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The structure of finite groups whose elements outside a normal subgroup have prime power orders

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The structure of groups in which every element has prime power order (CP-groups) is extensively studied. We first investigate the properties of group G such that each element of $G \setminus N$ has prime power order. It is proved that N is solvable or every non-solvable chief factor H/K of G satisfying $H \leqslant N$ is isomorphic to $PSL_2(3^f)$ with f a 2-power. This partially answers the question proposed by Lewis in 2023, asking whether $G \cong M_{10}$? Furthermore, we prove that if each element $x \in G\setminus N$ has prime power order and $\mathbf{C}_G(x)$ is maximal in G, then N is solvable. Relying on this, we give the structure of group G with normal subgroup N such that $\mathbf{C}_G(x)$ is maximal in G for any element $x \in G \setminus N$. Finally, we investigate the structure of a normal subgroup N when the centralizer $\mathbf{C}_G(x)$ is maximal in G for any element $x \in N \setminus \mathbf{Z}(N)$, which is a generalization of results of Zhao, Chen, and Guo in 2020, investigating a special case that $N = G$ for our main result. We also provide a new proof for Zhao, Chen, and Guo's results above.

Keywords: prime power order; centralizers; maximal subgroups; Frobenius groups; CP-groups

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1. Introduction

All groups are supposed to be finite. Recall that G is called a minimal non-abelian group if G is non-abelian but every proper subgroup of G is abelian. We denote by $F(G)$ the Fitting subgroup of G. The upper nilpotent series $\{F_i(G)\}_{i\geqslant 0}$ of a group G is defined recursively by $F_0(G) = 1$ and $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$ for $i \geqslant 1$. If G is a solvable group, then the smallest integer h such that $F_h(G) = G$ is called the Fitting length (or nilpotent length) of G and is denoted by $h(G)$. All unexplained notation and terminology are standard (see [**19**]).

A group G is called a CP-group if every element of G has prime power order. The question about CP-groups was first addressed by Higman in [**16**], who determined all solvable CP-groups. In [**15**], Heineken gave the general structure of non-solvable CP-groups and listed all simple CP-groups.

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Denote $H_{p^n}(G) := \langle x \in G | x^{p^n} \neq 1\rangle$ for some prime $p \mid |G|$. In [14], Hughes and Thompson determined the structure of group G if $H_p(G)$ is a proper subgroup of G and G is not a p-group. Some authors studied those groups G having a proper subgroup $H_{p^n}(G)$ where $n > 1$, for instance, [3, 5]. Clearly, each element of $G \setminus H_{p^n}(G)$ is a p-element. Motivated by the ideas above, we study the structure of a group G that satisfies the following:

Property (∗): Let G be a group and N be a proper normal subgroup of G. Assume that every element of $G \setminus N$ has prime power order.

Clearly, every CP-group G trivially satisfies property (∗). Our main theorem is:

THEOREM A. *Let G be a group and N be a non-trivial normal subgroup of G. Suppose that G and N satisfy property (∗). If N is non-solvable, then every non-solvable chief factor H/K of G satisfying $H \leqslant N$ is isomorphic to $PSL_2(3^f)$, where $f > 1$ is a 2-power. In particular, G/N is solvable.*

REMARK 1. In [28, Theorem 1], it is asserted that 'Let $\Delta(q)$ be a subgroup of the automorphism group of a finite simple group $L_2(q)$ generated by its inner automorphism group and by an automorphism $\varphi\delta$, where φ and δ are the generators for the groups of field and diagonal automorphisms of $L_2(q)$, respectively. If G is a finite generalized Frobenius group with an insoluble kernel F, then $|G : F| = 2$ and $G/\mathrm{Sol}(F)$ is isomorphic to $\Delta(q)$, where $q = 3^{2^l}$ for some natural number l. Here, $\mathrm{Sol}(F)$ denotes the largest solvable normal subgroup of F'.

In fact, the chief factor in theorem A is not necessarily a simple group. Let $F := S_1 \times S_2$, where $S_1 \cong S_2 \cong PSL_2(9)$ and let further φ_i, $\delta_i \in \mathrm{Aut}(S_i)$ be field and diagonal automorphisms of S_i, for $i = 1, 2$, respectively, and $u_i = \varphi_i\delta_i$. It is easy to see that every element in S_iu_i is a 2-element by [4]. Suppose that $G = F\langle u\rangle$, where $u = u_1u_2$. Obviously, $\mathrm{Sol}(F) = 1$ and $G \setminus F = Fu = (S_1u_1) \times (S_2u_2)$. Hence, every element in $G \setminus F$ is also a 2-element. But in this case, F is not a non-abelian simple group. Hence, there must be a mistake in [28, Theorem 1].

The key ingredient for proving theorem A is theorem B, which has interest on its own.

THEOREM B. *Let N be a non-abelian simple group and $Aut(N)$ be its automorphism group. If $u \in \mathrm{Aut}(N) \setminus N$ is an r-element for some prime r such that every element of Nu has prime power order, then $N \cong PSL_2(3^f)$ with integer f, $u = \delta\varphi$ is a product of a diagonal automorphism δ and a field automorphism φ of N. In particular, $o(\varphi) = f$ is a power of 2.*

REMARK 2. Although this theorem is proved in [21], our method is quite different since we consider the element orders of the coset of $Soc(G)$, where G is an almost simple group, while the proof of [21] is relying on theory of classical groups. Of course, both of the proofs depend on the classification of finite simple groups.

It is well-known that maximal subgroups play an important role in researching the structure of groups. For instance, a straightforward result asserts that G is solvable when all its maximal subgroups have prime indices. On the contrary, the influence of the centralizers of elements on the structure of groups is also studied extensively.

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For instance, the authors of [**6**, **25**] investigated groups in which the centralizer of any non-trivial element is nilpotent, while the authors of [**2**, **18**] studied groups with conditions on centralizers. As an application of Theorem A, we get the following result:

COROLLARY 1. *Let G be a group and $N \trianglelefteq G$. If every element $x \in G \setminus N$ has prime power order, and $\mathbf{C}_G(x)$ is maximal in G, then N is solvable.*

Furthermore, we investigate the structure of group G if the centralizer of each element in $G \setminus N$ is maximal in G, where N is a normal subgroup of G. Zhao, Chen, and Guo gave the structure of group G when $N = G$, and proved that:

THEOREM C ([**26**, Theorems A and B]). *Let G be a non-abelian group. Write $\overline{G} := G/\mathbf{Z}(G)$. If for any $x \in G \setminus \mathbf{Z}(G)$, $\mathbf{C}_G(x)$ is maximal in G, then $\overline{G}$ is either an elementary abelian p-group, or $\overline{G} = \overline{P} \rtimes \overline{Q}$ is an inner abelian group with $|\overline{P}| = p^a$ and $|\overline{Q}| = q$, where p and q are two different primes, and a is a positive integer.*

In this paper, we consider the case $\mathbf{Z}(G) < N$, and obtain that:

THEOREM D. *Let G be a group and N be a proper normal subgroup of G such that $\mathbf{Z}(G) < N$. If $\mathbf{C}_G(x)$ is maximal in G for every element $x \in G \setminus N$, then G is solvable with G/N abelian. Furthermore,*

(I) *If G is nilpotent, then G/N is a p-group for some prime p. Moreover, $G = P \times \mathbf{Z}(G)_{p'}$ and $\mathbf{C}_G(x) \trianglelefteq G$ for every $x \in P \setminus N$;*

(II) *G is non-nilpotent, then $|G : \mathbf{C}_G(x)| = r^a$ with prime r and positive integer a. Suppose that R is a Sylow r-subgroup of G and K is a Hall r'-subgroup of G, then one of the statements holds:*
- (1) *If G/N is a p-group for some prime p, we have*
  - (1.1) *If $r = p$, write $\overline{G} := G/\mathbf{O}_p(G)\mathbf{Z}(G)$. Then $\overline{G} = \overline{K} \rtimes \overline{P}$ is a Frobenius group with abelian kernel $\overline{K}$ and complement $\overline{P}$ of order p.*
  - (1.2) *If $r \neq p$, write $\widetilde{G} := G/\mathbf{Z}(G)_r$. Then*
    - (1.2.1) *If $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$, then the Fitting length $h(\widetilde{G}) = 3$ and $\widetilde{G}$ has the following normal series:*

$$\widetilde{1} \trianglelefteq \mathbf{O}_{r'}(\widetilde{G}) \trianglelefteq \mathbf{O}_{r',r}(\widetilde{G}) \trianglelefteq \widetilde{G} = \mathbf{O}_{r',r,r'}(\widetilde{G}),$$

where $\mathbf{O}_{r',r}(\widetilde{G})/\mathbf{O}_{r'}(\widetilde{G}) \cong \widetilde{R}$ is elementary abelian and $\widetilde{G}/\mathbf{O}_{r',r}(\widetilde{G})$ is a p-group;

    - (1.2.2) *Assume that $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. If $\mathbf{C}_G(x)_{p'} \nleq \mathbf{Z}(G)$, then the Fitting length $h(\widetilde{G}) = 2$ and $\widetilde{G}$ has the following normal series:*

$$\widetilde{1} \trianglelefteq \widetilde{R} \trianglelefteq \widetilde{N} \trianglelefteq \widetilde{G},$$

where $\widetilde{R}$ is an elementary abelian r-group and G/R is nilpotent.

    - (1.2.3) *Write $\widehat{G} := G/\mathbf{Z}(G)$. Assume that $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. If $\mathbf{C}_G(x)_{p'} \leqslant \mathbf{Z}(G)$, then $\widehat{G} = \widehat{R} \rtimes \widehat{P}$, where P is a Sylow p-subgroup of G*

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and R is a Sylow r-subgroup of G with $\widehat{R}$ elementary abelian. Let $N_p = N \cap P$. If $\widehat{N_p} \trianglelefteq \widehat{G}$, then $\widehat{G}/\widehat{N_p}$ is a Frobenius group; if $\widehat{N_p} \ntrianglelefteq \widehat{G}$, then $\mathbf{N}_{\widehat{G}}(\widehat{N_p}) = \widehat{P}$.

(2) *If $|\pi(G/N)| \geqslant 2$, then one of the following statements holds:*

(2.1) *Let $\overline{G} := G/\mathbf{O}_r(G)\mathbf{Z}(G)$. Then $\overline{G} = \overline{K} \rtimes \overline{R}$ is a Frobenius group with abelian kernel $\overline{K}$ and complement $\overline{R}$ of order r;*

(2.2) *Let $\overline{G} := G/\mathbf{Z}(G)$. Then $\overline{G} = \overline{R} \rtimes \overline{K}$ is a Frobenius group with $\overline{R}$ a minimal normal subgroup of $\overline{G}$ and $\overline{K}$ cyclic. In particular, $R \leqslant N$.*

On the contrary, in a very recent paper, Zhao *et al.* [**27**] investigated the structure of a normal subgroup N of G, when $\mathbf{C}_G(x)$ is maximal for every element $x \in N \setminus \mathbf{Z}(G)$. Being inspired by the idea above, in this paper, by using less elements, we study the structure of group G if $\mathbf{C}_G(x)$ is maximal for every element $x \in N \setminus \mathbf{Z}(N)$. We obtain:

THEOREM E. *Let G be a group and $N \trianglelefteq G$. Let $\overline{G} := G/\mathbf{Z}(N)$. If $\mathbf{C}_G(x)$ is maximal in G for every element $x \in N \setminus \mathbf{Z}(N)$, then one of the following statements holds:*

(1) *If $\overline{N}$ is nilpotent, then $\overline{N}$ is an elementary abelian p-group for some prime p;*

(2) *If $\overline{N}$ is non-nilpotent, then $\overline{N} = \overline{P} \rtimes \overline{Q}$ is a Frobenius group with an elementary abelian kernel $\overline{P}$ and complement $\overline{Q}$ with prime order. In particular, $\overline{P}$ is the minimal normal subgroup of $\overline{G}$.*

REMARK 3. In [**20**], Lewis raised an interesting question asserting that: G is a group, N is a normal subgroup, p is a prime, P is a Sylow subgroup so that $(G, P, P \cap N)$ is a Frobenius–Wielandt triple, $G = NP$, $\mathbf{O}_p(G) = 1$, and G is non-solvable. Is this enough to imply that $G \cong M_{10}$? Theorems A and B partially answered the question above.

REMARK 4. Corollary 1 can also be obtained by [**20**, Theorem 1.1]. It is worth to mention that our method and result are different from Lewis' since he determined the structure of group G while we are concerning on the information of the normal subgroup N of G.

REMARK 5. Both [**27**, Theorems A and B] and theorem C can be considered as corollaries of theorem 5.

2. Proof of theorem B

To show theorem A, we first prove theorem B. Here, we list some notation and lemmas, which will be used below.

We denote by $\omega(G)$ the set of the element orders of G. If $A \subseteq G$ is a subset of G, then $\omega(A)$ denotes the set of element orders of A, and $k \cdot \omega(A) = \{ka | a \in \omega(A)\}$.

If $\varepsilon \in \{+, -\}$, we may write ε instead of $\pm$ in arithmetic expressions. The notation used in this section is mainly borrowed from [**4**, **10**, **12**].

LEMMA 2.1 ([**22**, Lemma 2]). *Let p and q be two primes and m, n be natural numbers such that $p^m = q^n + 1$. Then one of the following statements holds:*

(1) *$n = 1$, m is a prime number, $p = 2$ and $q = 2^m - 1$ is a Mersenne prime;*

(2) *$m = 1$, n is a power of 2, $q = 2$ and $p = 2^n + 1$ is a Fermat prime;*

(3) *$p = n = 3$ and $q = m = 2$.*

LEMMA 2.2 ([**29**]). *Let a and n be integers greater than 1. Then there exists a primitive prime divisor of $a^n - 1$, that is a prime s dividing $a^n - 1$ and not dividing $a^i - 1$ for $1 \leqslant i \leqslant n - 1$, except if*

(1) *$a = 2$ and $n = 6$, or*

(2) *a is a Mersenne prime and $n = 2$.*

Proof of theorem B. Since every element of Nu has prime power order, we take $nu \in Nu$ an r_1-element for some element $n \in N$ and prime r_1. Note that $N\langle u\rangle = N\langle nu\rangle$. Then $N\langle u\rangle/N = N\langle nu\rangle/N$. Moreover, $N\langle u\rangle/N \cong \langle u\rangle/\langle u\rangle \cap N$ is an r-group and $\langle nu\rangle N/N \cong \langle nu\rangle/\langle nu\rangle \cap N$ is an r_1-group. This forces $r = r_1$. Consequently, we conclude that every element in Nu is an r-element.

If N is a sporadic simple group, we may take $N = M_{12}$ as an example. In this case, select $u \in \mathrm{Aut}(N) \setminus N$ a 2-element. By [**4**], there exists an element of order 10, against our assumption. By a similar reasoning, we rule out the case that N is a sporadic simple group.

Assume then that $N = A_n$ is an alternating group of degree n with $n \geqslant 5$ but $n \neq 6$. As $\mathrm{Out}(N) \cong C_2$, we select $u = (12)$. Take $x = (345) \in N$. Then $ux = xu \in Nu$ does not have prime power, also a contradiction.

Now, we consider $N = N(q)$ is a simple group of Lie type defined over a field of q-elements, where $q = p^f$ with p a prime. As $u \in \mathrm{Aut}(N)$, by [**9**, Theorem 2.5.1], we may write $u = \delta\varphi\tau$, where δ is a diagonal automorphism, φ is a field automorphism, and τ is a graph automorphism of N, respectively.

If u is a field or a graph-field automorphism of N, then by [**8**], $\mathbf{C}_N(u) = N(p^{f/t})$ is a group of the same type as N. Clearly, $\mathbf{C}_N(u) = N(p^{f/t})$ is not a prime power group by [**4**, Table 6]. Let $1 \neq u' \in \mathbf{C}_N(u)$ be an r'-element. Then $u'u = uu' \in Nu$ is not an r-element, contrary to our assumption. If u is a graph automorphism of N, according to [**4**], we see that $N \cong PSL_n^+(q)$ with $n \geqslant 3$; $P\Omega_5(q)$ with q even; $P\Omega_{2n}^+(q)$ with $n \geqslant 4$; $G_2(q)$ with $p = 3$; $F_4(q)$ with $p = 2$; or $E_6(q)$ with $3 \mid (q - 1)$. However, there will be a contradiction according to [**1**, **13**].

Consequently, $u = \delta\varphi\tau$ with $\delta \neq 1$. In the following, we will analyse it case by case.

Case 1. $u = \delta$.
As $\delta \neq 1$, according to [**4**, Table 5], we see that $N \cong PSL_n^+(q)$ with $n \geqslant 2$; $PSL_n^-(q)$ with $n \geqslant 3$; $PSp_{2n}(q)$ with $n \geqslant 2$ and q odd; $P\Omega_{2n+1}(q)$ with $n \geqslant 3$ and q odd; $P\Omega_{2n}^{\varepsilon}(q)$ with $n \geqslant 4$; $E_6^{\varepsilon}(q)$ with $3 \mid (q - \varepsilon 1)$ or $E_7(q)$ with q odd.

Assume first $N \cong PSL_n^+(q)$. Write $d := o(u)$. Easily, $r \mid d$. If $n = 2$, then $Nu = PGL_2(q) \setminus PSL_2(q)$ and $2 \mid (2, q-1)$. As we see that q is odd and $r = 2$. It follows by [**11**, Lemma 2.1] that both $(q + (\varepsilon 1))$ and $(q - (\varepsilon 1))$ are powers of r. Lemma 2.1 indicates that $q = p = 3$. However, in this case, $N \cong PSL_2(3)$ is not a simple group, a contradiction. Hence, $n \geqslant 3$. Then $G = N \rtimes \langle u \rangle \leqslant PGL_n^\varepsilon(q)$. Let $T := \langle x_0 \rangle$ be a Singer subgroup of $PGL_n^\varepsilon(q)$ such that $S_0 := T \cap N$ is a Singer subgroup of N. By [**17**, Theorem 2.7.3], $|T| = (q^n - (\varepsilon 1)^n)/(q - \varepsilon 1)$ and $|T \cap N| = (q^n - (\varepsilon 1)^n)/(d(q - (\varepsilon 1)))$. Note that $PGL_n^\varepsilon(q) = PSL_n^\varepsilon(q)\langle x_0 \rangle$ and $x_0^d \in PSL_n^\varepsilon(q)$, without loss of generality, we may assume that $u \in T$. Then u centralizes $T \cap N$. It follows that $(q^n - (\varepsilon 1)^n)/(d(q - (\varepsilon 1)))$ is an r-power. If $n = 3$ and $q = 2$, we have $N \cong PSL_3(2) \cong PSL_2(7)$, which is a contradiction. Hence, $(n, q) \neq (3, 2)$. If $\varepsilon = +$, according to lemma 2.1, $q^n - 1$ has a prime divisor p_1 such that $p_1 \neq r$ and $p_1 \nmid q - 1$. Hence, $p_1 \mid (q^n - 1)/(d(q - 1))$, a contradiction. If $\varepsilon = -$, we can also rule out this case by a similar argument as above.

Now, consider $N \cong P\Omega_{2n}^\varepsilon(q)$. In this case, $r \mid d = (4, q^n - \varepsilon)$, indicating $r = 2$, and thus q is odd. By [**9**, Theorem 2.5.12], we see that all diagonal automorphisms of the same order are conjugate in $\mathrm{Out}(N)$. By [**12**, Figs. 1, 2, 3], $\omega(Nu) = \omega(PCSO_{2n}^\varepsilon(q) \setminus PSO_{2n}^\varepsilon(q))$ or $\omega(Nu) = \omega(PSO_{2n}^\varepsilon(q) \setminus P\Omega_{2n}^\varepsilon(q))$. By [**12**, Lemma 2.9], each element in $\omega(PCSO_{2n}^\varepsilon(q) \setminus PSO_{2n}^\varepsilon(q))$ is not an r-number for $n \geqslant 4$, contrary to our assumption. By [**12**, Lemmas 2.6 and 2.4], there exists an element in $\omega(Nu) = \omega(PSO_{2n}^\varepsilon(q) \setminus P\Omega_{2n}^\varepsilon(q))$, whose order is not an r-number, also contradiction.

Furthermore, according to [**10**], we can rule out the cases $N \cong P\Omega_{2n+1}(q)$ and $PSp_{2n}(q)$. The remaining cases can be ruled out by [**30**].

Case 2. $u = \delta\varphi$ with $\delta \neq 1$ and $\varphi \neq 1$.
Let $S := \mathrm{Inndiag}(N)$. Then S is a simple group of Lie type, which is defined over a field of q-elements, where $q = p^f$ with prime p. Let $\delta_0 := \delta_0(q)$ be a generator of S/N (other than $N \cong P\Omega_{2n}^+(q)$ with n even) such that $\delta = \delta_0^i$ for some positive integer i, and φ_0 be a generator of the field automorphism group of N.

Since $\delta \neq 1$, by [**4**, Table 5], we see that $N \cong PSL_n^\varepsilon(q)$ with $n \geqslant 2$ and $\varepsilon \in \{+, -\}$; $P\Omega_{2n}^\varepsilon(q)$ with $n \geqslant 4$, or $P\Omega_{2n+1}(q)$ with $n \geqslant 3$ and q odd, $PSp_{2n}(q)$, with $n \geqslant 3$ and q odd, $E_6^\varepsilon(q)$ with $3 \mid (q - \varepsilon 1)$ or $E_7(q)$ with q odd.

First consider $N \cong PSL_n^\varepsilon(q)$. If $n \geqslant 3$, by [**11**, Lemma 3.3], we have $\omega(uN) = k \cdot \omega(\tau^\alpha \delta(q_0) PSL_n^\varepsilon(q_0))$, where τ is a graph automorphism of N, and $\alpha = 0$ if $\varepsilon = +$ and $\alpha = 1$ if $\varepsilon = -$. Suppose $o(\varphi) = k$. Then k is an r-power. Assume first that $\varepsilon = +$. Then $\omega(uN) = k \cdot \omega(\delta(q_0) PSL_n(q_0))$. Furthermore, $((q_0^n - 1)i)/(d(q_0 - 1)) \in \omega(\delta(q_0) PSL_n(q_0))$, forcing that $((q_0^n - 1)i)/(d(q_0 - 1))$ is an r-power. By applying a similar argument as in case 1, we get a contradiction. Assume now that $\varepsilon = -$. Then $\omega(uN) = k \cdot \omega(\tau\delta(q_0) PSL_n(q_0))$. Since $\mathrm{Out}(N) \cong \langle \delta_0 \rangle \rtimes (\langle \varphi_0 \rangle \times \langle \tau \rangle)$, we obtain that τ is conjugate to $\tau\delta$. Hence, $\omega(\tau\delta(q_0) PSL_n(q_0)) = \omega(\tau PSL_n(q_0))$. By [**11**, Lemma 4.7], we see that $\omega(\tau PSL_n(q_0)) = 2 \cdot \omega(PSp_n(q_0))$ if $p \neq 2$, against [**11**, Lemma 2.2]. Hence, $p = 2$. Since $r \mid k$ and $k \mid (n, q + 1)$, we get that r is odd. Note that $\langle \delta_0 \rangle \rtimes \langle \tau \rangle$ is a dihedral group of $2d$ with d odd. Hence, $u \in \langle \delta_0 \rangle$. That is to say, u is a diagonal automorphism of N, contrary to our assumption.

As a result, $n = 2$. That is, $N \cong PSL_2(q)$ with q odd. Easily, $r = 2$. Let $M := PGL_2(q)$. Then $\mathbf{C}_M(\varphi) = PGL_2(q_0)$, where $q_0 = p^{f/k}$. Clearly, $\delta \in \mathbf{C}_M(\varphi)$.

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It follows that $PGL_2(q_0) \setminus PSL_2(q_0) \subseteq Nu$. Hence, every element in $PGL_2(q_0) \setminus PSL_2(q_0)$ is a 2-element. By [**11**, Lemma 2.1], both $q_0 - 1$ and $q_0 + 1$ are 2-power. By lemma 2.1, we get $q_0 = p = 3$ and $f = k$. Therefore, $N \cong PSL_2(3^f)$, where f is a 2-power, as required.

If $N \cong P\Omega^{\varepsilon}_{2n}(q)$, then either $(2, q-1) = 2$ or $(4, q^n - \varepsilon 1) = 2,\ 4$, forcing that q is odd. Assume first that $o(\delta) = 4$. By [**10**, Theorem 2.5.12], we see that $\delta\varphi$ is conjugate to $\delta^3\varphi$ in $\mathrm{Out}(N)$. It follows that $\omega(uN) = \omega(N\langle\delta\rangle\varphi \setminus (N\langle\delta^2\rangle\varphi))$. Assume first that $\varepsilon = +$. By [**12**, Lemma 1.2 and Fig. 2], we have that $\omega(uN) = k \cdot \omega(PCSO^+_{2n}(q_0) \setminus PSO^+_{2n}(q_0))$. Therefore, both $q_0^{n-1} - 1$ and $(q_0^{n-2} + 1)(q_0 - 1)$ are in $\omega(PCSO^{\varepsilon}_{2n}(q_0) \setminus PSO^{\varepsilon}_{2n}(q_0))$ by [**12**, Lemma 2.9]. Hence, both $q_0^{n-1} - 1$ and $(q_0^{n-2} + 1)(q_0 - 1)$ are r-power, which is a contradiction. Assume now that $\varepsilon = -$. Then $\delta^{\varphi} = \delta$ and n is odd. It follows that u is a 2-element. Let $W := PCSO^-_{2n}(q)$. Then $\mathbf{C}_W(\varphi) = PCSO^-_{2n}(q_0)$, where $q_0 = p^{f/k}$. Clearly, $\delta \in \mathbf{C}_W(\varphi)$. Therefore, we obtain that $\mathbf{C}_W(\varphi)\varphi \setminus \mathbf{C}_N(\varphi)\langle\delta^2\rangle\varphi \subseteq Nu$. This shows that $\mathbf{C}_W(\varphi) \setminus \mathbf{C}_N(\varphi)\langle\delta^2\rangle = PCSO^-_{2n}(q_0) \setminus PSO^-_{2n}(q_0)$ is a 2-element, contrary to [**12**, Lemma 2.9].

Assume now that $o(\delta) = 2$. By [**12**, Lemma 1.2 and Figs. 1, 2, 3], we have that $\omega(uN) = k \cdot \omega(PCSO^+_{2n}(q_0) \setminus PSO^+_{2n}(q_0))$ if $(4, q^n - 1) = 2$; $\omega(\varphi(PSO^+_{2n}(q) \setminus P\Omega^+_{2n}(q)))$ if n is odd and $(q^n - 1, 4) = 4$; $\omega(\varphi(PCSO^+_{2n}(q) \setminus PSO^+_{2n}(q)))$, or $\omega(\varphi(PSO^+_{2n}(q) \setminus P\Omega^+_{2n}(q)))$ if $n > 4$ is even. If $(4, q^n - 1) = 2$, by [**12**, Lemma 2.9], we have that $q_0^{n-1} - 1 \in \omega(PCSO^+_{2n}(q_0) \setminus PSO^+_{2n}(q_0))$. It is easy to get a contradiction by lemma 2.2. If n is odd and $(q^n - 1, 4) = 4$, then $\omega(\varphi(PSO^+_{2n}(q) \setminus P\Omega^+_{2n}(q))) \supseteq \omega(\varphi PSO^+_{2n}(q)) \setminus \omega(\varphi P\Omega^+_{2n}(q))$. By [**12**, Lemma 1.2], we have $\omega(\varphi(PSO^+_{2n}(q) \setminus P\Omega^+_{2n}(q))) \supseteq k \cdot (\omega(PSO^+_{2n}(q_0)) \setminus \omega(P\Omega^+_{2n}(q_0)))$. By [**12**, Lemmas 2.4 and 2.6], $(q_0^n - 1)/2 \in \omega(PSO^+_{2n}(q_0)) \setminus \omega(P\Omega^+_{2n}(q_0))$. It follows that $(q_0^n - 1)/2$ is an r-power. Since $n \geqslant 3$ and q_0 is odd, there exists some odd prime $p_1 \mid (q_0^n - 1)$ with $p_1 \neq r$ by lemma 2.2, a contradiction. By the same reason, we can also rule out the case $n \geqslant 4$ is even.

If $N \cong P\Omega_{2n+1}(q)$ with $n \geqslant 3$ and q odd, then $Nu = N\langle\delta\rangle\varphi \setminus N\varphi$. By [**10**, Lemma 2.8], we have that $\omega(Nu) \supseteq \omega(N\langle\delta\rangle\varphi) \setminus \omega(N\varphi) = k \cdot (\omega(\mathrm{Inndiag}(P\Omega_{2n+1}(q_0)) \setminus \omega(P\Omega_{2n+1}(q_0)))$. By [**10**, Lemma 2.1], we get that $p(q_0^{n-1} \pm 1) \in \omega(\mathrm{Inndiag}(P\Omega_{2n+1}(q_0)) \setminus \omega(P\Omega_{2n+1}(q_0))$. By hypothesis, $p(q_0^{n-1} \pm 1)$ is an r-power, a contradiction. By the same reason, we can also rule out the case $N \cong PSp_{2n}(q)$ with $n \geqslant 3$ and q odd.

If $N \cong E^{\varepsilon}_6(q)$, then $u^{\tau} = \delta^2\varphi$. It follows that $\omega(\mathrm{Inndiag}(N)\varphi) = \omega(M\varphi) \cup \omega(N\delta\varphi)$. Hence, $\omega(N\delta\varphi) = \omega(\mathrm{Inndiag}(N)\varphi) \setminus \omega(N\varphi)$. By [**30**, (3) and (5)], we have that $\omega(N\delta\varphi) = \omega(\mathrm{Inndiag}(N)\varphi) \setminus \omega(N\varphi) = k \cdot (\omega(\mathrm{Inndiag}(E^{\varepsilon}_6(q_0)) \setminus \omega(E^{\varepsilon}_6(q_0)))$. By [**30**, Lemmas 1 and 3], we see that $q^6 + \varepsilon q^3 + 1,\ (q^2 - 1)(q^4 + 1) \in \omega(\mathrm{Inndiag}(E^{\varepsilon}_6(q_0)) \setminus \omega(E^{\varepsilon}_6(q_0))$. By hypothesis, we obtain that both $q^6 + \varepsilon q^3 + 1$ and $(q^2 - 1)(q^4 + 1)$ are r-power. Since $3 \mid (q^6 + \varepsilon q^3 + 1)$, we get that $r = 3$. As $(q^2 - 1)(q^4 + 1)$ is a 3-power, then both $q^2 - 1$ and $q^4 + 1$ are 3-power, a contradiction. Similarly, we may rule out the case $N \cong E_7(q)$.

Case 3. $u = \delta\tau$ with $\delta \neq 1$, $\tau \neq 1$.
By [**4**, Table 5], we obtain that $N \cong PSL_n(q)$ with $n \geqslant 3$, $P\Omega^+_{2n}(q)$ with $n \geqslant 4$, or $E_6(q)$. If $N \cong PSL_n(q)$, then $\delta^{\tau} = \delta^{-1}$ and thus $o(u) = o(\delta\tau) = 2$, yielding $r = 2$. Assume first that q is even. Then $(n, q-1)$ is odd. In this case, τ is conjugate to $\tau\delta_0^i$ for every i in $\mathrm{Out}(N)$, where δ_0 is the generator of the diagonal automorphism

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group of N in $\mathrm{Out}(N)$. Therefore, $\omega(Nu) = \omega(N\tau) = \omega(PGL_n(q)\tau)$. By [1, 19.9], we have that $\mathbf{C}_N(\tau)$ is isomorphic to $PSp_n(q)$ if n is even, and to $P\Omega_n(q)$ if n is odd. Hence, there exists an element in Nu whose order is not a prime power, a contradiction. By the same reason, by applying [13], there is also a contradiction for the case q is odd.

If $N \cong P\Omega^+_{2n}(q)$, we have that $(2,\, q-1) = 2$ or $(4,\, q^n-1) > 1$, which follows that q is odd. If $\mathrm{Inndiag}(N)/N$ is cyclic, then $\delta^\tau = \delta^{-1}$. Therefore, $o(u) = o(\delta\tau) = 2$, by the same reason as above, we can also get a contradiction. If $\mathrm{Inndiag}(N)/N \cong C_2 \times C_2$, first we may consider the case $n > 4$. By [9, Theorem 2.5.12] and [12, Fig. 3], we have $\omega(Nu) = \omega(PTO^+_{2n}(q_0) \setminus PSO^+_{2n}(q_0))$, against [12, Lemma 2.8]. Now, we consider the case $n = 4$. According to [9, Theorem 2.5.12], we see that $o(\tau) = 2$ since $o(Nu) \geqslant 4$ is a 2-power. By the same reason above, there is also a contradiction.

If $N \cong E_6(q)$, then $\delta^\tau = \delta^{-1}$. It follows that $o(u) = 2$ and u is conjugate to τ in $\mathrm{Out}(N)$. Hence, $\omega(Nu) = \omega(N\tau)$. Since $\mathbf{C}_N(\tau) = F_4(q)$, we get that Nu contains a non-r-element, a contradiction.

Case 4. $u = \delta\varphi\tau$ with $\delta \neq 1$, $\varphi \neq 1$, and $\tau \neq 1$.
Let φ_0 be a generator of the field automorphism group of N such that $\varphi = \varphi_0^{f/k}$ and δ_0 a generator of the diagonal automorphism group of N such that $\delta = \delta_0^i$ for some positive integer i. Let $q = q_0^k$. By [4, Table 5], we have $N \cong PSL_n(q)$ with $n \geqslant 3$, $P\Omega^+_{2n}(q)$ with $n \geqslant 4$, or $E_6(q)$ with $3 \mid (q-1)$.

If $N \cong PSL_n(q)$, then, by [11, Lemma 3.3] and hypothesis, we see that k is an r-power and every element of $\omega(\delta(-q_0)PSU_n(q_0))$ is also an r-element. By [11, Lemma 2.1], we see that $\delta(-q_0)PSU_n(q_0)$ if $r = 2$ or $\delta(q_0)PSL_n(q_0)$ if $r > 2$, has an element of order $((q_0^n - (\varepsilon 1)^n)i)/((n,\, q_0 + \varepsilon 1)(q_0 - \varepsilon 1))$. By hypothesis, this order is an r-power. By a similar argument as in case 1, there is a contradiction.

If $P\Omega^+_{2n}(q)$ with $n \geqslant 4$, then q is odd. By [12], there is also a contradiction.

The remaining case is $N \cong E_6(q)$. In this case, $3 \mid (q-1)$. Since $N\langle u\rangle/N$ is an r-group, we see that u is a field or graph, or a graph-field automorphism of N up to conjugation, the final contradiction. □

3. Proofs of theorem A and corollary 1

THEOREM A. *Let G be a group and N be a non-trivial normal subgroup of G. Suppose that G and N satisfy property $(*)$. If N is non-solvable, then chief factor H/K of G satisfying $H \leqslant N$ is isomorphic to $PSL_2(3^f)$, where $f > 1$ is a 2-power. In particular, G/N is solvable.*

Proof. Suppose on the contrary that G is a counter-example of minimal order. We first assert that chief factor H/K of G satisfying $H \leqslant N$ is also a non-solvable chief factor of $N\langle u\rangle$. Let M_1/N_1 be a non-solvable chief factor of G satisfying $M_1 \leqslant N$. Write $M_1/N_1 = F_1 \times \cdots \times F_t$, where F_i are isomorphic non-abelian simple groups with integer $t \geqslant 1$. Assume that N/N_1 acts transitively on $\Omega = \{F_1, \ldots, F_t\}$. Then M_1/N_1 is also a chief factor of N and thus is a chief factor of $N\langle u\rangle$, we are done. Assume that N/N_1 does not act transitively on Ω. Therefore, $t > 1$. Then there is an element $N_1 w \in G/N_1 \setminus N/N_1$ such that $N_1\langle w\rangle$ acts non-trivially on Ω. Otherwise, every element in $G/N_1 \setminus N/N_1$ acts trivially on Ω, which indicates that

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N/N_1 acts transitively on Ω, a contradiction. Hence, there is an orbit Ω_1 of $N_1\langle w\rangle$ on Ω having size greater than 1. Without loss of generality, we may assume that $\Omega_1 = \{F_1, \ldots, F_s\}$ with $s > 1$. By hypothesis, we get that N_1w is an r_1-element for some prime r_1. Let $1 \neq f_1 \in F_1$ be an r_1'-element. Then there is an element $N_1w^{j_i}$ such that $f_1^{N_1w^{j_i}} \in F_i$ where j_i is a positive integer. Now, we get that $N_1w_0 := \prod_{i=1}^{s} f_1^{(N_1w)^{j_i}}$ is centralized by N_1w. In this case, N_1w_0w does not have prime power order and thus w_0w is not a prime power order element, contrary to the assumption of the theorem as $w_0w \in G \setminus N$. Consequently, we conclude that M_1/N_1 is a chief factor of N. So is of $N\langle u\rangle$, as required.

Take $1 \neq u \in G \setminus N$. If $N\langle u\rangle < G$, by assumption, every non-solvable chief factor H/K of $N\langle u\rangle$ satisfying $H \leqslant N$ is isomorphic to $PSL_2(3^f)$, where $f > 1$ is a 2-power. As chief factor H/K of G satisfying $H \leqslant N$ is also a chief factor of $N\langle u\rangle$, the theorem holds, against our assumption. As a result, $G = N\langle u\rangle$.

Let $M > 1$ be a minimal normal subgroup of G, which is contained in N. Clearly, each element in $G/M \setminus N/M$ has prime power order. This shows that G/M and N/M satisfy property (∗). Let $(M_1/M)/(M_2/M)$ be a chief factor of G/M such that $M_1/M \leqslant N/M$. Then M_1/M_2 is also a chief factor of G satisfying in $M_1 \leqslant N$. By induction, $(M_1/M)/(M_2/M)$ isomorphic to $PSL_2(3^f)$, where some $f > 1$ is a 2-power, so is M_1/M_2. In the following, we focus on the chief factors which are contained in M. We only need to consider the case that M is non-solvable.

In this case, M is a non-solvable minimal normal subgroup of G. Therefore, we may write $M = S_1 \times \cdots \times S_t$, where t is a positive integer and S_i are isomorphic non-abelian simple groups for all $i \in \{1, \ldots, t\}$. Assume that $t > 1$. Easily, $\langle u\rangle$ acts transitively on $\{S_1, \ldots, S_t\}$. Let $o(u) = m$ and $1 \neq x \in S_1$ be a q-element for some odd prime q distinct from p, where p is a prime divisor of m. Then $y = \prod_{i=1}^{m} x^{u^i}$ is centralized by u. That is, the order of $uy \in G\backslash N$ is divisible by pq. This contradiction deduces that $t = 1$. Consequently, M is a non-abelian simple group, and $u \in \mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ is the automorphism group of M.

Since $Mu \subseteq Nu \subseteq G \setminus N$, we see that the order of every element in Mu has prime power order. By theorem B, we get that $M \cong PSL_2(3^f)$, where f is a 2-power and u is the product of a field and a diagonal automorphism of $PSL_2(3^f)$. Therefore, chief factor H/K of G satisfying $H \leqslant N$ is isomorphic to $PSL_2(3^f)$, with $f > 1$ a 2-power, as required.

Let W be a maximal solvable normal subgroup of G contained in N and N_0/M be a chief factor of G with $N_0 \leqslant N$. Then $N_0/W \cong PSL_2(3^{f_1})$, where f_1 is a 2-power. Let $\widetilde{G} := G/W$. Take any prime power element $e \in G \setminus N$. Then $\widetilde{N_0}\widetilde{e} \subseteq \widetilde{N}\widetilde{e} \subseteq G \setminus N$. It follows that every element in $\widetilde{N_0}\widetilde{e}$ has prime power order. By theorem B, we get that the order of e is a 2-power. By the arbitrariness of e, we get that G/N is a 2-group, contrary our assumption that G/N is non-solvable. Consequently, G/N is solvable. □

Corollary 1. *Let G be a group and $N \trianglelefteq G$. If every element $x \in G \setminus N$ has prime power order, and $\mathbf{C}_G(x)$ is maximal in G, then N is solvable.*

Proof. Suppose on the contrary that N is non-solvable. By theorem A, G has a non-solvable chief factor $M_2/M_1 \cong PSL_2(3^f)$, where f is a 2-power such that $M_2 \leqslant N$

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and M_1 is solvable. Write $\overline{G} := G/M_1$. Since $\overline{M_2}\overline{u} \subseteq \overline{N}\overline{u} \subseteq \overline{G} \setminus \overline{N}$, we see that each element in $\overline{M_2}\overline{u}$ has prime power order. By theorem B, $\overline{u}$ is a product of a field automorphism and a diagonal automorphism of $\overline{M_2}$, and $\overline{M_2} \cong PSL_2(3^f)$ with $f > 1$ a 2-power. Easily, $\overline{u}$ is a 2-element. Without loss of generality, we may consider u is a 2-element satisfying $G = M_2\langle u\rangle$ and thus $\overline{G} = \overline{M_2}\langle\overline{u}\rangle$.

Now, consider $P := \mathbf{C}_G(u)$. Since every element of $G \setminus N$ is a 2-element, we see that P must be a 2-group. The maximality of P indicates that P is a Sylow 2-subgroup of G. If $M_1 \nleqslant P$, we have that $G = PM_1$, indicating that G is solvable, a contradiction. Hence, $M_1 \leqslant P$. Then $\overline{P}$ is a maximal subgroup of $\overline{G}$. Let $P_0 := P \cap M_2$. Then $P_0 \in \mathrm{Syl}_2(M_2)$, forcing $\overline{P_0} \in \mathrm{Syl}_2(\overline{M_2})$. Let $|\overline{P_0}| = 2^a$. If $\overline{P_0}$ is maximal in $\overline{M_2}$, then by [**17**, Theorem 2.8.27], $2^a = 3^f \pm 1$. By Lemma 2.1, it follows that $a = 3$ and $f = 2$. Therefore, $\overline{M_2} \cong PSL_2(3^2)$. However, according to [**4**], we see that $\overline{P_0}$ is not maximal in $\overline{M_2}$. This contradiction forces that $\overline{P_0}$ is not maximal in $\overline{M_2}$. Furthermore, as $\overline{M_2} \leqslant \overline{G} \leqslant \mathrm{Aut}(\overline{M_2})$ and $\overline{M_2} \cong PSL_2(3^f)$, we obtain that $\overline{G} \cong M_{10} \cong PSL_2(9) \cdot \langle\overline{u}\rangle$ and $\overline{P} \cong C_8 \rtimes C_2$ or D_{16} by [**7**, Theorem 1.1]. In this case, $\overline{u} \notin \mathbf{Z}(\overline{P})$, the final contradiction completes the proof. □

4. Theorem D and its proof

For the reader's convenience, we restate theorem D.

THEOREM 4.1. *Let G be a group and N be a proper normal subgroup of G such that $\mathbf{Z}(G) < N$. If $\mathbf{C}_G(x)$ is maximal in G for every element $x \in G \setminus N$, then G is solvable with G/N abelian. Furthermore,*

(I) *If G is nilpotent, then G/N is a p-group for some prime p. Moreover, $G = P \times \mathbf{Z}(G)_{p'}$ and $\mathbf{C}_G(x) \trianglelefteq G$ for every $x \in P \setminus N$;*

(II) *G is non-nilpotent, then $|G : \mathbf{C}_G(x)| = r^a$ with prime r and positive integer a. Suppose that R is a Sylow r-subgroup of G and K is a Hall r'-subgroup of G, then one of the statements holds:*
- (1) *If G/N is a p-group for some prime p, we have*
  - (1.1) *If $r = p$, write $\overline{G} := G/\mathbf{O}_p(G)\mathbf{Z}(G)$. Then $\overline{G} = \overline{K} \rtimes \overline{P}$ is a Frobenius group with abelian kernel $\overline{K}$ and complement $\overline{P}$ of order p.*
  - (1.2) *If $r \neq p$, write $\widetilde{G} := G/\mathbf{Z}(G)_r$. Then*
    - (1.2.1) *If $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$, then the Fitting length $h(\widetilde{G}) = 3$ and $\widetilde{G}$ has the following normal series:*

$$\widetilde{1} \trianglelefteq \mathbf{O}_{r'}(\widetilde{G}) \trianglelefteq \mathbf{O}_{r',r}(\widetilde{G}) \trianglelefteq \widetilde{G} = \mathbf{O}_{r',r,r'}(\widetilde{G}),$$

where $\mathbf{O}_{r',r}(\widetilde{G})/\mathbf{O}_{r'}(\widetilde{G}) \cong \widetilde{R}$ is elementary abelian and $\widetilde{G}/\mathbf{O}_{r',r}(\widetilde{G})$ is a p-group;

- (1.2.2) *Assume that $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. If $\mathbf{C}_G(x)_{p'} \nleqslant \mathbf{Z}(G)$, then the Fitting length $h(\widetilde{G}) = 2$ and $\widetilde{G}$ has the following normal series:*

$$\widetilde{1} \trianglelefteq \widetilde{R} \trianglelefteq \widetilde{N} \trianglelefteq \widetilde{G},$$

where $\widetilde{R}$ is an elementary abelian r-group and G/R is nilpotent.

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(1.2.3) *Write* $\widehat{G} := G/\mathbf{Z}(G)$. *Assume that* $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. *If* $\mathbf{C}_G(x)_{p'} \leqslant \mathbf{Z}(G)$, *then* $\widehat{G} = \widehat{R} \rtimes \widehat{P}$, *where* P *is a Sylow* p*-subgroup of* G *and* R *is a Sylow* r*-subgroup of* G *with* $\widehat{R}$ *elementary abelian. Let* $N_p = N \cap P$. *If* $\widehat{N_p} \trianglelefteq \widehat{G}$, *then* $\widehat{G}/\widehat{N_p}$ *is a Frobenius group; if* $\widehat{N_p} \ntrianglelefteq \widehat{G}$, *then* $\mathbf{N}_{\widehat{G}}(\widehat{N_p}) = \widehat{P}$.

(2) *If* $|\pi(G/N)| \geqslant 2$, *then one of the following statements holds:*

(2.1) *Let* $\overline{G} := G/\mathbf{O}_r(G)\mathbf{Z}(G)$. *Then* $\overline{G} = \overline{K} \rtimes \overline{R}$ *is a Frobenius group with abelian kernel* $\overline{K}$ *and complement* $\overline{R}$ *of order* r*;*

(2.2) *Let* $\overline{G} := G/\mathbf{Z}(G)$. *Then* $\overline{G} = \overline{R} \rtimes \overline{K}$ *is a Frobenius group with* $\overline{R}$ *a minimal normal subgroup of* $\overline{G}$ *and* $\overline{K}$ *cyclic. In particular,* $R \leqslant N$.

Proof. Let $x \in G \setminus N$ be an arbitrary element. Then there exists a component of x, say x_1, such that $x_1 \in G \setminus N$. Note that $\mathbf{C}_G(x) \leqslant \mathbf{C}_G(x_1)$ and both $\mathbf{C}_G(x)$ and $\mathbf{C}_G(x_1)$ are maximal in G. Then $\mathbf{C}_G(x) = \mathbf{C}_G(x_1)$. Without loss of generality, we may consider x as a p-element for some prime p. Furthermore,

Step 1. $\mathbf{C}_G(x) = \mathbf{C}_G(x)_p \times \mathbf{C}_G(x)_{p'}$, where $\mathbf{C}_G(x)_p$ is the Sylow p-subgroup of $\mathbf{C}_G(x)$ and $\mathbf{C}_G(x)_{p'}$ is the Hall p'-subgroup of $\mathbf{C}_G(x)$ with $\mathbf{C}_G(x)_{p'} \leqslant \mathbf{Z}(\mathbf{C}_G(x))$. In particular, $\mathbf{C}_G(x)$ is nilpotent.

For every p'-element $v \in \mathbf{C}_G(x)$, we always have $xv \in G \setminus N$. By lemma 5.1, it follows that $\mathbf{C}_G(vx) = \mathbf{C}_G(x) \cap \mathbf{C}_G(v)$. Note that $\mathbf{C}_G(x)$ and $\mathbf{C}_G(xv)$ are maximal subgroups of G. Then $\mathbf{C}_G(vx) = \mathbf{C}_G(x) \leqslant \mathbf{C}_G(v)$, yielding $v \in \mathbf{Z}(\mathbf{C}_G(x))$. As a result, $\mathbf{C}_G(x) = \mathbf{C}_G(x)_p \times \mathbf{C}_G(x)_{p'}$, where $\mathbf{C}_G(x)_p$ is the Sylow p-subgroup of $\mathbf{C}_G(x)$ and $\mathbf{C}_G(x)_{p'} \leqslant \mathbf{Z}(\mathbf{C}_G(x))$. Clearly, $\mathbf{C}_G(x)$ is nilpotent.

Step 2. G/N is abelian.
For every $y \in G \setminus N$, we see that $\mathbf{C}_G(y)$ is maximal in G and $\mathbf{C}_G(y)N/N \leqslant \mathbf{C}_{G/N}(yN)$. If $N \leqslant \mathbf{C}_G(y)$, then $y \in \mathbf{C}_G(N)$; if $N \nleqslant \mathbf{C}_G(y)$, then $G = \mathbf{C}_G(y)N$, forcing $\mathbf{C}_{G/N}(yN) = G/N$. Therefore, $yN \in \mathbf{Z}(G/N) := Z/N$, yielding to $y \in Z$. As a result, $G = Z \cup \mathbf{C}_G(N)$, which implies that $G = Z$ or $G = \mathbf{C}_G(N)$. Since $\mathbf{Z}(G) < N$, we obtain that $G = Z$. Hence, $G/N = Z/N = \mathbf{Z}(G/N)$. Consequently, G/N is abelian, as required.

Step 3. G is solvable.
Assume false. If there exists a $2'$-element $x_0 \in G \setminus N$ having prime power order, then by step 1, we see that $\mathbf{C}_G(x_0)$ is nilpotent. Write $\mathbf{C}_G(x_0) = T_0 \times U_0$, where T_0 is the Sylow 2-subgroup of $\mathbf{C}_G(x_0)$ and U_0 is the Hall $2'$-subgroup of $\mathbf{C}_G(x_0)$. By [**23**, Theorem 1], we obtain that $U_0 \trianglelefteq G$, $\mathbf{Z}(U_0) \leqslant \mathbf{Z}(G)$, and $G/\mathbf{Z}(U_0) \cong G/U_0 \times U_0/\mathbf{Z}(U_0)$.

Note that G is non-solvable. So is G/U_0. As $T_0U_0/U_0 = \mathbf{C}_G(x_0)/U_0$ is a maximal 2-subgroup of G/U_0, we assert that T_0U_0/U_0 must be a Sylow 2-subgroup of G/U_0, which indicates that T_0U_0/U_0 is not normal in G/U_0 as G/U_0 is non-solvable. On the contrary, by step 1, T_0 is abelian, so is T_0U_0/U_0. Hence, $\mathbf{N}_{G/U_0}(T_0U_0/U_0) \geqslant \mathbf{C}_{G/U_0}(T_0U_0/U_0) \geqslant T_0U_0/U_0$, which yields to $\mathbf{N}_{G/U_0}(T_0U_0/U_0) = \mathbf{C}_{G/U_0}(T_0U_0/U_0)$. By [**19**, Theorem 7.2.1], we see that G/U_0

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has a normal 2-complement, against the fact that G/U_0 is non-solvable. Consequently, each element in $G \setminus N$ is a 2-element. By corollary 1, N is solvable, so is G by step 2, which is a contradiction.

Step 4. If G is nilpotent, then G/N is a p-group and $G = P \times \mathbf{Z}(G)_{p'}$ with $P \in \mathrm{Syl}_p(G)$. Furthermore, $\mathbf{C}_G(x)$ is a normal maximal subgroup of G for any $x \in P \setminus N$ satisfying $|G/\mathbf{C}_G(x)| = p$.
Assume that G is nilpotent. If $|\pi(G/N)| \geqslant 2$, then there exist two distinct primes p, $q \in \pi(G/N)$. Select $w \in P \setminus N$ a p-element and $v \in Q \setminus N$ a q-element, where P and Q are Sylow p-subgroup and Sylow q-subgroup of G, respectively. In this case, $wv = vw \in G \setminus N$, showing that $\mathbf{C}_G(v), \mathbf{C}_G(w)$, and $\mathbf{C}_G(vw)$ are all maximal subgroups of G. On the contrary, lemma 5.1 indicates that $\mathbf{C}_G(wv) = \mathbf{C}_G(w) \cap \mathbf{C}_G(v)$, which forces that $\mathbf{C}_G(wv) = \mathbf{C}_G(w) = \mathbf{C}_G(v)$. Write $G = P \times Q \times W$, where W is the Hall $\{p, q\}'$-subgroup of G. Clearly, $P \times W \leqslant \mathbf{C}_G(v)$ and $P \times Q \leqslant \mathbf{C}_G(w)$, forcing $v \in \mathbf{Z}(G) < N$. This contradiction deduces $|\pi(G/N)| = 1$. Moreover, G/N is a p-group. Let $N_{p'}$ be the Hall p'-subgroup of N. As $N_{p'}$ char $N \trianglelefteq G$, we obtain that $N_{p'}$ is a normal Hall p'-subgroup of G. Therefore, $G = P \times N_{p'}$, where P is the Sylow p-subgroup of G.

We claim that $N_{p'} \leqslant \mathbf{Z}(G)$. If not, select $y \in N_{p'} \setminus \mathbf{Z}(G)$. Suppose that $x \in P \setminus N$. By lemma 5.1, we see that $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$. Since $\mathbf{C}_G(x)$ and $\mathbf{C}_G(xy)$ are maximal subgroups of G, we get that $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \leqslant \mathbf{C}_G(y)$. Moreover, $\mathbf{C}_G(x) = \mathbf{C}_G(y)$ as $y \notin \mathbf{Z}(G)$. Clearly, $N_{p'} \leqslant \mathbf{C}_G(x)$ and $P \leqslant \mathbf{C}_G(y)$, we get $y \in \mathbf{Z}(G)$, a contradiction. Consequently, $N_{p'} \leqslant \mathbf{Z}(G)$, leading that $G/\mathbf{Z}(G)$ is a p-group. Notice that $\mathbf{C}_G(x)/\mathbf{Z}(G)$ is a maximal subgroup of $G/\mathbf{Z}(G)$. Then $\mathbf{C}_G(x)$ is a maximal normal subgroup of G. Moreover, $|G/\mathbf{C}_G(x)| = |(G/\mathbf{Z}(G))/(\mathbf{C}_G(x)/\mathbf{Z}(G))| = p$, as required.

Step 5. The conclusion when G is non-nilpotent.
In the following, we consider the case that G is non-nilpotent. We will divide the proof into two cases depending on $|\pi(G/N)| = 1$ or not.

Case 1. $|\pi(G/N)| = 1$.
In this case, G/N is a p-group. Let $w \in G \setminus N$ be an arbitrary p-element. By assumption, $\mathbf{C}_G(w)$ is a maximal subgroup of G, implying $|G : \mathbf{C}_G(w)| = r^a$ as G is solvable, where r is a prime and a is a positive integer.

Subcase 1.1. $r = p$.
Then $|G : \mathbf{C}_G(w)| = p^a$. By step 1, $\mathbf{C}_G(w) = P_w \times K_w$ is nilpotent with Sylow p-subgroup P_w and abelian Hall p'-subgroup K_w. Clearly, $K := K_w$ is a Hall p'-subgroup of G. Let P be a Sylow p-subgroup of G containing P_w. Then $P_w \lneq P$, leading $\mathbf{N}_G(P_w) > \mathbf{C}_G(w)$. Since $\mathbf{C}_G(w)$ is maximal in G, we have $P_w \trianglelefteq G$, and thus $\mathbf{C}_G(P_w) \trianglelefteq G$. Note that $K \leqslant \mathbf{C}_G(P_w) \leqslant \mathbf{C}_G(w)$. As $\mathbf{C}_G(w)$ is nilpotent, so is $\mathbf{C}_G(P_w)$. Hence, $K \trianglelefteq G$ because K char $\mathbf{C}_G(P_w) \trianglelefteq G$. As a result, $K = \mathbf{O}_{p'}(G)$ and $G = K \rtimes P$.

Clearly, $P \ntrianglelefteq G$, since otherwise, $G = P \times K$ with $K \leqslant \mathbf{Z}(G)$, implying that G is nilpotent, against our assumption. Hence, $\mathbf{O}_p(G) \lneq P$. Along with the fact that $\mathbf{C}_G(w) = P_w \times K \leqslant \mathbf{O}_p(G) \times \mathbf{O}_{p'}(G) \lneq G$. The maximality of $\mathbf{C}_G(w)$ indicates that $\mathbf{C}_G(w) = \mathbf{O}_p(G) \times \mathbf{O}_{p'}(G) \trianglelefteq G$. In particular, $|G : \mathbf{C}_G(w)| = p$ and $|P : \mathbf{O}_p(G)| = p$.

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Let $N_0 := \mathbf{O}_p(G)\mathbf{Z}(G)$. Then $N_0 \trianglelefteq G$. Let further $\overline{G} := G/N_0$. Easily, $|\overline{P}| = p$. We show that $\overline{G} = \overline{K} \rtimes \overline{P}$ is a Frobenius group with abelian kernel $\overline{K}$ and complement $\overline{P}$. Otherwise, there must exist $k \in K \setminus N_0$ and $y \in P \setminus N_0$ such that $[k, y] \in N_0$. Since $K \trianglelefteq G$, we see that $[k, y]$ is a p'-element, forcing $[k, y] \in \mathbf{Z}(G)$. Then $1 = [k^{o(k)}, y] = [k, y]^{o(k)}$ and $1 = [k, y^{o(y)}] = [k, y]^{o(y)}$, forcing $[k, y] = 1$. Recall that $K = \mathbf{O}_{p'}(G)$ is abelian and $y \notin \mathbf{O}_p(G)$, we have $\mathbf{C}_G(k) \geqslant \langle \mathbf{C}_G(w), y\rangle > \mathbf{C}_G(w)$. The maximality of $\mathbf{C}_G(w)$ forces $k \in \mathbf{Z}(G) \leqslant N_0$, against the choice of k. Hence, statement (1.1) of the theorem holds.

Subcase 1.2. $r \neq p$.
In this case, $w \in G\backslash N$ is a p-element such that $|G : \mathbf{C}_G(w)| = r^a$ is a p'-number. By step 1, $\mathbf{C}_G(w) = P_w \times K_w$ is nilpotent with Sylow p-subgroup P_w and abelian Hall p'-subgroup K_w, showing that $P := P_w$ is a Sylow p-subgroup of G.

Assume first $P \trianglelefteq G$. Then $\mathbf{C}_G(P) \trianglelefteq G$. As $K_w \leqslant \mathbf{C}_G(P) \leqslant \mathbf{C}_G(w)$ and $\mathbf{C}_G(w)$ is nilpotent, we see that K_w char $\mathbf{C}_G(P) \trianglelefteq G$, yielding $K_w \trianglelefteq G$. Hence, $\mathbf{C}_G(w) = P \times K_w \leqslant P \times \mathbf{O}_{p'}(G)$. Let K be a Hall p'-subgroup of G containing K_w. If $K \trianglelefteq G$, then $G = P \times K$. Under this situation, $K \leqslant \mathbf{C}_G(w)$, forcing $G = \mathbf{C}_G(w)$. This contradiction indicates that $\mathbf{C}_G(w) = P \times \mathbf{O}_{p'}(G) \trianglelefteq G$ with $|K/\mathbf{O}_{p'}(G)| = r$.

Let $N_0 := \mathbf{O}_{p'}(G)(P \cap N)$. Then $N_0 \trianglelefteq G$. Write $\overline{G} := G/N_0$. Then $\overline{N} = \overline{K} \trianglelefteq \overline{G}$ as G/N is a p-group. Moreover, $\overline{G} = \overline{P} \times \overline{K}$ is a $\{p, r\}$-group. Take $z \in G \setminus N_0$ a $\{p, r\}$-element. Write $z = ab$, where $a \in P \setminus N_0$ is the p-part and $b \in K \setminus N_0$ is the r-part of z, respectively. By step 1, we see that $\mathbf{C}_G(a) = P_a \times K_a$, where P_a is the Sylow p-subgroup of $\mathbf{C}_G(a)$ and K_a is the Hall p'-subgroup of $\mathbf{C}_G(a)$. Note that $a \in \mathbf{C}_G(w)$. Then $K_w \leqslant K_a$. Analogously, $K_a \leqslant K_w$ since $w \in \mathbf{C}_G(a)$. This deduces that $b \in K_a = K_w = \mathbf{O}_{p'}(G) \leqslant N_0$, against our assumption. As a consequence, $P \ntrianglelefteq G$.

Recall that $\mathbf{C}_G(w) = P \times K_w$ with K_w abelian. Then $\mathbf{C}_G(w) \leqslant \mathbf{C}_G(K_w)$, leading to $\mathbf{C}_G(K_w) = \mathbf{C}_G(w)$ or $K_w \leqslant \mathbf{Z}(G)$ by the maximality of $\mathbf{C}_G(w)$. Assume first the former holds. Let R_0 be the Sylow r-subgroup of K_w and R be a Sylow r-subgroup of G such that $R_0 \lneqq R$. This indicates that $\mathbf{C}_G(w) < \mathbf{N}_G(R_0)$. Furthermore, the maximality of $\mathbf{C}_G(w)$ forces $R_0 \trianglelefteq G$, and thus $\mathbf{C}_G(R_0) \trianglelefteq G$.

On the contrary, K_w is abelian, implying $\mathbf{C}_G(w) \leqslant \mathbf{C}_G(R_0)$. Again by the maximality of $\mathbf{C}_G(w)$, we see that either $R_0 \leqslant \mathbf{Z}(G)$ or $\mathbf{C}_G(w) = \mathbf{C}_G(R_0)$. If the latter holds, then $\mathbf{C}_G(R_0) = \mathbf{C}_G(w) \trianglelefteq G$. Since $\mathbf{C}_G(w)$ is nilpotent, we obtain that $P \trianglelefteq G$, against our assumption.

As a result, $R_0 \leqslant \mathbf{Z}(G)$. Write $\widetilde{G} := G/R_0$. Then $\widetilde{G} = \widetilde{R}\mathbf{C}_{\widetilde{G}}(\widetilde{w})$. Since $\mathbf{C}_{\widetilde{G}}(\widetilde{w}) = \mathbf{C}_G(w)/R_0$ is maximal in $\widetilde{G}$, we have that $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$ or $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. Assume first that $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$. Since G is solvable, we have $\mathbf{O}_{r'}(\widetilde{G}) > \widetilde{1}$. We assert that $\widetilde{P} \ntrianglelefteq \widetilde{G}$. Since otherwise, $P \times R_0 \trianglelefteq G$, leading $P \trianglelefteq G$, contrary to our assumption.

Consequently, $\widetilde{P} \nleq \mathbf{O}_{r'}(\widetilde{G})$. Note that $\widetilde{P}\mathbf{O}_{r'}(\widetilde{G}) \leqslant \mathbf{C}_{\widetilde{G}}(\widetilde{w})$ and $\mathbf{C}_{\widetilde{G}}(\widetilde{w})$ is nilpotent. Then $\widetilde{J} = \widetilde{P} \times \widetilde{L}$ with $\widetilde{L}$ abelian, where L is a Hall $\{p, r\}'$-subgroup of $\mathbf{O}_{r'}(\widetilde{G})$. Easily, $\widetilde{L} \trianglelefteq \widetilde{G}$ and $\mathbf{C}_{\widetilde{G}}(\widetilde{L}) \geqslant \mathbf{C}_{\widetilde{G}}(\widetilde{w})$. By the maximality of $\mathbf{C}_{\widetilde{G}}(\widetilde{w})$, we have $\mathbf{C}_{\widetilde{G}}(\widetilde{L}) = \mathbf{C}_{\widetilde{G}}(\widetilde{w})$ or $\widetilde{L} \leqslant \mathbf{Z}(\widetilde{G})$. Assume first that $\mathbf{C}_{\widetilde{G}}(\widetilde{L}) = \mathbf{C}_{\widetilde{G}}(\widetilde{w})$ holds. In this situation, $\mathbf{C}_{\widetilde{G}}(\widetilde{L}) \trianglelefteq \widetilde{G}$, we get $\widetilde{P} \trianglelefteq \widetilde{G}$, against the argument in the previous paragraph. Hence, $\widetilde{L} \leqslant \mathbf{Z}(\widetilde{G})$. Therefore, $\mathbf{O}_{r'}(\widetilde{G}) = \mathbf{O}_p(\widetilde{G}) \times \mathbf{Z}(\widetilde{G})_{p'}$. Furthermore, $\mathbf{C}_{\widetilde{G}}(\mathbf{O}_{r'}(\widetilde{G})) \leqslant \mathbf{O}_{r'}(\widetilde{G})$, which indicates that the Hall $\{p, r\}'$-subgroup of $\mathbf{O}_{r'}(\widetilde{G})$

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is also a Hall $\{p, r\}'$-subgroup of $\widetilde{G}$. Hence, $\mathbf{C}_{\widetilde{G}}(\widetilde{w}) = \widetilde{P} \times \mathbf{Z}(\widetilde{G})_{p'}$. Without loss of generality, we may assume that $\mathbf{Z}(\widetilde{G})_{p'} = \widetilde{1}$.

In this case, $\mathbf{C}_{\widetilde{G}}(\widetilde{w}) = \widetilde{P}$, and thus $\widetilde{G} = \widetilde{R}\widetilde{P}$, leading that $\widetilde{G}$ is a $\{p, r\}$-group. Recall that $\mathbf{O}_r(\widetilde{G}) = 1$, we have $\mathbf{O}_p(\widetilde{G}) \neq 1$. As a result, $\mathbf{O}_p(\widetilde{G}) < \widetilde{P}$ since $\widetilde{P} \ntrianglelefteq G$. Note that $\mathbf{O}_p(\widetilde{G}/\mathbf{O}_p(\widetilde{G})) = 1$. We see that $\mathbf{O}_r(\widetilde{G}/\mathbf{O}_p(\widetilde{G})) \neq 1$ as $\widetilde{G}$ is solvable. Therefore, $\widetilde{G}/\mathbf{O}_p(\widetilde{G}) = \mathbf{O}_{p,r}(\widetilde{G})/\mathbf{O}_p(\widetilde{G}) \rtimes \mathbf{C}_{\widetilde{G}}(\widetilde{w})/\mathbf{O}_p(\widetilde{G})$ by the maximality of $\mathbf{C}_{\widetilde{G}}(\widetilde{w})/\mathbf{O}_p(\widetilde{G})$. Furthermore, $\mathbf{C}_{\widetilde{G}}(\widetilde{w})/\mathbf{O}_p(\widetilde{G})$ acts irreducibly on $\mathbf{O}_{p,r}(\widetilde{G})/\mathbf{O}_p(\widetilde{G})$ and $\mathbf{O}_{p,r}(\widetilde{G})/\mathbf{O}_p(\widetilde{G})$ is an elementary abelian r-group. Since $\mathbf{O}_{p,r}(\widetilde{G})/\mathbf{O}_p(\widetilde{G}) \cong \widetilde{R}$, we get that $\widetilde{R}$ is elementary abelian. Clearly, $\widetilde{G}/\mathbf{O}_p(\widetilde{G})$ is not nilpotent. Hence, the Fitting length $h(\widetilde{G}) = 3$. In this case, $\widetilde{G}$ has the following normal series:

$$\widetilde{1} \trianglelefteq \mathbf{O}_p(\widetilde{G}) \trianglelefteq \mathbf{O}_{p,r}(\widetilde{G}) \trianglelefteq \widetilde{G} = \mathbf{O}_{p,r,p}(\widetilde{G}),$$

where $\mathbf{O}_{p,r}(\widetilde{G})/\mathbf{O}_p(\widetilde{G})$ is an elementary abelian Sylow r-group and $\widetilde{G}/\mathbf{O}_{p,r}(\widetilde{G})$ is a p-group, as required in Statement (1.2.1).

Assume now that $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. Then $\widetilde{G} = \mathbf{O}_r(\widetilde{G}) \rtimes \mathbf{C}_{\widetilde{G}}(\widetilde{w})$. By the same reason as above, $\mathbf{C}_{\widetilde{G}}(\widetilde{w})$ acts irreducibly on $\mathbf{O}_r(\widetilde{G})$ and $\mathbf{O}_r(\widetilde{G})$ is an elementary abelian r-group. In this case, $\widetilde{G}$ has the following normal series:

$$\widetilde{1} \trianglelefteq \widetilde{R} \trianglelefteq \widetilde{N} \trianglelefteq \widetilde{G}$$

where $\widetilde{R}$ is an elementary abelian r-group and G/R is nilpotent, statement (1.2.2) holds.

Now, we consider the case that $K_w \leqslant \mathbf{Z}(G)$. Let $l \in G \setminus N$ be an arbitrary primary element. As $\mathbf{Z}(G) < N$, then $l \notin \mathbf{Z}(G)$. By assumption, $\mathbf{C}_G(l)$ is maximal in G. Recall that G is solvable. Therefore, $|G : \mathbf{C}_G(l)|$ is a prime power. If $|G : \mathbf{C}_G(l)|$ is a p-power, we are done according to case 1. As a result, $|G : \mathbf{C}_G(l)|$ is a q-power with prime q distinct from p. Moreover, $\mathbf{C}_G(l)^g = P^g \times K_w \leqslant \mathbf{C}_G(l)$ for some $g \in G$, forcing $|G : \mathbf{C}_G(l)|$ is a power of r. By [**24**, Lemma 2.5], $l \in \mathbf{O}_{r,r'}(G)$, yielding $G \setminus N \subseteq \mathbf{O}_{r,r'}(G)$. Furthermore, $G = \mathbf{O}_{r,r'}(G)$, implying that G has a normal Sylow r-subgroup R.

Let $\widehat{G} := G/\mathbf{Z}(G)$. Since $\mathbf{C}_G(l)$ is maximal in G and $\mathbf{C}_{\widehat{G}}(\widehat{l}) \geqslant \mathbf{C}_G(l)/\mathbf{Z}(G)$, we obtain that $\mathbf{C}_{\widehat{G}}(\widehat{l}) = \widehat{G}$ or $\mathbf{C}_{\widehat{G}}(\widehat{l}) = \mathbf{C}_G(l)/\mathbf{Z}(G)$. Note that $\mathbf{C}_G(l) = P^g \times K_w$, we conclude that $\mathbf{C}_{\widehat{G}}(\widehat{l}) = \mathbf{C}_G(l)/\mathbf{Z}(G)$ and thus $\mathbf{C}_{\widehat{G}}(\widehat{l}) = \widehat{P}$ is maximal in $\widehat{G}$. Then $\pi(\widehat{G}) = \{p, r\}$, and $\widehat{P}$ acts on $\widehat{R}$ irreducibly. Moreover, $\widehat{R}$ is the minimal normal subgroup of $\widehat{G}$, and thus $\widehat{R}$ is elementary abelian. Let $N_p = N \cap P$. Then $\widehat{N_p} \trianglelefteq \widehat{P}$. If $\widehat{N_p} \trianglelefteq \widehat{G}$, then $\widehat{G}/\widehat{N_p}$ is a Frobenius group. If $\widehat{N_p} \ntrianglelefteq \widehat{G}$, then $\mathbf{N}_{\widehat{G}}(\widehat{N_p}) = \widehat{P}$, statement (1.2.3) holds.

Case 2. $|\pi(G/N)| \geqslant 2$.
Let $\pi := \pi(G/N)$. Recall that G/N is abelian. There must exist an element $wN \in G/N$ such that $\pi(wN) = \pi$. Without loss, we may consider $w \in G \setminus N$ is an element with $\pi(w) = \pi$. Suppose that w_p, $w_q \in G \setminus N$ is the p-part and the q-part of w, respectively. By lemma 5.1, we have $\mathbf{C}_G(w) \leqslant \mathbf{C}_G(w_p) \cap \mathbf{C}_G(w_q)$. Note that all of $\mathbf{C}_G(w)$, $\mathbf{C}_G(w_p)$, and $\mathbf{C}_G(w_q)$ are maximal subgroups of G. This indicates that

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$\mathbf{C}_G(w) = \mathbf{C}_G(w_p) = \mathbf{C}_G(w_q)$. In particular, $\mathbf{C}_G(w)$ is abelian according to step 1. Recall that G is solvable and $\mathbf{C}_G(w)$ is a maximal subgroup of G, indicating that $|G : \mathbf{C}_G(w)| = r^a$, where r is a prime and $a > 0$ is an integer.

Write $\mathbf{C}_G(w) = K \times R_w$, where K is the abelian Hall r'-subgroup of G and R_w is the Sylow r-subgroup of $\mathbf{C}_G(w)$. Let R be a Sylow r-subgroup of G such that $R_w < R$. Easily, $\mathbf{N}_G(R_w) \gneq \mathbf{C}_G(w)$. The maximality of $\mathbf{C}_G(w)$ yields to $R_w \trianglelefteq G$. Moreover, $\mathbf{C}_G(w) \leqslant \mathbf{C}_G(R_w) \trianglelefteq G$. Again by the maximality of $\mathbf{C}_G(w)$, we get $\mathbf{C}_G(R_w) = \mathbf{C}_G(w)$ or $R_w \leqslant \mathbf{Z}(G)$.

Subcase 2.1. $\mathbf{C}_G(R_w) = \mathbf{C}_G(w)$.
In this case, $\mathbf{C}_G(w) \trianglelefteq G$ and $|G : \mathbf{C}_G(w)| = r$. Furthermore, $K \trianglelefteq G$ since K is the Hall r'-subgroup of abelian group $\mathbf{C}_G(w)$, implying $G = K \rtimes R$. Notice that $\mathbf{C}_G(w) = K \times R_w \leqslant \mathbf{O}_{r'}(G) \times \mathbf{O}_r(G)$. Since G is non-nilpotent and $\mathbf{C}_G(w)$ is maximal, we see that $R_w = \mathbf{O}_r(G)$ with $|R : R_w| = r$.

Let $N_0 := \mathbf{Z}(G)\mathbf{O}_r(G)$ and $\overline{G} := G/N_0$. Then $\overline{G} = \overline{K} \rtimes \overline{R}$ with $|\overline{R}| = r$. Assume that there exists an $\{r, t\}$-element $\overline{e} \in \overline{G}$ for some prime $t \neq r$. We may assume that $e \in G \setminus N_0$ is an $\{r, t\}$-element. Write $e = e_1e_2$, where $e_1 \in R \setminus N_0$ and $e_2 \in K \setminus N_0$ are the r-part and the t-part of e, respectively. In this case, $e_1 \in \mathbf{C}_G(e_2) = \mathbf{C}_G(w)$, forcing $e_1 \in \mathbf{O}_r(G)$. This contradiction shows that $\overline{G}$ is a Frobenius group with abelian kernel $\overline{K}$ and a complement $\overline{R}$ of order r, statement (2.1) of the theorem holds.

Subcase 2.2. $R_w \leqslant \mathbf{Z}(G)$.
Recall that $w \in G \setminus N$ with $\pi(w) = \pi$. We assert that $r \notin \pi$. If not, assume that $w_r \in G \setminus N$ is the r-part of w. Easily, $w_r \in \mathbf{C}_G(w) = K \times R_w$, forcing $w_r \in R_w \leqslant \mathbf{Z}(G) < N$, which is a contradiction. As a result, $r \nmid |G/N|$, leading $R \leqslant N$. Moreover, $G = \mathbf{C}_G(w)N = \mathbf{C}_G(w)R$.

Write $\widetilde{G} := G/\mathbf{Z}(G)$. Then $\widetilde{G} = \widetilde{\mathbf{C}_G(w)}\widetilde{R}$. Easily, $\widetilde{\mathbf{C}_G(w)}$ is a maximal r'-subgroup of $\widetilde{G}$, implying $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$ or $\widetilde{R}$. Assume first $\mathbf{O}_r(\widetilde{G}) = \widetilde{1}$. Then $\mathbf{O}_{r'}(\widetilde{G}) > \widetilde{1}$ since G is solvable. In particular, $\widetilde{K} \leqslant \mathbf{C}_{\widetilde{G}}(\mathbf{O}_{r'}(\widetilde{G})) \leqslant \mathbf{O}_{r'}(\widetilde{G}) \leqslant \widetilde{K}$ since $\widetilde{K}$ is an abelian Hall r'-subgroup of $\widetilde{G}$, yielding $\mathbf{C}_{\widetilde{G}}(\mathbf{O}_{r'}(\widetilde{G})) = \mathbf{O}_{r'}(\widetilde{G}) = \widetilde{K}$. As a result, $K \trianglelefteq G$ and $\mathbf{C}_G(w) \trianglelefteq G$.

Consequently, $\widetilde{G} = \widetilde{\mathbf{C}_G(w)} \rtimes \widetilde{R}$. We prove that $\widetilde{G}$ is a Frobenius group. Suppose false, there exists an $\{r, t\}$-element $\widetilde{e} \in \widetilde{G}$ for some prime $t \neq r$. We may assume that e is a $\{r, t\}$-element. Write $e = e_1e_2$, where $e_1 \in R \setminus \mathbf{Z}(G)$ and $e_2 \in K \setminus \mathbf{Z}(G)$ are the r-part and the t-part of e, respectively. Note that $e_1 \in \mathbf{C}_G(e_2) = K \times R_w = \mathbf{C}_G(w)$. This contradiction shows that $\widetilde{G}$ is a Frobenius group with abelian kernel $\widetilde{\mathbf{C}_G(w)}$. Moreover, $\widetilde{\mathbf{C}_G(w)}$ is maximal in $\widetilde{G}$ indicates that $\widetilde{R}$ is of order r, statement (2.1) of the theorem holds.

Now, we consider $\mathbf{O}_r(\widetilde{G}) = \widetilde{R}$. Write $\widetilde{G} = \widetilde{R} \rtimes \widetilde{\mathbf{C}_G(w)}$. Since $\widetilde{\mathbf{C}_G(w)}$ is maximal in $\widetilde{G}$, we see that $\widetilde{\mathbf{C}_G(w)}$ acts irreducibly on $\widetilde{R}$. By the same argument in the previous paragraph, we conclude that $\widetilde{G}$ is a Frobenius group with kernel $\widetilde{R}$ and complement $\widetilde{\mathbf{C}_G(w)}$. Furthermore, $\widetilde{R}$ is a minimal normal subgroup of $\widetilde{G}$ and thus $\widetilde{R}$ is elementary abelian by the maximality of $\widetilde{\mathbf{C}_G(w)}$. Also $\widetilde{\mathbf{C}_G(w)}$ is cyclic as $\widetilde{\mathbf{C}_G(w)}$ is abelian, implying that G/N is cyclic, statement (2.2) of the theorem holds. □

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5. Proofs of theorems E and C

To prove theorem E and corollary 2, here we list several lemmas, which will be used in the sequel.

Lemma 5.1. *Let G be a group. If $x, y \in G$ such that $[x, y] = 1$ and $(o(x), o(y)) = 1$, then $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$.*

Lemma 5.2 ([**15**, Proposition 2]). *Let G be a non-solvable CP-group. Then there exist normal subgroups B and C of G such that $1 \trianglelefteq B \trianglelefteq C \trianglelefteq G$, where B is a 2-subgroup of G, C/B is a non-abelian simple group, and G/C is a p-group for some prime p. In particular, G/C is either cyclic or a generalized quaternion group.*

Lemma 5.3 ([**15**, Proposition 3]). *If G is a non-abelian simple CP-group, then G is isomorphic to one of the following groups: $PSL_2(q)$, for $q = 5, 7, 8, 9, 17$, $PSL_3(4)$, $Sz(8)$, or $Sz(32)$.*

Proof of theorem E. First consider $\overline{N}$ is nilpotent. Then N is also nilpotent. If there exist two distinct primes $p_1, p_2 \in \pi(\overline{N})$, we may take $a_1 \in P_1 \setminus \mathbf{Z}(N)$ and $a_2 \in P_2 \setminus \mathbf{Z}(N)$, where P_1 and P_2 are Sylow p_1 and p_2-subgroups of N, respectively. By lemma 5.1, $\mathbf{C}_G(a_1a_2) = \mathbf{C}_G(a_1) \cap \mathbf{C}_G(a_2) \leqslant \mathbf{C}_G(a_i)$ for $i = 1, 2$. As all of $\mathbf{C}_G(a_1a_2)$, $\mathbf{C}_G(a_1)$, $\mathbf{C}_G(a_2)$ are maximal in G, we have $\mathbf{C}_G(a_1a_2) = \mathbf{C}_G(a_1) = \mathbf{C}_G(a_2)$. This indicates that $P_1 \leqslant \mathbf{C}_G(a_2) = \mathbf{C}_G(a_1)$, forcing $a_1 \in \mathbf{Z}(N)$. This contradiction deduces that $\overline{N}$ is a p-group.

Let $P \in \mathrm{Syl}_p(N)$. Then $\overline{N} = \overline{P}$. On the contrary, P char $N \trianglelefteq G$, implying $P \trianglelefteq G$ and thus $\Phi(P) \leqslant \Phi(G) \cap N$. Along with the fact that $\Phi(G) \leqslant \mathbf{C}_G(u)$ for any $u \in N \setminus \mathbf{Z}(N)$, we obtain that $\Phi(P) \leqslant \mathbf{C}_N(u)$, yielding $\Phi(P) \leqslant \mathbf{Z}(N)$ by the choice of u. In this case, $\overline{N} = \overline{P} = P/\mathbf{Z}(N) \cong (P/\Phi(P)/(\mathbf{Z}(N)/\Phi(P))$ is elementary abelian, statement (1) of the theorem holds.

Now, we consider that $\overline{N}$ is non-nilpotent. Let $x \in N \setminus \mathbf{Z}(N)$ be an arbitrary element. Write $x = x_1 \cdots x_s$, where $x_1, \ldots, x_s$ are distinct components of x. Since $x \notin \mathbf{Z}(N)$, without loss of generality, we may consider $x_1 \notin \mathbf{Z}(N)$. By lemma 5.1, we see that $\mathbf{C}_G(x) = \mathbf{C}_G(x_1) \cap \cdots \cap \mathbf{C}_G(x_s) \leqslant \mathbf{C}_G(x_1)$. Since both $\mathbf{C}_G(x)$ and $\mathbf{C}_G(x_1)$ are maximal subgroups of G, we have $\mathbf{C}_G(x) = \mathbf{C}_G(x_1)$. Consequently, x can be assumed to be a p-element with prime p.

In the following, we distinguish the proof into two cases:

Case 1. $\overline{\mathbf{C}_N(v)}$ is an r-group, for any r-element $v \in N \setminus \mathbf{Z}(N)$ with prime r.

Step 1. $\overline{N}$ is a CP-group.
Assume false. Then there exists an element $\overline{z} \in \overline{N}$ of order $q^a r^b$, where $q, r \in \pi(\overline{N})$ and $a, b > 0$ are positive integers. By assumption, $\overline{\mathbf{C}_N(z^{q^a})}$ is an r-group. Notice that $\overline{z} \in \overline{\mathbf{C}_N(z^{q^a})}$, which is a contradiction because $qr \mid o(\overline{z})$.

Step 2. $\overline{N}$ is solvable.
Assume on the contrary that $\overline{N}$ is non-solvable. By lemma 5.2, $\overline{N}$ has normal subgroups $\overline{B}$, $\overline{C}$ such that $\overline{1} \trianglelefteq \overline{B} \trianglelefteq \overline{C} \trianglelefteq \overline{N}$, where $\overline{B}$ is a 2-group, $\overline{C}/\overline{B}$ is non-abelian and simple, and $\overline{N}/\overline{C}$ is a q_1-group for some prime q_1, which is cyclic or generalized quaternion.

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Suppose that $\overline{B} \neq \overline{1}$. Let $v \in N \setminus \mathbf{Z}(N)$ be a q-element for some odd prime q. Then $\mathbf{C}_G(v)$ is maximal in G such that $B \nleqslant \mathbf{C}_G(v)$ by step 1. Therefore, $G = B\mathbf{C}_G(v)$ as $\mathbf{C}_G(v)$ is maximal in G, yielding $N = B\mathbf{C}_N(v)$. In this case, $|\overline{N} : \overline{\mathbf{C}_N(v)}| = |N : \mathbf{C}_N(v)| = |B : \mathbf{C}_B(v)|$ is a 2-number. This shows that $|\overline{N}|$ has exactly two prime divisors, against the fact that $\overline{N}$ is non-solvable.

As a result, $\overline{B} = \overline{1}$, and $\overline{C}$ is a non-abelian simple CP-group. By lemma 5.3, $\overline{C}$ is one of the following groups: $PSL_2(q)$, for $q = 5, 7, 8, 9, 17$, $PSL_3(4), Sz(8)$, or $Sz(32)$. Recall that $\overline{G}/\mathbf{C}_{\overline{G}}(\overline{C}) \leqslant \mathrm{Aut}(\overline{C})$. As $\overline{C} \cap \mathbf{C}_{\overline{G}}(\overline{C}) = \overline{1}$ and $\overline{C} \times \mathbf{C}_{\overline{G}}(\overline{C}) \trianglelefteq \overline{G}$, we see that $\mathbf{C}_{\overline{G}}(\overline{C}) = \overline{1}$ by step 1 and thus $\overline{C} \leqslant \overline{G} \leqslant \mathrm{Aut}(\overline{C})$. Moreover, for any $z \in C \setminus \mathbf{Z}(N)$, we see that $\mathbf{C}_{\overline{G}}(\overline{z}) \geqslant \overline{\mathbf{C}_G(z)}$ and $\overline{z} \notin \mathbf{Z}(\overline{N})$. Since $\mathbf{C}_G(z)$ is maximal in G, we obtain that $\mathbf{C}_{\overline{G}}(\overline{z}) = \overline{\mathbf{C}_G(z)}$ is also maximal in $\overline{G}$.

If $\overline{C} \cong PSL_2(5)$, then $\overline{G} \leqslant S_5$. However, by [4], $\mathbf{C}_{\overline{G}}(\overline{v})$ is not maximal in $\overline{G}$ for a 5-element $v \in C \setminus \mathbf{Z}(N)$, a contradiction. By the same reason, we can rule out the cases $\overline{C} \cong PSL_2(q)$, when $q = 7, 8, 9, 17$, and $Sz(8)$ or $Sz(32)$. For the remaining case $\overline{C} \cong PSL_3(4)$, we can find an element $u \in C \setminus \mathbf{Z}(N)$ with order 2 such that $\overline{\mathbf{C}_G(v)}$ is not maximal in $\overline{G}$ according to [4], also a contradiction.

Step 3. The conclusion of case 1.
Let $\overline{S}$ be a minimal normal subgroup of $\overline{G}$ contained in $\overline{N}$. Then $\overline{S}$ is an elementary abelian s-group for some prime s. Since $\overline{N}$ is non-nilpotent, there must exist an s_1-element $a \in N \setminus \mathbf{Z}(N)$ with $s_1 \neq s$. By assumption, $\overline{\mathbf{C}_N(a)}$ is an s_1-subgroup, indicating that $\overline{S} \nleqslant \overline{\mathbf{C}_G(a)}$, and thus $S \nleqslant \mathbf{C}_G(a)$. Hence, $G = S\mathbf{C}_G(a)$ by the maximality of $\mathbf{C}_G(a)$, yielding $\overline{N} = \overline{S}\overline{\mathbf{C}_N(a)}$. In particular, $\overline{N} = \overline{S} \rtimes \overline{\mathbf{C}_N(a)}$ is a Frobenius group with complement $\overline{\mathbf{C}_N(a)}$ by step 1.

We show that $|\overline{\mathbf{C}_N(a)}| = s_1$. For every $a_1 \in \mathbf{C}_N(a) \setminus \mathbf{Z}(N)$, the similar argument in the previous paragraph deduces that $\overline{N} = \overline{S} \rtimes \overline{\mathbf{C}_N(a_1)}$ is also a Frobenius group. Since both $\overline{\mathbf{C}_N(a)}$ and $\overline{\mathbf{C}_N(a_1)}$ are Frobenius complement of $\overline{N}$, and $\overline{a_1} \in \overline{\mathbf{C}_N(a)} \cap \overline{\mathbf{C}_N(a_1)} \neq \overline{1}$, we have $\overline{\mathbf{C}_N(a)} = \overline{\mathbf{C}_N(a_1)}$, forcing $\overline{\mathbf{C}_N(a)}$ is abelian. By [17, Theorem 5.8.7], $\overline{\mathbf{C}_N(a)}$ is cyclic.

Let $1 \neq \overline{d} \in \overline{S}$. Without loss, we assume that $d \in S \setminus \mathbf{Z}(N)$. Then $\mathbf{C}_G(d)$ is maximal in G by hypothesis, forcing that $\overline{\mathbf{C}_G(d)}$ is maximal in $\overline{G}$. In particular, $\overline{G} = \overline{N}\overline{\mathbf{C}_G(d)}$. On the contrary, $\overline{\mathbf{C}_G(d)} \leqslant \mathbf{C}_{\overline{G}}(\overline{d})$ and $\overline{d} \notin \mathbf{Z}(\overline{N})$, we obtain that $\overline{S} \leqslant \mathbf{C}_{\overline{G}}(\overline{d}) = \overline{\mathbf{C}_G(d)}$. In this case, $\overline{G}/\overline{S} = \overline{N}/\overline{S} \rtimes \overline{\mathbf{C}_G(d)}/\overline{S}$. The maximality of $\overline{\mathbf{C}_G(d)}/\overline{S}$ indicates that $\overline{N}/\overline{S}$ is a minimal normal subgroup of $\overline{G}/\overline{S}$, so $\overline{N}/\overline{S}$ has prime power order. Recall that $\overline{N}/\overline{S} \cong \overline{\mathbf{C}_N(a)}$ is cyclic. Then $|\overline{\mathbf{C}_N(a)}| = s_1$, as required.

Case 2. There exists a p-element $x \in N \setminus \mathbf{Z}(N)$ such that $\overline{\mathbf{C}_N(x)}$ is not a p-group.

Step 4. $\mathbf{C}_N(x) = P_x \times H_x$, where $P_x \in \mathrm{Syl}_p(\mathbf{C}_N(x))$ and H_x is an abelian Hall p'-subgroup of $\mathbf{C}_N(x)$.
Let $y \in \mathbf{C}_N(x) \setminus \mathbf{Z}(N)$ be an arbitrary p'-element. By lemma 5.1, we have $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y) \leqslant \mathbf{C}_G(x)$. Notice that $\mathbf{C}_G(xy)$, $\mathbf{C}_G(x)$, $\mathbf{C}_G(y)$ are all maximal in G. We have $\mathbf{C}_G(xy) = \mathbf{C}_G(x) = \mathbf{C}_G(y)$, yielding $y \in \mathbf{Z}(\mathbf{C}_G(x))$. Moreover, $y \in \mathbf{Z}(\mathbf{C}_N(x))$. As a result, $\mathbf{C}_N(x) = P_x \times H_x$, where $P_x \in \mathrm{Syl}_p(\mathbf{C}_N(x))$ and H_x is an abelian Hall p'-subgroup of $\mathbf{C}_N(x)$.

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Step 5. $\mathbf{C}_N(x) \leqslant \mathbf{Z}(\mathbf{C}_G(x))$.
By the assumption of case 2, we see that $H_x \nleqslant \mathbf{Z}(N)$. Take a q-element $v \in H_x \setminus \mathbf{Z}(N)$ with prime q. A similar argument as in the previous paragraph deduces that $\mathbf{C}_N(v) = Q_v \times K_v$, where $Q_v \in \mathrm{Syl}_q(\mathbf{C}_N(v))$ and K_v is an abelian Hall q'-subgroup of $\mathbf{C}_N(v)$. Notice that $x \in K_v$ and $v \in H_x$. Then $\mathbf{C}_N(v) \leqslant \mathbf{C}_N(x)$ and $\mathbf{C}_N(x) \leqslant \mathbf{C}_N(v)$. In particular, $\mathbf{C}_N(v) = \mathbf{C}_N(x) \leqslant \mathbf{Z}(\mathbf{C}_G(x))$.

Step 6. $\mathbf{C}_N(x) \cap \mathbf{C}_N(y) = \mathbf{Z}(N)$ for any $y \in N \setminus \mathbf{C}_N(x)$.
Assume false. Then there exists an element $z \in (\mathbf{C}_N(x) \cap \mathbf{C}_N(y)) \setminus \mathbf{Z}(N)$. Since $\mathbf{C}_N(x) \leqslant \mathbf{Z}(\mathbf{C}_G(x))$, we see that $\mathbf{C}_G(z) \geqslant \langle \mathbf{C}_G(x), y\rangle > \mathbf{C}_G(x)$. As $\mathbf{C}_G(x)$ is maximal in G, it follows that $\mathbf{C}_G(z) = G$, that is, $z \in \mathbf{Z}(N)$, a contradiction.

Step 7. The contradiction of case 2.
By step 5, $\mathbf{C}_N(x) \trianglelefteq \mathbf{C}_G(x)$, which implies that $\mathbf{N}_G(\mathbf{C}_N(x)) \geqslant \mathbf{C}_G(x)$. Consequently, $\mathbf{N}_G(\mathbf{C}_N(x)) = \mathbf{C}_G(x)$ or $\mathbf{C}_N(x) \trianglelefteq G$ as $\mathbf{C}_G(x)$ is maximal in G.

Assume first that $\mathbf{C}_N(x) \trianglelefteq G$. Then P_x is a normal subgroup of G according to step 4. Let $z \in N \setminus \mathbf{C}_N(x)$ be a primary element. Then $\mathbf{C}_G(z)$ is maximal in G. Moreover, $P_x \nleqslant \mathbf{C}_G(z)$ by step 6. As a result, $G = P_x\mathbf{C}_G(z)$, implying $N = P_x\mathbf{C}_N(z)$. Furthermore, by step 6, $\mathbf{C}_N(x) = \mathbf{Z}(N)P_x$, showing that $\overline{\mathbf{C}_N(x)}$ is a p-group, contrary to our assumption.

Hence, $\mathbf{N}_G(\mathbf{C}_N(x)) = \mathbf{C}_G(x)$, forcing $\mathbf{N}_N(\mathbf{C}_N(x)) = \mathbf{C}_N(x)$. Since $\mathbf{N}_{\overline{G}}(\overline{\mathbf{C}_N(x)}) \geqslant \overline{\mathbf{N}_G(\mathbf{C}_N(x))} = \overline{\mathbf{C}_G(x)}$, we get that $\mathbf{N}_{\overline{G}}(\overline{\mathbf{C}_N(x)}) = \overline{G}$ or $\mathbf{N}_{\overline{G}}(\overline{\mathbf{C}_N(x)}) = \overline{\mathbf{C}_G(x)}$. If the former holds, then $\mathbf{C}_N(x) \trianglelefteq G$, against our assumption. Hence, $\mathbf{N}_{\overline{G}}(\overline{\mathbf{C}_N(x)}) = \overline{\mathbf{C}_G(x)}$, yielding $\mathbf{N}_{\overline{N}}(\overline{\mathbf{C}_N(x)}) = \overline{\mathbf{C}_N(x)}$.

We claim that for any $g \in N \setminus \mathbf{C}_N(x)$, we always have $\mathbf{C}_N(x)^g \cap \mathbf{C}_N(x) = \mathbf{Z}(N)$. Let $d \in (\mathbf{C}_N(x)^g \cap \mathbf{C}_N(x)) \setminus \mathbf{Z}(N)$. Note that $\mathbf{C}_N(x) \leqslant \mathbf{Z}(\mathbf{C}_G(x))$. Then $\mathbf{C}_G(d) \geqslant \langle \mathbf{C}_G(x)^g, \mathbf{C}_G(x)\rangle$. Since $\mathbf{C}_G(x)$ is maximal and $\mathbf{C}_G(x)^g \neq \mathbf{C}_G(x)$, we have that $d \in \mathbf{Z}(G)$ and thus $d \in \mathbf{Z}(N)$, a contradiction. By [**17**, Theorem 5.7.6], $\overline{N}$ is a Frobenius group with a complement $\overline{\mathbf{C}_N(x)}$. Write $\overline{N} = \overline{T_x} \rtimes \overline{\mathbf{C}_N(x)}$, where $\overline{T_x}$ is the Frobenius kernel of $\overline{N}$. Let $\overline{Q} \in \mathrm{Syl}_q(\overline{T_x})$. Note that $\overline{\mathbf{C}_G(x)}$ is maximal in $\overline{G}$. Then $\overline{G} = \overline{Q} \rtimes \overline{\mathbf{C}_G(x)}$ by step 6. The maximality of $\overline{\mathbf{C}_G(x)}$ indicates that $\overline{T_x} = \overline{Q}$ is a minimal normal subgroup of $\overline{G}$. In particular, $\overline{T_x}$ is abelian.

Take $y \in T_x \setminus \mathbf{Z}(N)$. Then $N \nleqslant \mathbf{C}_G(y)$. The maximality of $\mathbf{C}_G(y)$ implies that $G = N\mathbf{C}_G(y)$, and thus $\overline{G}/\overline{T_x} \cong \overline{N}/\overline{T_x}\ \overline{\mathbf{C}_G(y)}/\overline{T_x}$. Notice that $\overline{\mathbf{C}_G(y)}/\overline{T_x}$ is a maximal subgroup of $\overline{G}/\overline{T_x}$. Then $\overline{N}/\overline{T_x}$ must be a minimal normal subgroup of $\overline{G}/\overline{T_x}$, forcing that $\overline{N}/\overline{T_x}$ has prime power order. However, $\overline{N}/\overline{T_x} \cong \overline{\mathbf{C}_N(x)}$ does not have prime power order, the final contradiction completes the proof. □

As an application of theorem E, here we give a new proof of theorem C.

Proof of theorem C. Take $G = N$ in theorem E. Then $\overline{G}$ is either an elementary abelian p-group for some prime p, or $\overline{G} = \overline{P} \rtimes \overline{Q}$ is a Frobenius group, with Frobenius kernel $\overline{P}$ and Frobenius complement $\overline{Q}$. In particular, $\overline{P}$ is the minimal normal subgroup of $\overline{G}$ and $\overline{Q}$ is of prime order.

For any $1 \neq x \in P \setminus \mathbf{Z}(G)$, $\mathbf{C}_G(x)$ is maximal in G, which implies that $\overline{\mathbf{C}_G(x)}$ is maximal in $\overline{G}$. Note that $\overline{G}$ is a Frobenius group and $\overline{P} \leqslant \overline{\mathbf{C}_G(x)}$, it follows that $\overline{P} = \overline{\mathbf{C}_G(x)}$ and $|\overline{Q}| = q$ is a prime. Let $v \in Q \setminus \mathbf{Z}(G)$. Then $\mathbf{C}_G(v)$ is maximal in

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G. Note that Q is abelian, we get that $\overline{\mathbf{C}_G(v)} = \overline{Q}$ is maximal in $\overline{G}$. As a result, each subgroup of $\overline{G}$ is contained in $\overline{P}$ or $\overline{Q}^{\overline{g}}$ for some $\overline{g} \in \overline{G}$, showing that G is a minimal non-abelian group. □

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