}_2$.

The first two statements are clear from what has already been established. For the third, one must show that a regular nilpotent element is always contained in a unique Borel subalgebra.

Remark The Springer resolution has a nice symplectic geometric interpretation. Namely there is a natural identification

$$\widetilde{\mathcal{N}} \cong T^*\mathcal{F}\ell$$

as G-spaces, giving $\widetilde{\mathcal{N}}$ the structure of a Hamiltonian G-space. The map ρ is identified with the moment map

$$T^*\mathcal{F}\ell \to \mathfrak{g}^* \cong \mathfrak{g}.$$

Example 5.19 If $\mathfrak{g} = \mathfrak{sl}_2$, then $\widetilde{\mathcal{N}}$ is the total space of the line bundle $\mathcal{O}(-2)$ over $\mathcal{F}\ell(2) = \mathbb{P}^1$. The map

$$\rho : \widetilde{\mathcal{N}} \to \mathcal{N}$$

just crushes the zero section $\mathbb{P}^1 \subseteq \widetilde{\mathcal{N}}$ to a point.

The following result establishes the key algebro-geometric properties of nilpotent Springer fibres.

Theorem 5.20 (See e.g. [4], Corollary 3.3.24, Remark 3.3.26) *The Springer fibres $\mathcal{F}\ell^x$ for $x \in \mathcal{N}$ are connected and equidimensional (i.e. all the irreducible components have the same dimension). The dimension $d(x)$ is given by the following formula:*

$$d(x) = \frac{1}{2}\dim(C_G(x) - r) = \dim \mathcal{F}\ell - \frac{1}{2}\dim G \cdot x,$$

where $C_G(x)$ denotes the stabilizer and $G \cdot x$ denotes the orbit for the adjoint G-action.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

Component Groups

There is one more ingredient required to precisely state the Springer correspondence for arbitrary $\mathfrak{g}$. For each $x \in \mathfrak{g}$ let $A_G(x) = C_G(x)/C_G(x)^\circ$ denote the component group of the centralizer of x in G. It is a finite group.

One shows that the natural action of $C_G(x)$ on $\mathcal{F}\ell^x$ descends to an action of $A_G(x)$ on $H^*(\mathcal{F}\ell^x)$ (preserving the grading). If σ denotes an irreducible representation of $A_G(x)$ and V is any other representation, we denote by

$$V_\sigma := \mathrm{Hom}_{A_G(e)}(\sigma, V)$$

the corresponding multiplicity space.

Remark The group $A_G(x)$ really depends on the choice of group G, not just the Lie algebra $\mathfrak{g}$. In practice, for the purposes of the Springer correspondence, we can take $G = G_{ad}$, the adjoint group, for which the group $A_G(x)$ is smallest.

Remark Recall that any finite dimensional representation of a finite group is a direct sum of irreducible representations. The multiplicity space precisely measures the multiplicity of the given irreducible σ in this decomposition.

The Statement of the Springer Correspondence for Semisimple Lie Algebras

We may now finally state the following:

Theorem 5.21 (See e.g. [4], Theorem 3.5.7) *Let $x \in \mathcal{N}$ be a nilpotent element.*

1 There is an action of $\mathbb{W}$ on $H^*(\mathcal{F}\ell^x)$, preserving the grading, and commuting with the action of $A_G(x)$.
2 For each irreducible representation σ of $A_G(x)$, the multiplicity space

$$H^{2d}(\mathcal{F}\ell^x)_\sigma$$

is (either zero, or is) an irreducible representation of $\mathbb{W}$. Moreover, up to isomorphism, every irreducible representation appears in this way for a unique pair (x, σ) up to G-conjugation.

Remark Not every irreducible representation of $A_G(x)$ appears in this correspondence. If we allow for G to be the simply connected form then the correspondence is already not one-to-one for $\mathfrak{sl}_2$. There is a beautiful generalization of the Springer correspondence due to Lusztig [14] (called, surprisingly, the generalized Springer correspondence), which accounts for these missing elements in terms of representations of certain other Weyl groups associated to other root systems.

Remark Assuming that all the representations of $A_G(x)$ are defined over $\mathbb{Q}$ (which is the case if we take G to be the adjoint form) then we get that all the representations of $\mathbb{W}$ are also defined over $\mathbb{Q}$. This was not known for all Weyl groups prior to Springer's work.

Example 5.22 (The zero orbit) If $x = 0 \in \mathfrak{g}$ then the Springer fibre $\mathcal{F}\ell^0$ is the entire flag variety $\mathcal{F}\ell$. In this case, the action of $\mathbb{W}$ on $H^*(\mathcal{F}\ell)$ can be described more explicitly as follows. Let H be a maximal torus in G and B a Borel subgroup containing H. There is a map

$$p : G/H \to G/B \cong \mathcal{F}\ell.$$

On the one hand G/H is naturally acted on by $\mathbb{W} \cong W(\mathfrak{g}, \mathfrak{h}) = N_G(H)/H$. On the other hand the map p is a fibration with contractible fibres so induces an isomorphism $H^*(G/H) \cong H^*(G/B)$.

Remark Another way to see this fact is to identify G/B with G_{cpt}/H_{cpt} where G_{cpt} is a maximal compact subgroup of G and $H_{cpt} = H \cap G_{cpt}$ a maximal torus. Then we have $W = N_{G_{cpt}}(H_{cpt})/H_{cpt}$ acts directly on $\mathcal{F}\ell \cong G_{cpt}/H_{cpt}$; the catch is that this action is not holomorphic – it does not respect the complex structure! For example, in the case $\mathfrak{sl}_2$, this action is the antipodal action on $\mathbb{P}^1 \cong S^2$.

It is relatively easy to see that $H^*(G/B)$ has a basis indexed by $w \in \mathbb{W}$, where the degree is given by the length $\ell(w)$. In fact, there is a graded ring isomorphism to the *coinvariant algebra*

$$H^*(G/B) \cong \mathbb{C}[\mathfrak{h}]/\left(\mathbb{C}[\mathfrak{H}]_+^{\mathbb{W}}\right).$$

One verifies that $H^*(G/B)$ is isomorphic to the regular representation of $\mathbb{W}$. The top degree part $H^{2m}(G/B)$ carries the sign character of $\mathbb{W}$.

Example 5.23 (The regular orbit) At the other extreme if $x \in \mathcal{N}$ is regular, then $\mathcal{F}\ell^x \cong pt$. In this case $H^0(\mathcal{F}\ell^x)$ carries the trivial representation.

Example 5.24 (The subregular orbit) One can show that there is a unique G-orbit in $\mathcal{N}$ of dimension $2m-2$. This is called the subregular orbit. For $x \in \mathcal{N}$ subregular, the Springer fibre $\mathcal{F}\ell^x$ is 1-dimensional, i.e. a (complex) curve. It turns out that it is always a union of $\mathbb{P}^1$'s intersecting according to a certain graph. In the simply laced case (that is $\mathfrak{g}$ is one of the types A, D, or E in the Cartan–Killing–Dynkin classification) this graph is precisely the Dynkin diagram of $\mathfrak{g}$. In the non-simply laced case, one can associate another semisimple Lie algebra $\mathfrak{g}'$ which is simply laced, such that the Dynkin diagram of $\mathfrak{g}$ is obtained from that of $\mathfrak{g}'$ by "folding". Then the graph associated to $\mathcal{F}\ell^x$ is precisely the Dynkin diagram of $\mathfrak{g}'$. Moreover, the diagram automorphism giving rise to the folding is precisely implemented by the action of $A_G(x)$. In general, one can show that the Springer representation $H^2(\mathcal{F}\ell^x)^{A_G(x)}$ associated to subregular x and the trivial representation of $A_G(x)$ is isomorphic to the reflection representation $\mathfrak{h}$.

5.2 Springer Theory via Convolution

The goal for this lecture Last time, we claimed that there is a natural action of the Weyl group on the cohomology of Springer fibres even though the Weyl group does not act on the Springer fibres themselves. So where does this action come from?

In this lecture we will discuss one approach to this problem using convolution in Borel–Moore homology. We will divide the problem into two steps:

1 Construct an algebra A which naturally acts on the cohomology of Springer fibres.
2 Find an algebra isomorphism $\mathbb{Q}[W] \cong A$.

The first part of the lecture will be spent discussing the general properties of Borel–Moore homology. For further details, see [4] Chapter 2.6.

5.2.1 Generalities on Borel–Moore Homology

The Definition

Borel–Moore homology is a certain homology theory for topological spaces. For simplicity, in this section the word *space* shall refer to a suitably nice topological space, say homeomorphic to the complement of a sub CW-complex in a CW complex. Most of the spaces we will consider will be complex algebraic varieties, which all satisfy this condition. If X is a space, $H^*(X)$ (respectively $H_*(X)$) will always denote the singular cohomology (respectively homology) with rational coefficients.

Informally, one can think of a Borel–Moore k-chain on a space X as a possibly non-compact version of an ordinary k-chain. If X is compact then a Borel–Moore chain is just an ordinary chain (and thus Borel–Moore homology agrees with ordinary homology). Borel–Moore homology arises naturally in the study of Poincaré duality in the following form:

Theorem 5.25 (Poincaré duality for Borel–Moore homology) *If X is a smooth oriented manifold of dimension d (not necessarily compact), there is an isomorphism*

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

$$H_k^{BM}(X) \cong H^{d-k}(X).$$

Equivalently, there is a perfect pairing (called the intersection pairing*)*

$$H_k^{BM}(X) \otimes H_{d-k}(X) \to \mathbb{Q}.$$

There are a few different approaches towards giving a precise definition of Borel–Moore homology. We list some of these below:

1 A singular Borel–Moore chain may be defined as a locally finite sum of singular simplices (i.e. possibly infinite sums which are finite when intersected with any compact subset).

2 If $X \hookrightarrow M$ is an embedding into a closed oriented n-manifold, then

$$H_k^{BM}(X) = H^{n-k}(M, M-X).$$

3 We have

$$H_k^{BM}(X) = H_k(X_+, \{\infty\}),$$

where $X_+ = X \cup \{\infty\}$ is the one-point compactification.

4 $H_*^{BM}(X)$ is the sheaf (hyper)cohomology of the Verdier dualizing complex ω_X (more on this in the next lecture).

Example 5.26 Using any of the above as a definition, we may compute

$$H_k^{BM}(\mathbb{R}^n) = \begin{cases} \mathbb{Q} & \text{if } k = n; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, Borel–Moore homology (like its dual notion, compactly supported cohomology) is not a homotopy invariant (though it is of course a homeomorphism invariant).

Example 5.27 Let us consider the space

$$X = S^1 \times \mathbb{R}.$$

We have:

$*$	$H_*^{BM}(X)$	$H_*(X)$
0	0	$\mathbb{Q}$
1	$\mathbb{Q}$	$\mathbb{Q}$
2	$\mathbb{Q}$	0

One can represent the generators of these groups as (locally finite) cycles. Namely, the generator for H_2^{BM} is the entire space X and the generator for H_1^{BM} is the vertical line $\{*\} \times \mathbb{R}$. Note that this 1-cycle is transverse to the generator for $H_1(X)$. This reflects the perfection of the intersection pairing in the Poincaré duality theorem. See Figure 5.7.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

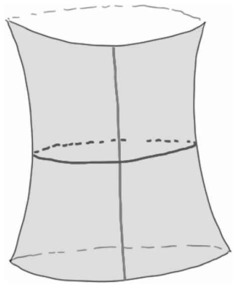


Figure 5.7 The cylinder.

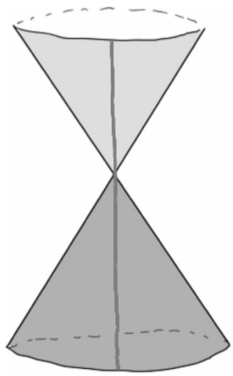


Figure 5.8 The cone.

Example 5.28 Now let $Y = S^1 \times \mathbb{R}/S^1 \times \{0\}$ be a cone. Then we have:

$*$	$H_*^{BM}(Y)$	$H_*(Y)$
0	0	$\mathbb{Q}$
1	$\mathbb{Q}$	0
2	$\mathbb{Q}^2$	0

This time there are two generators in H_2^{BM} represented by the two components of the cone. Note that $H_*^{BM}(Y)$ is quite large even though Y is contractible! See Figure 5.8.

Functoriality

Recall that for a map $f : X \to Y$ of topological spaces, we get an induced push-forward map on homology and an induced pullback map on cohomology. In

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

more categorical terms, homology is *covariantly* functorial and cohomology is *contravariantly* functorial. For Borel–Moore homology (as for compactly supported cohomology) the functoriality is slightly more complicated: sometimes there is a pullback, sometimes there is a pushforward according to the nature of the map f.

Here are the key examples of this functoriality. Most of these can be proved using either definition 2 or 3 below – see Chriss–Ginzburg [4], Chapter 2.6. Alternatively, one can use definition 4 together with the six operations formalism to be discussed later.

Proposition 5.29 *Suppose $f : X \to Y$ is a map of spaces.*

1. *If $f : X \to Y$ is a proper map (i.e. the preimage of a compact set is compact), there is a pushforward map:*
$$f_* : H^{BM}(X) \to H^{BM}(Y).$$
2. *If $f : X \to Y$ is an open embedding there is a restriction map:*
$$f^! : H^{BM}(Y) \to H^{BM}(X).$$
3. *If $f : X \to Y$ is an oriented fibration of relative complex dimension d (that is, a locally trivial fibration, whose fibres are oriented d-manifolds, and the transition maps preserve the orientation), there is a pullback map:*
$$f^! : H_k^{BM}(Y) \to H_{k+d}^{BM}(X).$$
4. *If $f : X \to Y$ is an oriented embedding of a manifold of codimension d (i.e. the normal bundle is oriented), then there is a pullback map:*
$$f^! : H_k^{BM}(Y) \to H_{k-d}^{BM}(X).$$

Remark We will need something a little bit stronger than the last point. Suppose we have a cartesian diagram of spaces (this means that $\widetilde{X}$ is isomorphic to the fibre product $X \times_Y \widetilde{Y}$ such that the maps $\widetilde{f}$ and $\widetilde{g}$ become identified with the projections):

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle \widetilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Suppose also that f is an oriented embedding of a submanifold of codimension d, so that $f^!$ makes sense. Then there is a pullback map for the base change:

$$\widetilde{f}^! : H_k^{BM}(\widetilde{Y}) \to H_{k-d}^{BM}(\widetilde{X}).$$

Properties and Structures in Borel–Moore Homology

This kind of functoriality may seem strange at first, but it manifests quite naturally in certain situations.

Long exact sequence of an open-closed decomposition For example, suppose $i : Z \hookrightarrow X$ is the inclusion of a closed subset and $j : U = X - Z \hookrightarrow X$ is the inclusion of its open complement. Then i is proper and j is an open embedding. Thus we get a sequence of maps as follows.

$$H_*^{BM}(Z) \xrightarrow{i_*} H_*^{BM}(X) \xrightarrow{j^!} H_*^{BM}(U)$$

In fact, one can further show that these maps come from a short exact sequence of complexes at the chain level. Thus there is an associated long exact sequence at the level of homology:

$$\ldots H_{k+1}^{BM}(U) \xrightarrow{\partial} H_k^{BM}(Z) \xrightarrow{i_*} H_k^{BM}(X) \xrightarrow{j^!} H_k^{BM}(U) \xrightarrow{\partial} H_{k+1}^{BM}(Z) \to \ldots$$

Remark Suppose we can partition X as a union

$$X = \bigsqcup_\alpha X_\alpha$$

of locally closed subsets X_α, with the property that the closure of each X_α is a union of X_β. Suppose also that $H_*^{BM}(X)$ is concentrated in entirely even degrees. Then repeatedly applying the long exact sequence of an open-closed decomposition, and noting that all the boundary maps must vanish, we obtain:

$$H_*^{BM}(X) = \bigoplus_\alpha H_*^{BM}(X_\alpha).$$

In particular, this works when X_α is an affine paving, i.e. each X_α is isomorphic to an affine space $\mathbb{C}^k$. This is the case for flag varieties – the affine paving is given by Schubert cells. Less obviously, this is also true for Springer fibres – see [5].

Fundamental classes Suppose U is an oriented d-manifold. Then item 3 of Proposition 5.29 gives us a map:

$$p^! : \mathbb{Q} = H_0^{BM}(pt) \to H_d^{BM}(U).$$

The element $[U] := p^!(\mathbb{Q}) \in H_d^{BM}(U)$ is called the *fundamental class* of the manifold U.

Now suppose that $U \subseteq X$ is embedded as an open dense subset such that the complement $X - U$ has (real) codimension 2 in X (for example U could be the smooth locus of an irreducible complex algebraic variety). Then, by the long exact sequence associated to the open closed decomposition, the restriction map

$$H_d^{BM}(X) \to H_d^{BM}(U)$$

is an isomorphism. It follows that there is a unique element $[X] \in H_d^{BM}(X)$ which maps to $[U]$. We will also refer to this as the fundamental class of X.

Example 5.30 Suppose Z is an algebraic variety of pure complex dimension n, that is, all the irreducible components $Z_1, \dots, Z_k$ of Z have dimension n. Then the fundamental classes $[Z_1], \dots, [Z_k]$ form a basis for $H_{2n}^{BM}(Z)$.

Specialization Given a suitable family of spaces X_t, it is possible to specialize Borel–Moore classes from the generic fibre to the special fibre. More precisely, let us fix a manifold S with basepoint s_0. Suppose we have a map of spaces $f : X \to S$ such that it is a locally trivial fibration over $S^* = S - \{s_0\}$. Thus we have a commutative diagram:

$$\begin{array}{ccccc} X_0 & \hookrightarrow & X & \hookleftarrow & X^* \\ \downarrow & & \downarrow f & & \downarrow \\ \{s_0\} & \hookrightarrow & S & \hookleftarrow & S^* \end{array}$$

Then there is a natural map:

$$\mathrm{Sp}_{s \to 0} : H_k^{BM}(X^*) \to H_{k-d}^{BM}(X_0).$$

Let us explain how this works in the case $S = [0, \infty)$, $s_0 = 0$ (in fact the general map is constructed by reducing to this case). We assume $X^* \cong X_1 \times (0, \infty)$ is trivialized. In this setting the specialization map is just the boundary map in the long exact sequence associated to the open-closed decomposition $X = X^* \cup X_0$:

$$H_k^{BM}(X^*) \to H_{k-1}^{BM}(X^0).$$

Remark If we assume that $X^* \cong X_1 \times (0, \infty)$ is trivialized then we can interpret specialization as the map

$$H_{k-1}^{BM}(X_1) \to H_k^{BM}(X_1 \times (0, \infty)) \cong H_k^{BM}(X^*) \to H^{BM}(X_0).$$

Thus we are "specializing" a cycle in a generic fibre X_1 to the special fibre X_0.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

5.2.2 Convolution Algebras

The set-up Suppose X is a smooth manifold of dimension d with a proper map $f : X \to Y$. We define $Z = X \times_Y X$. Then we have a commutative diagram:

$$\begin{array}{ccccc} Z \times Z & \xleftarrow{\quad s \quad} & Z \times_X Z & \xrightarrow{\quad r \quad} & Z \\ \| & & \| & & \| \\ (X \times_Y X) \times (X \times_Y X) & \xleftarrow[(p_{12},p_{23})]{} & X \times_Y X \times_Y X & \xrightarrow[p_{13}]{} & X \times_Y X \end{array}$$

Here s is the base change of the diagonal embedding $X \to X \times X$ (of codimension d) and r is proper. Thus the functoriality of Borel–Moore homology defines for us a linear map called *convolution*:

$$* = r_* s^! : H_*^{BM}(Z) \otimes H_*^{BM}(Z) \cong H_*^{BM}(Z \times Z) \to H_{*-d}^{BM}(Z).$$

We denote by $H_*^{BM}(X)[-d]$ the graded vector space where we shift the grading so that $H_d^{BM}(Z)$ lies in degree 0.

The following result can be proved by hand in an elementary fashion, but it also falls out once enough functoriality machinery has been developed (the object Z itself is a monoid object in a suitable category of correspondences, which is a source category for the Borel–Moore homology functor).

Proposition 5.31 *The convolution product $*$ gives $H^{BM}(Z)[-d]$ the structure of a graded associative algebra.*

Semismall Morphisms

Let us assume for the moment that $f : X \to Y$ is a morphism of algebraic varieties. In general, $Z = X \times_Y X$ may be singular and reducible (in the algebro-geometric sense), with components of various dimensions $d = \dim_{\mathbb{R}}(X) \leqslant n \leqslant 2d = \dim_{\mathbb{R}}(X \times X)$. In particular the graded algebra $H_*^{BM}(X)[-d]$ may have graded components of positive and negative degrees.

If it happens that the dimension of Z is equal to the dimension of X (the minimal possible), we say that the map $f : X \to Y$ is *semismall*. In that case, we see that $H_*^{BM}(Z)[-d]$ is supported in positive degrees. Moreover, the degree zero component $H_d^{BM}(Z)$ has a basis given by the fundamental classes of irreducible components of Z. This will be the case in the example of interest to us.

Examples of Convolution

Example 5.32 (The double of a closed oriented manifold) Consider the case when $Y = pt$ and thus X is a compact d-manifold. In this case $Z = X \times X$ and we have

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

$$H_*^{BM}(Z)[-d] \cong H_*(X) \otimes H_*(X)[-d] \cong H_*(X) \otimes H^*(X) \cong \mathrm{End}(H^*(X)).$$

Here the first isomorphism is by the Künneth theorem and the fact that the Borel–Moore homology agrees with ordinary homology for compact spaces. The second isomorphism is by Poincaré duality. One can check that the convolution structure on $H_*^{BM}(X \times X)[-d]$ corresponds to composition of endomorphisms.

The following example will be useful for our study of the Springer correspondence.

Example 5.33 (Galois covers) Suppose a finite group W acts freely and properly discontinuously on an oriented d-manifold X and let $f : X \to Y = X/W$ be the quotient map. Then there is an identification:

$$Z = X \times_{X/W} X \cong W \times X.$$

In this case the convolution algebra gets identified with the *smash product*:

$$H_*^{BM}(Z)[-d] \cong H^*(X) \sharp \mathbb{Q}[W].$$

Here, the smash product means the algebra whose underlying vector space is $H^*(X) \otimes \mathbb{Q}[W]$ and the multiplication follows the rule as for semidirect products, that when you commute an element of w past a class in $H^*(X)$ you act by w on that class.

The Convolution Action on the Fibre Homology

Recall that $f : X \to Y$ is a proper map of spaces with X an oriented d-manifold. Now let us fix a point $y \in Y$ and consider the fibre $X_y = f^{-1}(y)$, a compact space. Consider the diagram

$$\begin{array}{ccccc} Z \times X_y & \xleftarrow{\quad s' \quad} & Z \times_X X_y & \xrightarrow{\quad r' \quad} & X_y \\ \| & & \| & & \| \\ (X \times_Y X) \times (X \times_Y \{y\}) & \xleftarrow{(p_{12}, p_{23})} & X \times_Y X \times_Y \{y\} & \xrightarrow{\quad p_{13} \quad} & X \times_Y \{y\} \end{array}$$

Again we have that r' is proper and s' is a base-change of the diagonal embedding of X. As above, we obtain a map:

$$H_*^{BM}(Z)[-d] \otimes H_*(X_y) \to H_*(X_y).$$

Again, this can be upgraded to the following statement.

Proposition 5.34 *The map defined above equips $H_*(X_y)$ with the structure of a graded $H_*(Z)[-d]$-module.*

5.2.3 The Steinberg Variety

Big vs Small

Recall from Section 5.1.3 the diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{N}} & \hookrightarrow & \widetilde{\mathfrak{g}} \\ \rho \downarrow & & \downarrow \pi \\ \mathcal{N} & \hookrightarrow & \mathfrak{g} \end{array}$$

The maps π, ρ are proper, $\widetilde{\mathfrak{g}}$ is smooth of (complex) dimension $d = \dim \mathfrak{g} = 2m + r$ and $\widetilde{\mathcal{N}}$ is smooth of (complex) dimension $2m = \dim \mathcal{N}$.

We define the *big Steinberg variety*

$$\mathrm{St}(\mathfrak{g}) := \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} = \{(x, \mathfrak{b}_1, \mathfrak{b}_2) \in \mathfrak{g} \times \mathcal{F}\ell \times \mathcal{F}\ell \mid x \in \mathfrak{b}_1 \cap \mathfrak{b}_2\},$$

and the *small (or nilpotent) Steinberg variety*

$$\mathrm{St}(\mathcal{N}) := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} = \{(x, \mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{N} \times \mathcal{F}\ell \times \mathcal{F}\ell \mid x \in \mathfrak{b}_1 \cap \mathfrak{b}_2\}.$$

According to the results of 5.2.2 we have:

Proposition 5.35 *Convolution equips $H_*^{BM}(\mathrm{St})[-2d]$ and $H_*^{BM}(\mathrm{St}(\mathcal{N}))[-4m]$ with a graded algebra structure. Moreover both algebras act canonically on the homology of Springer fibres $H_*(\mathcal{F}\ell^x)$ for $x \in \mathfrak{g}$.*

Let

$$A(\mathfrak{g}) = H_{2d}^{BM}(\mathrm{St}(\mathfrak{g})),$$

and

$$A(\mathcal{N}) = H_{4m}^{BM}(\mathrm{St}(\mathcal{N})).$$

These are algebras with respect to convolution. Each one naturally acts on the homology of Springer fibres. We will see that both of these algebras are in fact isomorphic to $\mathbb{Q}[\mathbb{W}]$, giving the desired action on the homology of Springer fibres.

Remark The way we will present things in this lecture, the isomorphisms are compatible and thus the actions of $\mathbb{W}$ defined using either $A(\mathfrak{g})$ or $A(\mathcal{N})$ are the same. However, we will see in the next lecture that there is another choice for the second isomorphism which causes the two actions to differ by the sign representation of $\mathbb{W}$.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

The Components of the Steinberg Variety

Recall that we have a G-equivariant map:

$$s = (s_1, s_2) : \mathrm{St}(\mathfrak{g}) \to \mathcal{F}\ell \times \mathcal{F}\ell,$$

which takes a triple $(x, \mathfrak{b}_1, \mathfrak{b}_2)$ to the pair of flags $(\mathfrak{b}_1, \mathfrak{b}_2)$. Thus $\mathrm{St}(\mathfrak{g})$ is partitioned according to the relative position of $\mathfrak{b}_1$ and $\mathfrak{b}_2$. Accordingly, for each $w \in \mathbb{W}$, we define

$$\mathrm{St}_w(\mathfrak{g}) = s^{-1}((\mathcal{F}\ell \times \mathcal{F}\ell)_w) = \{(x, \mathfrak{b}_1, \mathfrak{b}_2) \mid \mathfrak{b}_1 \text{ and } \mathfrak{b}_2 \text{ are in relative position } w\}$$

and similarly, define $\mathrm{St}_w(\mathcal{N})$.

Stratum by stratum, the Steinberg varieties are relatively easy to understand:

Proposition 5.36 *The projection morphisms*

$$s_w(\mathfrak{g}) : St_w(\mathfrak{g}) \to (\mathcal{F}\ell \times \mathcal{F}\ell)_w$$

and

$$s_w(\mathcal{N}) : St_w(\mathcal{N}) \to (\mathcal{F}\ell \times \mathcal{F}\ell)_w$$

naturally carry the structure of a vector bundle. The fibre of $s_w(\mathfrak{g})$ (respectively, $s_w(\mathcal{N})$) over a pair $(\mathfrak{b}_1, \mathfrak{b}_2)$ is $\mathfrak{b}_1 \cap \mathfrak{b}_2$ (respectively, $\mathfrak{n}(\mathfrak{b}_1) \cap \mathfrak{n}(\mathfrak{b}_2)$).

In particular, $\mathrm{St}_w(\mathfrak{g})$ *(respectively,* $\mathrm{St}_w(\mathcal{N})$*) is a smooth connected variety of dimension $d =$* $\dim(\mathfrak{g})$ *(respectively, of dimension* $\dim(\mathcal{N}) = 2m$*) for each w.*

Remark Recall that there is an isomorphism $\widetilde{\mathcal{N}} \cong T^*\mathcal{F}\ell$. Thus $\mathrm{St}(\mathcal{N}) = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$ sits inside $T^*(\mathcal{F}\ell \times \mathcal{F}\ell)$. As such the strata $\mathrm{St}_w(\mathcal{N})$ are identified with the conormal bundles to the orbits $(\mathcal{F}\ell \times \mathcal{F}\ell)_w$.

It follows from the proposition that $\mathrm{St}(\mathfrak{g})$ (respectively, $\mathrm{St}(\mathcal{N})$) itself is equidimensional of dimension d (respectively, $2m$) and the irreducible components are given by the stratum closures. In the terminology introduced in 5.2.2, the Springer resolution is *semismall*.

In particular, the algebras $A(\mathfrak{g})$ and $A(\mathcal{N})$ have bases given by fundamental classes of their components. We denote these bases by

$$\Lambda_w \in A(\mathfrak{g})$$

and

$$T_w \in A(\mathcal{N}),$$

as w ranges over the Weyl group $\mathbb{W}$.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

Convolution on the Big Steinberg

Theorem 5.37 *The fundamental classes* Λ_w *define an isomorphism of algebras* $\mathbb{Q}[\mathbb{W}] \cong A(\mathfrak{g})$. *In other words, we have*

$$\Lambda_v * \Lambda_w = \Lambda_{vw},$$

for all $v, w \in \mathbb{W}$.

Theorem 5.37 is proved by looking at the open subset

$$\mathrm{St}(\mathfrak{g}^{rs}) := \widetilde{\mathfrak{g}}^{rs} \times_{\mathfrak{g}^{rs}} \widetilde{\mathfrak{g}}^{rs}.$$

Recall from Proposition 5.17 that the morphism

$$\pi^{rs} : \widetilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$$

is a $\mathbb{W}$-Galois cover. Following Example 5.33 we see that there is a natural algebra isomorphism:

$$\mathbb{Q}[\mathbb{W}] \cong A(\mathfrak{g}^{rs}) := H_{2d}^{BM}(\mathrm{St}(\mathfrak{g}^{rs})).$$

On the other hand, one observes that the restriction map $A(\mathfrak{g}^{rs}) \to A(\mathfrak{g})$ is an isomorphism of algebras, respecting the fundamental classes of the components, which completes the argument.

Convolution on the Nilpotent Steinberg

While Theorem 5.37 gives an action of $\mathbb{W}$ on the homology of Springer fibres, in order to say something about the nilpotent Springer fibres, we need to understand how this action restricts over the nilpotent cone.

One might first hope then that the linear isomorphism

$$\mathbb{Q}[\mathbb{W}] \to A(\mathcal{N})$$
$$w \mapsto T_w$$

given by the basis T_w induces an algebra isomorphism. Unfortunately (or perhaps fortunately, as this fact underlies a lot of interesting mathematics!) this map is not an algebra isomorphism. That is,

$$T_v * T_w \neq T_{vw}$$

in general.

To obtain a basis that is compatible with convolution, one must specialize the basis Λ_w from the regular semisimple locus to the nilpotent Steinberg. This procedure is explained in detail in Chriss-Ginzburg [4], Chapter 3.4; we sketch some of the main ideas below.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

We let Λ_w^0 denote the elements of $A(\mathcal{N})$ obtained by specializing Λ_w from the big Steinberg. General properties of convolution in Borel–Moore homology can be applied to show that these elements respect the group multiplication in the desired manner. It remains to show that they form a basis. For this, we must compare them to the known basis given by the T_w.

Lemma 5.38 *For each $w, v \in \mathbb{W}$, let n_{vw} be defined by*

$$\Lambda_w^0 = \sum_{v \in \mathbb{W}} n_{vw} T_v.$$

Then

1 $n_{vw} = 0$ *if* $v \geqslant w$.
2 $n_{ww} = 1$ *for all* $w \in \mathbb{W}$.

Remark Though it is not obvious from the definitions, the numbers n_{vw} are in fact all non-negative integers.

The first claim in the lemma is easy to check: the specialization construction of Λ_w^0 takes place entirely in the closure $\overline{\mathrm{St}_w(\mathfrak{g})}$. The second claim is less clear, and requires a more careful analysis (see [4], Lemma 3.4.14).

The lemma implies that the matrix (n_{vw}) is upper triangular with 1's along the diagonal. In particular it is invertible. It follows that Λ_w^0 is a basis as required.

This leads to a proof of the following:

Theorem 5.39 *There is an algebra isomorphism:*

$$\mathbb{Q}[\mathbb{W}] \cong A(\mathcal{N}).$$

Once we have this, it is possible to directly prove the Springer correspondence, Theorem 5.20, using a similar kind of geometric analysis – see [4], Section 3.5. Alternatively, we will present another point of view next time using perverse sheaves.

5.3 Springer Theory via Perverse Sheaves

The goal for this lecture We would like to combine the homology of Springer fibres together with their $\mathbb{W}$-action into a single package. This package will be called the *Springer sheaf Spr* and it will live inside a certain category of perverse sheaves on $\mathcal{N}$.

This lecture may require you to take a bit more on faith. If you haven't had much exposure to things like sheaves and the derived category, I recommend

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

you take these things as a black box to begin with (you can enjoy opening up the box and tinkering at a later point).

5.3.1 The Constructible Derived Category

The Constructible Derived Category

In the previous lecture, our main tool was the Borel–Moore homology of a space

$$H_*^{BM}(X).$$

In this lecture our principal player is a certain category

$$D(X)$$

called the *constructible derived category of sheaves*.

We will not have time to discuss the precise definition of this category (see e.g. the books [6],[12] for details). Rather we will attempt to understand this by considering some natural classes of objects and some natural functors out of it.

Remark Very briefly, one can define $D(X)$ as the subcategory of the bounded derived category of sheaves of $\mathbb{Q}$-vector spaces on X whose cohomology sheaves are constructible. Here a sheaf is said to be constructible if there is a stratification $X = \bigsqcup_\alpha X_\alpha$ such that each cohomology sheaf restricted to X_α is a locally constant sheaf of finite rank.

The Case $X = pt$

One can identify the category $D(pt)$ with the category of finite dimensional graded vector spaces. (This is slightly cheating: really $D(pt)$ is the bounded derived category of complexes of vector spaces with finite dimensional cohomology. The identification with graded vector spaces is given by taking a complex to its cohomology.) We think of $D(pt)$ as the home for the (co)homology of a space X. So we have objects $H^*(X) \in D(pt)$ and $H_c^*(X)$ for each X. We can also consider the homology $H_*(X)$ and Borel–Moore homology $H_*^{BM}(X)$ as objects of $D(pt)$ by taking the negative grading (so, e.g. $H_i(X)$ is in degree $-i$).

Measurements: Sections, Stalks, Costalks

Now, given a general space X, there are a bunch of canonical functors to $D(pt)$. These come in two flavours.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

Sections We have the functors of *(derived) global sections* and *compactly supported (derived) global sections*:

$$R\Gamma_X, R\Gamma_{X,c} : D(X) \to D(pt).$$

More generally, if U is an open subset of X, there is a canonical restriction functor

$$(-)|_U : D(X) \to D(U),$$

and we can compose with the sections on U to get functors:

$$R\Gamma_U, R\Gamma_{U,c} : D(X) \to D(pt).$$

Stalks On the other hand, for any point $x \in X$, we have two functors called the *stalk* and *costalk*, respectively:

$$i_x^*, i_x^! : D(X) \to D(pt).$$

Given any object $\mathcal{F} \in D(X)$ we can attempt to understand it by analyzing its global sections, stalks and costalks.

Objects

Constant and dualizing sheaves Given a space X we have two basic objects:

- the *constant sheaf* $\mathbb{Q}_X$ and
- the *dualizing complex* ω_X.

Both are preserved under restriction to an open subset $U \subseteq X$. One can think that $\mathbb{Q}_X$ is representing local cochains on X and ω_X is representing local Borel–Moore chains. More precisely, we have the following isomorphisms (in fact, the left-hand side could be taken as a definition of the right-hand side).

$$\begin{aligned} R\Gamma_U(\mathbb{Q}_X) &\cong H^*(U), \\ R\Gamma_U(\omega_X) &\cong H_*^{BM}(U), \\ R\Gamma_{U,c}(\mathbb{Q}_X) &\cong H_c^*(U), \\ R\Gamma_{U,c}(\omega_X) &\cong H_*(U). \end{aligned}$$

The constant sheaf (respectively, dualizing complex) has the property that its stalks $i_x^*(\mathbb{Q}_X)$ (respectively, costalks $i_x^!(\omega_X)$) are isomorphic to the 1-dimensional vector space $\mathbb{Q} \in D(pt)$.

Local systems A local system $\mathcal{L}$ on X (also known as a locally constant sheaf) is an object of $D(X)$ which is a twisted form of the constant sheaf. The sections $R\Gamma_U$ and $R\Gamma_{U,c}$ measure the cohomology and compactly supported cohomology with local coefficients in $\mathcal{L}$.

For each $x \in X$, the stalk $i_x^*(\mathcal{L})$ is a single vector space L_x in degree 0, and it carries an action of the fundamental group $\pi_1(X,x)$. In fact, the category of local systems on a connected space X is equivalent to the category of representations of the fundamental group.

The objects of geometric origin Now, given a map of spaces

$$f : X \to Y,$$

we have certain objects $f_!(\mathbb{Q}_X)$ and $f_*(\omega_X)$ of $D(Y)$. These objects are designed to measure the various (co)homology theories on the fibres X_y of f. More precisely, for each $y \in Y$ we have the following isomorphisms.

$$i_y^* f_!(\mathbb{Q}_X) \cong H_c^*(X_y),$$
$$i_y^! f_*(\omega_X) \cong H_*^{BM}(X_y).$$

Example 5.40 Consider the case where X is a cylinder $S^1 \times \mathbb{R}$, and $f : X \to Y$ is obtained by pinching the subspace $S^1 \times \{0\}$ to a point $y_0 \in Y$, making a cone. Then $\mathcal{F} := f_!(\mathbb{Q}_X)$ can be thought of in the following way. Over the open subset $Y - \{y_0\}$ we get a copy of the constant sheaf $\mathbb{Q}_Y$. But over the special point y_0, the stalk of $\mathcal{F}$ is equivalent to $H^*(S^1)$.

The Formalism of the Six Operations

A neat way to package this stuff is via the so-called *six operations*. In general, if $f : X \to Y$ is a map of spaces, we have the following four functors (one should add to these the functors of internal Hom and tensor product to make six).

$$f_*, f_! : D(X) \leftrightarrows D(Y) : f^!, f^*.$$

We gather the fundamental properties of these functors here:

Proposition 5.41 *Suppose $f : X \to Y$ is a map of spaces.*

1 *The functor f^* is left adjoint to f_* and $f^!$ is right adjoint to $f_!$.*

2 *If f is proper, then we have a natural isomorphism $f_* \cong f_!$.*

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

3 Suppose we have a Cartesian diagram:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle \widetilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Then there are natural isomorphisms:

$$g^! f_* \cong \widetilde{f}_* \widetilde{g}^!,$$
$$g^* f_! \cong \widetilde{f}_! \widetilde{g}^*.$$

Remark These functors subsume all the objects and functors already defined. For example, if $p : X \to pt$ is the unique map, we have

$$\mathbb{Q}_X = p^*(\mathbb{Q}), \quad \omega_X = p^!(\mathbb{Q}),$$

and

$$R\Gamma_X = p_*, \quad R\Gamma_{X,c} = p_!.$$

If $j : U \hookrightarrow X$ is the inclusion of an open subset, we have $j^* = j^! = (-)|_U$.

Grading shift For each integer n we get an autoequivalence of $D(X)$,

$$\mathcal{F} \mapsto \mathcal{F}[n]$$

called *shifting degree by n*. All the functors we will consider will intertwine the operations of shifting degree.

In the case of $X = pt$, the functor $[n]$ has the effect of shifting the grading degree so that if $V = \bigoplus V_k$ is a graded vector space, the degree k part of $V[n]$ is V_{k+n}.

Verdier and Poincaré Duality

The six operation formalism offers a nice way of packaging the idea of Poincaré duality. Namely we have the following:

Theorem 5.42 (Poincaré duality) *Suppose X is a smooth oriented d-manifold. Then there is a canonical isomorphism $\omega_X \cong \mathbb{Q}_X[d]$.*

The more traditional forms of Poincaré duality can essentially be recovered from this fact, together with the properties of the six operations.

One advantage of this setting is that we can formulate a relative version of the above statement:

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

Theorem 5.43 (Relative Poincaré duality) *Suppose $f : X \to Y$ is a smooth oriented fibration of relative dimension d. Then there is a canonical isomorphism*

$$f^! \cong f^*[d].$$

The reader may have noticed a certain symmetry in this subject, between constant and dualizing, or ! and $*$. This symmetry is realized by a contravariant duality functor called the *Verdier duality functor*

$$\mathbb{D}_X : D(X) \to D(X)^{op},$$

such that $\mathbb{D}^2 \cong Id$. The basic property of this functor is that it exchanges the constant sheaf $\mathbb{Q}_X$ with the dualizing sheaf ω_X. More generally, given a map $f : X \to Y$ we have

$$\mathbb{D}_Y f_* \cong f_! \mathbb{D}_X,$$
$$\mathbb{D}_X f^* \cong f^! \mathbb{D}_Y.$$

The functor $\mathbb{D}_{pt}$ is just the usual duality for graded vector spaces.

5.3.2 Perverse Sheaves and Intersection Homology

Motivation

Now suppose X is a smooth algebraic variety of (pure, complex) dimension d (thus it is a smooth $2d$-manifold). Then we have seen that there is a Poincaré duality isomorphism

$$\omega_X \simeq \mathbb{Q}_X[2d].$$

To put it more symmetrically, we have:

$$\omega_X[-d] \simeq \mathbb{Q}_X[d].$$

Note that this grading shift is only possible on an even real dimensional manifold (e.g. a complex manifold).

Yet another way to express this fact is to say that for a smooth d-dimensional variety X the object $\mathbb{Q}_X[d]$ is canonically Verdier self-dual.

More generally, if $\mathcal{L}$ is a local system on a smooth variety X of dimension d, then we have

$$\mathbb{D}(\mathcal{L}[d]) \cong \mathcal{L}^\vee[d],$$

where $\mathcal{L}^\vee$ is the dual local system (corresponding to the dual representation of the fundamental group).

If X is singular, then this of course fails in general. However, one can still ask the following:

Question 1 Is there an object $IC_X \in D(X)$ such that:

1 IC_X is Verdier self-dual, i.e. $\mathbb{D}(IC_X) \simeq IC_X$?
2 If $U \subseteq X$ is a smooth, open, dense subvariety of X (e.g. the entire smooth locus), then $IC_X|_U \simeq \mathbb{Q}_U[d]$?

Moreover, can one make this construction suitably canonical and functorial?

It turns out the answer is: yes, there is a such an object. It is called the *intersection complex* (more precisely, we are using the so-called *middle perversity*, and shifting the grading so that all our complexes are perverse sheaves). We will present a characterization below.

Remark Historically, the complex $R\Gamma(IC_X)$, called the intersection complex, was defined by Goresky and MacPherson [8] using the concept of *perversity* and *allowable cycles*, where the manner in which the cycles intersect with the singularities is restricted in a particular fashion.

Characterization of Intersection Cohomology

Suppose $U \subseteq X$ is an open subvariety of X which is smooth of pure dimension d. Suppose we fix a local system $\mathcal{L}$ on U.

Theorem 5.44 *([9]) There is an object $IC_X(\mathcal{L})$ together with an isomorphism:*

$$IC_X(\mathcal{L})|_U \cong \mathcal{L}[d],$$

and such that:

1

$$\dim\{x \in X - U \mid H^k(i_x^* IC_X) \neq 0\} < -k,$$

2

$$\dim\{x \in X - U \mid H^k(i_x^! IC_X) \neq 0\} < k.$$

Moreover, the object $IC_X(\mathcal{L})$ is uniquely characterized by these properties (up to unique isomorphism in $D(X)$).

The object $IC_X(\mathcal{L})$ defined by the above theorem satisfies the desired Verdier duality property.

Theorem 5.45 ([9]) *Given $X, U, \mathcal{L}$ as above, we have*

$$\mathbb{D}(IC_X(\mathcal{L})) \cong IC_X(\mathcal{L}^\vee).$$

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

We define

$$IH_*(X;\mathcal{L}) = R\Gamma_{X,c} IC_X(\mathcal{L})[d]$$

and

$$IH_*^{BM}(X;\mathcal{L}) = R\Gamma_X IC_X(\mathcal{L})[d].$$

Note that here we have shifted the grading back to lie in the traditional (rather than perverse) degrees.

Corollary 5.46 (Poincaré duality for intersection homology) *There is a perfect pairing:*

$$IH_k^{BM}(X;\mathcal{L}) \otimes IH_{2d-k}(X;\mathcal{L}^\vee) \to \mathbb{Q}.$$

In particular, if X is proper and we take $\mathcal{L}$ to be trivial we get a perfect pairing:

$$IH_k(X) \otimes IH_{2d-k}(X) \to \mathbb{Q}.$$

Example 5.47 (The cone revisited) Let $Y = S^1 \times \mathbb{R}/S^1 \times \{0\}$, the cone. We have:

$*$	$H_*^{BM}(Y)$	$H_*(Y)$	$IH_*^{BM}(Y)$	$IH_*(Y)$
0	0	$\mathbb{Q}$	0	$\mathbb{Q}^2$
1	$\mathbb{Q}$	0	0	0
2	$\mathbb{Q}^2$	0	$\mathbb{Q}^2$	0

The chain generating H_1^{BM} is no longer "allowable" in intersection homology. Thus we are left with either the two fundamental classes in IH_2^{BM} or the classes of two points in IH_2 – see Figure 5.9.

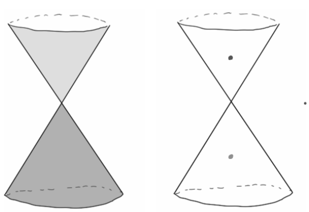


Figure 5.9 IH_*^{BM} and IH_* for the cone.

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

Definition of Perverse Sheaf

If one relaxes slightly the dimension bounds in the definition of the IC complex, one obtains the definition of a perverse sheaf:

Definition 5.48 An object $\mathcal{F}$ in $D(X)$ is called a *perverse sheaf* if:

1 $\dim\{x \in X \mid H^k(i_x^*\mathcal{F}) \neq 0\} \leqslant -k$ and
2 $\dim\{x \in X \mid H^k(i_x^!\mathcal{F}) \neq 0\} \leqslant k$.

We denote by $\mathrm{Perv}(X)$ the full subcategory of $D(X)$ whose objects are perverse sheaves.

Examples of perverse sheaves on X include the objects $IC_Z(\mathcal{L})$ for any closed subvariety Z of X and local system $\mathcal{L}$ on an open dense subset of Z.

Theorem 5.49 *The category $\mathrm{Perv}(X)$ is abelian and every object has finite length. The simple objects are given by $IC_Z(\mathcal{L})$, where Z is a closed subvariety, and $\mathcal{L}$ is an irreducible local system defined on a dense open subset of Z.*

Small and Semismall Maps

Recall from last time that we said a morphism $f : X \to Y$ of algebraic varieties was said to be *semismall* if the dimension of $X \times_Y X$ was equal to the dimension of X. This can be reformulated as follows:

Definition 5.50 Let X be a smooth variety of dimension d. A morphism $f : X \to Y$ of algebraic varieties is said to be *semismall* if

$$\dim\{y \in Y \mid \dim f^{-1}(y) \geqslant k\} \leqslant d - k,$$

for all $k \geqslant 0$, and *small* if

$$\dim\{y \in Y \mid \dim f^{-1}(y) \geqslant k\} \leqslant d - k,$$

for $k > 0$.

Notice the similarities between the definition of small (respectively, semismall) and IC complexes (respectively, perverse sheaves). This observation leads to the following key result.

Proposition 5.51 (See e.g. [11], Proposition 8.2.30) *Suppose $f : X \to Y$ is a proper morphism of algebraic varieties and that X is smooth of dimension d.*

1 If f is small, then

$$f_*\mathbb{Q}_X[d] \cong IC_Y(\mathcal{L}),$$

where $\mathcal{L} = (f|_U)_*\mathbb{Q}_U[d]$, and $U \subseteq X$ is an open dense subset such that $f|_U$ is a covering map.

2 *If f is semismall then there is an isomorphism:*

$$f_*\mathbb{Q}_X[d] \cong \bigoplus_{\alpha=1}^{n} IC_{Y_\alpha}(\mathcal{L}_\alpha),$$

where $Y_\alpha \subseteq Y$ are closed subvarieties together with irreducible local systems $\mathcal{L}_\alpha$ on the smooth locus Y_α^{sm} for each $\alpha = 1,\dots,n$.

5.3.3 The Springer Sheaf

Let us return again to the Lie theoretic setting of Section 5.1.2.

Big vs Small

We define the *big Springer sheaf*

$$\mathcal{S}_{\mathfrak{g}} := \pi_*\mathbb{Q}_{\tilde{\mathfrak{g}}}[d]$$

and the *small* or *nilpotent Springer sheaf*

$$\mathcal{S}_{\mathcal{N}} := \rho_*\mathbb{Q}_{\widetilde{\mathcal{N}}}[2m].$$

The stalks of $\mathcal{S}_{\mathfrak{g}}$ and of $\mathcal{S}_{\mathcal{N}}$ at nilpotent elements $x \in \mathcal{N}$ both record the homology of Springer fibres. In particular, by restriction, any endomorphism of $\mathcal{S}_{\mathfrak{g}}$ or $\mathcal{S}_{\mathcal{N}}$ defines an endomorphism of the homology of Springer fibres.

The basic properties of the six operations allow us to relate the endomorphism algebras of these objects to the convolution algebras considered in the previous lecture.

Proposition 5.52 *There are isomorphisms of graded algebras:*

$$H_*^{BM}(\mathrm{St}(\mathfrak{g}))[-2d] \cong R\mathrm{Hom}_{D(\mathfrak{g})}(\mathcal{S}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}}),$$
$$H_*^{BM}(\mathrm{St}(\mathcal{N}))[-4m] \cong R\mathrm{Hom}_{D(\mathcal{N})}(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{N}}).$$

In particular we get isomorphisms of algebras:

$$A(\mathfrak{g}) \cong \mathrm{Hom}_{D(\mathfrak{g})}(\mathcal{S}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}}),$$
$$A(\mathcal{N}) \cong \mathrm{Hom}_{D(\mathcal{N})}(\mathcal{S}_{\mathcal{N}}, \mathcal{S}_{\mathcal{N}}).$$

Thus we can rephrase Theorem 5.37 and Theorem 5.39 as statements about the endomorphism algebra of Springer sheaves.

The (Semi) smallness of the Springer Maps

The dimension formula for Springer fibres directly implies the following crucial result:

https://doi.org/10.1017/9781009093750.007 Published online by Cambridge University Press

Proposition 5.53 *1 The morphism*

$$\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$$

is small and its restriction π^{rs} to the regular semisimple locus is a $\mathbb{W}$-Galois covering.

2 The morphism

$$\rho : \widetilde{\mathcal{N}} \to \mathcal{N}$$

is semismall and birational (i.e. an isomorphism over an open set).

In particular, it follows that both $\mathcal{S}_{\mathfrak{g}}$ and $\mathcal{S}_{\mathcal{N}}$ are perverse sheaves.

The Structure of the Big Springer Sheaf

Let

$$\mathcal{K} = \pi^{rs}_{*}\mathbb{Q}_{\widetilde{\mathfrak{g}}^{rs}} \cong \mathcal{S}_{\mathfrak{g}}|_{\mathfrak{g}^{rs}}.$$

As π^{rs} is a $\mathbb{W}$-Galois cover, it follows that $\mathcal{K}$ is a local system on $\mathfrak{g}^{rs}$ of rank $|\mathbb{W}|$, and the endomorphisms of $\mathcal{K}$ are precisely the group algebra $\mathbb{Q}[\mathbb{W}]$. More or less equivalently, we have a decomposition

$$\mathcal{K} \cong \bigoplus_{L \in \mathrm{Irrep}(\mathbb{W})} L \otimes \mathcal{K}_L,$$

where $\mathcal{K}_L$ is an irreducible local system on $\mathfrak{g}^{rs}$ (of rank $\dim L$). The smallness of the map π then implies that endomorphisms extend uniquely from the regular semisimple locus, giving the following sheaf-theoretic version of Theorem 5.37:

Theorem 5.54 *We have a canonical isomorphism*

$$\mathrm{End}_{\mathrm{Perv}(\mathfrak{g})}(\mathcal{S}_{\mathfrak{g}}) \cong \mathbb{Q}[\mathbb{W}]$$

and the perverse sheaf $\mathcal{S}_{\mathfrak{g}}$ decomposes as a direct sum:

$$\mathcal{S}_{\mathfrak{g}} \cong \bigoplus_{L \in \mathrm{Irrep}(\mathbb{W})} L \otimes IC_{\mathfrak{g}}(\mathcal{K}_L).$$

The Structure of the Small Springer Sheaf

The semismallness of the map ρ implies that there is some decomposition

$$\mathcal{S}_{\mathcal{N}} = \bigoplus_i M_i \otimes IC_{Z_i}(\mathcal{E}_i)$$

as a direct sum of IC sheaves $IC_{Z_i}(\mathcal{E}_i)$, where the Z_i are closed subsets of $\mathcal{N}$, M_i are multiplicity vector spaces and $\mathcal{E}_i$ local systems on some open dense subset of Z_i. Moreover (by absorbing repeating factors into M_i), we may assume that the $IC_{Z_i}(\mathcal{E}_i)$ are pairwise non-isomorphic.

As ρ is G-equivariant, the closed subset $Z = \overline{\mathcal{O}}$ must be the closure of some G-orbit $\mathcal{O}$ and $\mathcal{E}$ must be a G-equivariant local system on $\mathcal{O}$. Note that G-equivariant local systems on an orbit $G \cdot x$ are precisely given by representations σ of $A_G(x)$. Thus, each of the factors $IC_{Z_i}(\mathcal{E}_i)$ above are of the form $IC_{\overline{G\cdot x}}(\sigma)$ for some pair (x, σ).

Reinterpreting the Springer Correspondence

Now suppose, for a moment, we assume Theorem 5.39, i.e. that there is an isomorphism

$$\mathbb{Q}[\mathbb{W}] \cong \mathrm{End}_{\mathrm{Perv}(\mathcal{N})}(\mathcal{S}_{\mathcal{N}}). \tag{5.3.1}$$

This then gives a precise enumeration of the decomposition of $\mathcal{S}_{\mathcal{N}}$ into simple objects: namely, there are pairwise non-isomorphic simple summands for each irreducible representation L, and the multiplicity space of each such summand is again given by L. In other words, there is an injective map of sets

$$L \mapsto (\mathcal{O}_L, \sigma_L)$$

from $\mathrm{Irrep}(\mathbb{W})$ to the set of pairs $(\mathcal{O}, \sigma)$ of a nilpotent orbit $\mathcal{O} = G \cdot x$ and an irreducible representation σ of $A_G(x)$. This is the Springer correspondence, reinterpreted in the language of perverse sheaves!

From here, it is not too hard to show (using the dimension formula) the traditional statement of the Springer correspondence, that L matches up with the σ_L-multiplicity space of $H^{2d(x)}(\mathcal{F}\ell^x)$ where $G \cdot x = \mathcal{O}_L$ (see e.g. Section 4.1 in Clausen's notes).

Two Parameterizations of the Springer Correspondence

Let $i_{\mathcal{N}} : \mathcal{N} \hookrightarrow \mathfrak{g}$ denote the inclusion. We have an isomorphism:

$$\mathcal{S}_{\mathcal{N}} \cong i_{\mathcal{N}}^{!}[r]\mathcal{S}_{\mathfrak{g}},$$

where r is the rank of the Lie algebra $\mathfrak{g}$ (i.e. the dimension of a Cartan subalgebra). Thus the functor $i_{\mathcal{N}}^{!}[r]$ induces an algebra homomorphism:

$$\mathrm{End}_{\mathrm{Perv}(\mathfrak{g})}(\mathcal{S}_{\mathfrak{g}}) \to \mathrm{End}_{\mathrm{Perv}(\mathcal{N})}(\mathcal{S}_{\mathcal{N}}). \tag{5.3.2}$$

It is possible to prove directly that this is an isomorphism; it is equivalent to proving that the simple objects $IC(\mathfrak{g}, \mathcal{K}_L)$ appearing in Theorem 5.54 restrict to pairwise non-isomorphic simple objects in $\mathrm{Perv}(\mathcal{N})$ via $i_{\mathcal{N}}^{!}[r]$ (namely, the objects $IC_{\overline{\mathcal{O}_L}}(\sigma_L)$ appearing above). This leads to the same isomorphism

$A(\mathfrak{g}) \cong A(\mathcal{N})$ as in Theorem 5.39, and thus the same Springer correspondence as in the previous lecture.

However, there is also another approach. The *Fourier transform* (or Fourier–Deligne transform) is a certain involutive endofunctor (the superscript mon denotes that we only consider the full subcategory of objects which are equivariant for the action of the scaling torus $\mathbb{C}^\times$):

$$\mathbb{F} : \mathrm{Perv}^{\mathrm{mon}}(\mathfrak{g}) \to \mathrm{Perv}^{\mathrm{mon}}(\mathfrak{g}).$$

It turns out that we have $\mathbb{F}(\mathcal{S}_\mathfrak{g}) \cong (\mathcal{S}_\mathcal{N})$ (where the latter is considered a perverse sheaf on $\mathfrak{g}$ via pushforward under the closed embedding). This leads to another isomorphism as in (5.3.2) and thus another identification $\mathbb{Q}[\mathbb{W}] \cong \mathrm{End}_{\mathrm{Perv}(\mathcal{N})}(\mathcal{S}_\mathcal{N})$ and finally to another Springer correspondence! It is possible to show that the two parameterizations of the Springer correspondence differ by the sign character of $\mathbb{W}$.

Bibliography

[1] Artin, Michael. 2011. *Algebra*. Boston, MA: Pearson Education.

[2] Baez, John C. 2001. The octonions. *Bulletin of the American Mathematical Society*, **39**(02), 145–206.

[3] Borho, Walter, and MacPherson, Robert. 1983. Partial resolutions of nilpotent varieties. Pages 23–74 of: *Analysis and topology on singular spaces, II, III (Luminy, 1981)*. Astérisque, vol. 101. Paris: Soc. Math. France.

[4] Chriss, Neil, and Ginzburg, Victor. 1997. *Representation theory and complex geometry*. Boston, MA: Birkhäuser Boston Inc.

[5] Concini, De, C., Lusztig, G. and Procesi, C. 1988. Homology of the Zero-Set of a Nilpotent Vector Field on a Flag Manifold. *Journal of the American Mathematical Society*, **1**(1), 15.

[6] Dimca, Alexandru. 2004. *Sheaves in Topology*.

[7] Ginsburg, V. 1987. "Lagrangian" construction for representations of Hecke algebras. *Advances in Mathematics*, **63**(1), 100–112.

[8] Goresky, Mark, and MacPherson, Robert. 1980. Intersection homology theory. *Topology*, **19**(2), 135–162.

[9] Goresky, Mark, and MacPherson, Robert. 1983. Intersection homology. II. *Invent. Math.*, **72**(1), 77–129.

[10] Hotta, Ryoshi, and Kashiwara, Masaki. 1984. The invariant holonomic system on a semisimple Lie algebra. *Inventiones Mathematicae*, **75**(2), 327–358.

[11] Hotta, Ryoshi, Takeuchi, Kiyoshi, and Tanisaki, Toshiyuki. 2008. *D-modules, perverse sheaves, and representation theory*. Progress in Mathematics, vol. 236. Boston, MA: Birkhäuser Boston Inc. Translated from the 1995 Japanese edition by Takeuchi.

[12] Kashiwara, Masaki, and Schapira, Pierre. 1990. *Sheaves on Manifolds*. Berlin, Heidelberg: Springer Berlin Heidelberg Imprint Springer.

[13] Kazhdan, David, and Lusztig, George. 1980. A topological approach to Springer's representations. *Advances in Mathematics*, **38**(2), 222–228.
[14] Lusztig, G. 1984. Intersection cohomology complexes on a reductive group. *Inventiones Mathematicae*, **75**(2), 205–272.
[15] Shoji, Toshiaki. 1988. Geometry of orbits and Springer correspondence. Pages 61–140 of: *Orbites unipotentes et représentations - I. Groupes finis et Algèbres de Hecke*. Astérisque, no. 168. Société mathématique de France.
[16] Spaltenstein, N. 1982. *Classes Unipotentes et Sous-groupes de Borel*. Springer-Verlag GmbH.
[17] Springer, T. A. 1976. Trigonometric sums, Green functions of finite groups and representations of Weyl groups. *Inventiones Mathematicae*, **36**(1), 173–207.
[18] Springer, T. A. 2008. *Linear Algebraic Groups*. Birkhauser Boston Inc.
[19] Springer, Tonny A. 1978. A construction of representations of Weyl groups. *Inventiones Mathematicae*, **44**(3), 279–293.