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Partial Euler Products on the Critical Line

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Abstract. The initial version of the Birch and Swinnerton-Dyer conjecture concerned asymptotics for partial Euler products for an elliptic curve L-function at $s = 1$. Goldfeld later proved that these asymptotics imply the Riemann hypothesis for the L-function and that the constant in the asymptotics has an unexpected factor of $\sqrt{2}$. We extend Goldfeld's theorem to an analysis of partial Euler products for a typical L-function along its critical line. The general $\sqrt{2}$ phenomenon is related to second moments, while the asymptotic behavior (over number fields) is proved to be equivalent to a condition that in a precise sense seems much deeper than the Riemann hypothesis. Over function fields, the Euler product asymptotics can sometimes be proved unconditionally.

1 Introduction

Let $E_{/\mathbf{Q}}$ be an elliptic curve. The Birch and Swinnerton-Dyer conjecture says, in part, that the rank of $E(\mathbf{Q})$ equals the order of vanishing of its L-function at $s = 1$. We write the L-function, for $\mathrm{Re}(s) > 3/2$, as

$$L(E,s) := \prod_{p|N} \frac{1}{1 - a_p p^{-s}} \cdot \prod_{(p,N)=1} \frac{1}{1 - a_p p^{-s} + p \cdot p^{-2s}} = \sum_{n \geq 1} \frac{a_n}{n^s},$$

where N is the conductor. By the elliptic modularity theorem [4, 7, 10, 30], $L(E,s)$ extends analytically to $\mathbf{C}$ and has a functional equation relating values at s and $2 - s$.

In the original form of their conjecture [2], Birch and Swinnerton-Dyer recognized the rank not as an order of vanishing, but as a growth exponent r:

$$\prod_{p \leq x} \frac{\#E_{\mathrm{ns}}(\mathbf{F}_p)}{p} \sim A(\log x)^r \tag{1.1}$$

for some constant $A > 0$, where $E_{\mathrm{ns}}(\mathbf{F}_p)$ is the set of nonsingular $\mathbf{F}_p$-rational points on a minimal Weierstrass model for E at p. Graphs illustrating (1.1) can be found in [22, p. 460]. Actually, the factors in (1.1) at $p|N$ were chosen differently by Birch and Swinnerton–Dyer, and are omitted in [22], but this does not affect (1.1) since A has not yet been specified.

Since, for all p, $\#E_{\mathrm{ns}}(\mathbf{F}_p)/p$ equals the reciprocal of the p-th local Euler factor in $L(E,s)$, at $s = 1$, (1.1) can be reciprocated and written in terms of partial Euler products as

$$\mathrm{Prod}(E,x) \sim \frac{C}{(\log x)^r}, \tag{1.2}$$

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where C is a nonzero constant and

$$\begin{aligned}\mathrm{Prod}(E,x) &= \prod_{\substack{p\mid N\\ p\le x}} \frac{1}{1-a_p/p}\cdot\prod_{\substack{(p,N)=1\\ p\le x}}\frac{1}{1-a_p/p+1/p}\\ &= \prod_{p\le x}\frac{1}{\#E_{\mathrm{ns}}(\mathbf{F}_p)/p}.\end{aligned}$$

The following theorem of Goldfeld [12] (which originally included an assumption of modularity) addresses consequences of the equivalent estimates (1.1) and (1.2).

Theorem 1.1 (Goldfeld) *Let $E_{/\mathbf{Q}}$ be an elliptic curve. If* $\mathrm{Prod}(E,x)\sim C/(\log x)^r$ *as $x\to\infty$, where $C>0$ and $r\in\mathbf{R}$, then $L(E,s)$ satisfies the Riemann hypothesis (*i.e., $L(E,s)\neq 0$ *for* $\mathrm{Re}(s)>1$*), $r=\mathrm{ord}_{s=1}L(E,s)$, and*

$$C=\frac{L^{(r)}(E,1)}{r!}\cdot\frac{1}{\sqrt{2}e^{r\gamma}}, \tag{1.3}$$

where $\gamma = .577215\ldots$ is Euler's constant.

The versions of Theorem 1.1 in [12] and [22] are equivalent to this, but look superficially different, *e.g.*, the letter C has a different meaning in each paper.

Since $\sqrt{2}e^{r\gamma}\ge\sqrt{2}>1$, the constant C in (1.2) does not equal the leading Taylor coefficient of $L(E,s)$ at $s=1$, which is quite surprising. When $r=0$, the theorem says in part: if the formal Euler product for $L(E,1)$ has a nonzero value, the value is $L(E,1)/\sqrt{2}$. Although Theorem 1.1 identifies the exponent r in (1.2) with an order of vanishing at $s=1$, it makes no identification with the rank of $E(\mathbf{Q})$.

The goal of this paper is to put Theorem 1.1 in a more general context, and explain how a large class of Euler products should behave on their critical line.

For instance, partial Euler products for $L(E,s)$ at a point on $\mathrm{Re}(s)=1$ other than 1 do not have a $\sqrt{2}$ factor in the analogue of Theorem 1.1. Partial Euler products at $s=1/2$ for the L-function of a nontrivial Dirichlet character have no $\sqrt{2}$ in the analogue of Theorem 1.1 unless the character is quadratic, when $\sqrt{2}$ appears in the numerator rather than in the denominator. We will show the appearance of $\sqrt{2}$ is generally governed by "second moments", in the following sense.

Theorem 1.2 *Let*

$$L(s)=\prod_{\mathfrak{p}}\frac{1}{(1-\alpha_{\mathfrak{p},1}\,\mathrm{N}\mathfrak{p}^{-s})\cdots(1-\alpha_{\mathfrak{p},d}\,\mathrm{N}\mathfrak{p}^{-s})}$$

be an Euler product over a number field, with $|\alpha_{\mathfrak{p},j}|\le 1$ and $\mathrm{Re}(s)>1$*. Assume $L(s)$ extends to a holomorphic function on* $\mathrm{Re}(s)\ge 1/2$ *and the second moment Euler product*

$$\prod_{\mathfrak{p}}\frac{1}{(1-\alpha_{\mathfrak{p},1}^2\,\mathrm{N}\mathfrak{p}^{-s})\cdots(1-\alpha_{\mathfrak{p},d}^2\,\mathrm{N}\mathfrak{p}^{-s})}$$

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extends to a holomorphic nonvanishing function on $\mathrm{Re}(s) = 1$ *except at one number* $1 + it_0$*, where the order of vanishing is* $R_0 \in \mathbf{Z}$*. For a complex number with real part* $1/2$*, say* $1/2 + it$*, assume*

$$\prod_{\mathrm{N}\mathfrak{p}\leq x} \frac{1}{(1-\alpha_{\mathfrak{p},1}\,\mathrm{N}\mathfrak{p}^{-(1/2+it)})\cdots(1-\alpha_{\mathfrak{p},d}\,\mathrm{N}\mathfrak{p}^{-(1/2+it)})} \sim \frac{C_t}{(\log x)^{r_t}}$$

as $x \to \infty$*, where* $C_t \in \mathbf{C}^\times$ *and* $r_t \in \mathbf{C}$*. Then, with* B_t *denoting the leading Taylor coefficient of* $L(s)$ *at* $s = 1/2 + it$*,* $C_t = B_t/e^{r_t\gamma}$ *if* $t \neq t_0/2$ *and* $C_t = B_t/\sqrt{2}^{R_0}e^{r_t\gamma}$ *if* $t = t_0/2$*.*

While Theorem 1.1 says (1.2) implies the Riemann hypothesis for $L(E, s)$, experience suggests a certain degree of caution. There are many numerical patterns which would imply the Riemann hypothesis for $\zeta(s)$ but eventually fail. Might (1.2) have a similar fate?

No. Let

$$\psi_E(x) = \sum_{\substack{p^k\leq x\\(p,N)=1}} (\alpha_p^k + \beta_p^k)\log p,$$

where α_p and β_p are the Frobenius eigenvalues at p. (Terms at p^k when $p|N$ can be ignored, as they do not affect what we are about to say.) We will prove in Theorem 6.3 a result about L-functions which has the following as a special case.

Theorem 1.3 *Equation* (1.2) *is equivalent to*

$$\psi_E(x) = o(x\log x). \tag{1.4}$$

The Riemann hypothesis for $L(E, s)$ is equivalent to $\psi_E(x) = O(x(\log x)^2)$, so in a precise sense (1.2) is much deeper than the Riemann hypothesis for $L(E, s)$ according to what is known today. The estimate (1.4) is plausible. Indeed, letting $\psi(x) = \sum_{p^k\leq x}\log p$ as usual, the Riemann hypothesis for $\zeta(s)$ is equivalent to $\psi(x) - x = O(\sqrt{x}(\log x)^2)$ while Montgomery [16, p. 16] has given a reason to believe the true order of magnitude of $\psi(x) - x$ is at most $\sqrt{x}(\log\log\log x)^2$. This suggests a comparable upper bound for $\psi_E(x)$ may be around $x(\log\log\log x)^2$, which would imply (1.4).

When this paper was finished, I learned from K. Murty that W. Kuo and R. Murty [15] found and proved Theorem 1.3 independently, by a different method. We will discuss this further at the end of Section 6.

The paper is organized as follows. In Section 2, we introduce some basic notation and terminology for the Euler products we will be treating. In Section 3, we collect several preliminary computations. Section 4 isolates the results which lead to the appearance of $\sqrt{2}$. In Section 5, we generalize Theorem 1.1 to other Euler products. Up to this point we can avoid explicit formulas (in the sense of analytic number theory), so Theorem 1.1 can be proved without them. This simplifies Goldfeld's proof. In Section 6, we prove the equivalence of (1.2) and (1.4) using several explicit formulas. We discuss in Section 7 how Dirichlet series associated to Euler products should behave on the critical line. In Section 8, we treat function fields, where sharper statements are possible. In particular, we prove the function field version of (1.2).

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2 Notation and Terminology

Fix a number field K and a positive integer d. A *normalized Euler product* over K of degree at most d is a product over the primes of K,

$$L(s) := \prod_{\mathfrak{p}} \frac{1}{(1-\alpha_{\mathfrak{p},1}\,\mathrm{N}\mathfrak{p}^{-s})\cdots(1-\alpha_{\mathfrak{p},d}\,\mathrm{N}\mathfrak{p}^{-s})}, \tag{2.1}$$

where $|\alpha_{\mathfrak{p},1}|,\dots,|\alpha_{\mathfrak{p},d}| \le 1$. (The L-function of an elliptic curve $E_{/\mathbf{Q}}$ fits (2.1) when written in the form $L(E, s+1/2)$.) Any finite product

$$\prod_{\mathrm{N}\mathfrak{p}\le x} \frac{1}{(1-\alpha_{\mathfrak{p},1}\,\mathrm{N}\mathfrak{p}^{-s})\cdots(1-\alpha_{\mathfrak{p},d}\,\mathrm{N}\mathfrak{p}^{-s})},$$

taken over $\mathfrak{p}$ with norm up to some bound x, will be called a *partial Euler product* for $L(s)$.

The bounds $|\alpha_{\mathfrak{p},j}| \le 1$ make the Euler product absolutely convergent and nonvanishing on $\mathrm{Re}(s) > 1$. We do allow some $\alpha_{\mathfrak{p},j}$ to equal 0, as happens in natural examples. Also in natural examples, we usually expect $|\alpha_{\mathfrak{p},j}| = 1$ for all but finitely many $\mathfrak{p}$, but we do not assume this condition holds. This gives our notion of normalized Euler product a certain looseness (*e.g.*, $L(s+c)$ is also normalized for any $c > 0$), which will be restricted when we start using a condition on second moments (in Definition 4.4) systematically. We will consider $\mathrm{Re}(s) = 1/2$ to be the critical line, although a functional equation which justifies this terminology will not be introduced until Section 6.

Since this paper concerns Euler products outside domains of absolute convergence, we must be more careful than usual about calculations with partial sums and products. In particular, we need to be attentive to the order of terms and any rearrangement of terms. It will always be understood that terms indexed by ideals in K (such as an Euler product over K) are arranged according to increasing norm, with terms indexed by ideals of equal norm being taken as one collective term:

$$\prod_{\mathfrak{p}} = \lim_{x\to\infty}\prod_{\mathrm{N}\mathfrak{p}\le x}, \quad \sum_{\mathfrak{p}}\sum_{k\ge 1} = \lim_{x\to\infty}\sum_{\mathrm{N}\mathfrak{p}\le x}\sum_{k\ge 1}, \quad \sum_{\mathfrak{p}^k} = \lim_{x\to\infty}\sum_{\mathrm{N}\mathfrak{p}^k\le x}.$$

It is essential to distinguish between the two sums $\sum_{\mathfrak{p}}\sum_k$ and $\sum_{\mathfrak{p}^k}$. For instance, this distinction is related to the $\sqrt{2}$ in Goldfeld's theorem.

An *ordinary* Dirichlet series is a series $\sum_{n\ge 1} a_n n^{-s}$ indexed by positive integers in increasing order. Writing $L(s)$ in (2.1) as an ordinary Dirichlet series $\sum a_n n^{-s}$, the condition that all $|\alpha_{\mathfrak{p},j}| \le 1$ is equivalent to both $a_n = O_\varepsilon(n^\varepsilon)$ for all $\varepsilon > 0$ and to $|a_{p^k}| \le \binom{d+k-1}{k}$ for prime powers $p^k > 1$.

The factors in $L(s)$ can be collected into a normalized Euler product over the primes in any subfield of K. While the partial sums for the Dirichlet series of $L(s)$ over K and over $\mathbf{Q}$ are the same, the partial Euler products for $L(s)$ over K and over $\mathbf{Q}$ are different (when $K \ne \mathbf{Q}$). This difference does not affect analytic properties of $L(s)$ (such as having an analytic continuation and a functional equation), but does change

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the meaning of auxiliary constructions (like a symmetric square). For instance, we will see at the end of Section 5 that the $\sqrt{2}$ phenomenon in Goldfeld's theorem disappears when $E_{/\mathbf{Q}}$ is a CM curve and we look at partial Euler products over primes in the field of complex multiplication.

Given $L(s)$ as in (2.1), its *second moment* (over K) is

$$L_2(s) = \prod_{\mathfrak{p}} \frac{1}{(1-\alpha_{\mathfrak{p},1}^2 \,\mathrm{N}\mathfrak{p}^{-s}) \cdots (1-\alpha_{\mathfrak{p},d}^2 \,\mathrm{N}\mathfrak{p}^{-s})}. \tag{2.2}$$

In practice, this second moment is the ratio of a symmetric and exterior square L-function over K. Note that we do not mean by "second moment" the Dirichlet series $\sum a_n^2 n^{-s}$ (*cf.* [17]). Also note that for $K \neq \mathbf{Q}$, the second moment of $L(s)$ is not really determined by the holomorphic function $L(s)$, but by the data $\alpha_{\mathfrak{p},j}$.

3 Preliminary Calculations

We study Euler products through their logarithms. For $\mathrm{Re}(s) > 1$, the logarithm of $L(s)$ in (2.1) is, by definition,

$$\log L(s) := \sum_{\mathfrak{p}} \sum_{k \geq 1} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k \,\mathrm{N}\mathfrak{p}^{ks}} = \sum_{\mathfrak{p}^k} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k \,\mathrm{N}\mathfrak{p}^{ks}}. \tag{3.1}$$

A more accurate notation would be $(\log L)(s)$.

Passing from the double sum to the single sum requires a rearrangement of terms, which is harmless for $\mathrm{Re}(s) > 1$. What can we say on the larger region $\mathrm{Re}(s) \geq 1/2$?

Lemma 3.1 *For each s with $\mathrm{Re}(s) \geq 1/2$,*

$$\sum_{\mathrm{N}\mathfrak{p} \leq x} \sum_{k \geq 1} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k \,\mathrm{N}\mathfrak{p}^{ks}} = \sum_{\mathrm{N}\mathfrak{p}^k \leq x} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k \,\mathrm{N}\mathfrak{p}^{ks}} + \sum_{\sqrt{x} < \mathrm{N}\mathfrak{p} \leq x} \frac{\alpha_{\mathfrak{p},1}^2 + \cdots + \alpha_{\mathfrak{p},d}^2}{2 \,\mathrm{N}\mathfrak{p}^{2s}} + o(1)$$

as $x \to \infty$. When $\mathrm{Re}(s) > 1/2$, the second sum on the right is $o(1)$ as $x \to \infty$.

Proof Subtracting the first sum on the right from the sum on the left gives

$$\sum_{\mathrm{N}\mathfrak{p} \leq x} \sum_{\substack{k \\ \mathrm{N}\mathfrak{p}^k > x}} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k \,\mathrm{N}\mathfrak{p}^{ks}} = \sum_{\sqrt{x} < \mathrm{N}\mathfrak{p} \leq x} \sum_{k \geq 2} + \sum_{\mathrm{N}\mathfrak{p} \leq \sqrt{x}} \sum_{\substack{k \\ \mathrm{N}\mathfrak{p}^k > x}} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k \,\mathrm{N}\mathfrak{p}^{ks}}. \tag{3.2}$$

The second sum on the right side of (3.2) is bounded by

$$d \sum_{\mathrm{N}\mathfrak{p} \leq \sqrt{x}} \sum_{k > \log_{\mathrm{N}\mathfrak{p}} x} \frac{1}{\log_{\mathrm{N}\mathfrak{p}}(x) \cdot \mathrm{N}\mathfrak{p}^{k/2}} \leq \frac{d}{2} \sum_{\mathrm{N}\mathfrak{p} \leq \sqrt{x}} \frac{1}{\sqrt{x}(1 - 1/\sqrt{2})},$$

which is $O(\pi_K(\sqrt{x})/\sqrt{x}) = O(1/\log x)$, where $\pi_K(x)$ is the number of primes in K with norm at most x. In the first sum on the right side of (3.2), the series over $k \geq 3$ is bounded by

$$\sum_{\sqrt{x}\leq \mathrm{N}\mathfrak{p}\leq x} \sum_{k\geq 3} \frac{d}{k\,\mathrm{N}\mathfrak{p}^{k/2}} \leq \frac{d}{3} \sum_{\sqrt{x}\leq \mathrm{N}\mathfrak{p}\leq x} \frac{1}{\mathrm{N}\mathfrak{p}^{3/2}(1-1/\sqrt{2})},$$

which tends to 0 since $\zeta_K(3/2) < \infty$. (The indices are restricted below as well as above.) Now we see the right side of (3.2) equals

$$\sum_{\sqrt{x}<\mathrm{N}\mathfrak{p}\leq x} \frac{\alpha_{\mathfrak{p},1}^2 + \cdots + \alpha_{\mathfrak{p},d}^2}{2\,\mathrm{N}\mathfrak{p}^{2s}} + o(1),$$

which tends to 0 if $\mathrm{Re}(2s) > 1$ by comparison with $\zeta_K(2s)$. ∎

An extension of the last part of Lemma 3.1 to the line $\mathrm{Re}(s) = 1/2$ is in Theorem 4.9.

Lemma 3.2 *Fix $d \geq 1$. For each prime $\mathfrak{p}$ of K, let $\gamma_{\mathfrak{p},1}, \dots, \gamma_{\mathfrak{p},d}$ be complex numbers in the open unit disk, and assume $\max_i |\gamma_{\mathfrak{p},i}| \to 0$ as $\mathrm{N}\mathfrak{p} \to \infty$. Then the asymptotic relation*

$$\prod_{\mathrm{N}\mathfrak{p}\leq x} \frac{1}{(1-\gamma_{\mathfrak{p},1})\cdots(1-\gamma_{\mathfrak{p},d})} \sim \frac{C}{(\log x)^r}$$

as $x \to \infty$, for some $C \in \mathbf{C}^\times$ and $r \in \mathbf{C}$, is equivalent to

$$-\sum_{\mathrm{N}\mathfrak{p}\leq x} (\log(1-\gamma_{\mathfrak{p},1}) + \cdots + \log(1-\gamma_{\mathfrak{p},d})) = -r \log\log x + C' + o(1)$$

for some C' with $e^{C'} = C$. Here $\log$ is the principal branch of the logarithm.

Proof Since the number of $\mathfrak{p}$ with any norm is bounded (by $[K:\mathbf{Q}]$), this is a simple extension of the standard fact that the product of a (linearly ordered) sequence of complex numbers tending to 1 converges if and only if the corresponding series of principal branch logarithms converges [1, pp. 191–192]. ∎

Lemma 3.2 lets questions about Euler products over number fields be reduced to an analysis of sums. While this is trivial, it is false over function fields. We will see a counterexample to Lemma 3.2 in the function field case in Example 8.3.

Theorem 3.3 *For $s_0 \in \mathbf{C}$ with $\mathrm{Re}(s_0) > 1/2$, the following are equivalent:*

(1) *The limit*

$$\lim_{x\to\infty} \prod_{\mathrm{N}\mathfrak{p}\leq x} \frac{1}{(1-\alpha_{\mathfrak{p},1}\,\mathrm{N}\mathfrak{p}^{-s_0})\cdots(1-\alpha_{\mathfrak{p},d}\,\mathrm{N}\mathfrak{p}^{-s_0})}$$

is nonzero.

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(2) *The limit*

$$\lim_{x\to\infty} \sum_{\mathrm{N}\mathfrak{p}^k \le x} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\,\mathrm{N}\mathfrak{p}^{ks_0}}$$

exists.

In this case, the ordinary Dirichlet series for $\log L(s)$ *converges for* $s = s_0$ *and for* $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. *Also,* $L(s)$ *has an analytic continuation to* $\mathrm{Re}(s) > \mathrm{Re}(s_0)$, *where*

$$L(s) = \lim_{x\to\infty} \prod_{\mathrm{N}\mathfrak{p}\le x} \frac{1}{(1-\alpha_{\mathfrak{p},1}\,\mathrm{N}\mathfrak{p}^{-s})\cdots(1-\alpha_{\mathfrak{p},d}\,\mathrm{N}\mathfrak{p}^{-s})} \tag{3.3}$$

and $L(s) \neq 0$. *Equation* (3.3) *is valid at* $s = s_0$.

Proof The equivalence of (1) and (2) follows from Lemmas 3.1 and 3.2 with $r = 0$. (The only point to check is the switch from $\sum_{\mathfrak{p}} \sum_k$ to $\sum_{\mathfrak{p}^k}$.)

Assuming the equivalent conditions hold at s_0, we can use (2) to see that the ordinary Dirichlet series for $\log L(s)$ converges at s_0 and therefore also when $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. Exponentiating with Lemma 3.1 leads to convergence of the Euler product on this region, and Abel's theorem applied to $\log L(s)$ lets us extend (3.3) to s_0 from the right, *i.e.*, as $s \to s_0^+$. ∎

Loosely speaking, Theorem 3.3 says the Euler product for $L(s)$ when $\mathrm{Re}(s) > 1/2$ is not expected to contain any surprises.

Example 3.4 Having just used Abel's theorem for Dirichlet series, we might as well point out that there is no simple Abel's theorem for Euler products. Goldfeld's theorem with $r = 0$ already illustrates the idea, but the hypothesis (1.2) for elliptic curves over $\mathbf{Q}$ is unprovable at present. For a family of unconditional examples, choose $m \ge 1$. For $\mathrm{Re}(s) > 1$,

$$\frac{\zeta(ms-(m-1))}{\zeta(s)} = \prod_p \frac{1-p^{-s}}{1-p^{m-1-ms}}.$$

At $s = 1$, the right side equals 1. Using the left side, we see the limit of the right side as $s \to 1^+$ is $1/m$. The case $m = 2$ is in [31].

Example 3.5 Abel's theorem fails not only for Euler products, but for products of the form $\prod(1 + a_n x^n)$. In 1908, Hardy [13, pp. 263–271] gave an example of a real sequence $\{a_n\}$ such that $\prod(1 + a_n)$ converges and

$$\prod_{n\ge1}(1+a_nx^n) \to 2\prod_{n\ge1}(1+a_n)$$

as $x \to 1^-$. Here there is an extra 2, instead of an extra $1/\sqrt{2}$ as in Goldfeld's theorem. We will return to this example at the end of Section 5.

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4 The role of $\sqrt{2}$

To provide context for the $1/\sqrt{2}e^{r\gamma}$ in Goldfeld's theorem, we discuss in this section how $\sqrt{2}$ arises from a series rearrangement. (A role for γ in Euler product asymptotics is well-known.) We start with a Tauberian theorem for logarithmic singularities, whose proof is omitted.

Theorem 4.1 *Let $f(s) = \sum a_n n^{-s}$, where $a_n \geq 0$, and assume f extends holomorphically to $\mathrm{Re}(s) = 1$ except for a simple pole at $s = 1$.*

Let $g(s) = \sum b_n n^{-s}$, where $|b_n| \leq C a_n$ for some constant C. Assume g extends holomorphically to $\mathrm{Re}(s) = 1$ except at $s = 1$, where there is a logarithmic singularity:

$$g(s) = r\log(s-1) + G(s),$$

for $\mathrm{Re}(s) \geq 1$ with $s \neq 1$, where $G(s)$ is holomorphic on $\mathrm{Re}(s) \geq 1$. Then

$$\sum_{n\leq x} \frac{b_n}{n} = -r\log\log x - r\gamma + G(1) + o(1), \quad \sum_{n\geq 1} \frac{b_n}{n^{1+it}} = g(1+it)$$

for real $t \neq 0$.

While this holds for $r = 0$, the interesting case for us is $r \neq 0$.

Example 4.2 Let $f(s) = \zeta_K(s)$ and $g(s) = \log\zeta_K(s) = \sum_{\mathfrak{p}^k} 1/k\,\mathrm{N}\mathfrak{p}^{ks}$. By Theorem 4.1,

$$\sum_{\mathrm{N}\mathfrak{p}^k\leq x} \frac{1}{k\,\mathrm{N}\mathfrak{p}^k} = \log\log x + \gamma + \log\rho_K + o(1),$$

where $\rho_K = \mathrm{Res}_{s=1}\,\zeta_K(s)$. We can replace $\sum_{\mathrm{N}\mathfrak{p}^k\leq x}$ with $\sum_{\mathrm{N}\mathfrak{p}\leq x}\sum_k$, by Lemma 3.1. Then exponentiating implies

$$\prod_{\mathrm{N}\mathfrak{p}\leq x} \frac{1}{1 - 1/\mathrm{N}\mathfrak{p}} \sim \rho_K e^{\gamma}\log x, \tag{4.1}$$

which is a classical formula of Mertens for $K = \mathbf{Q}$. (For a stronger relation between the two sides of (4.1), see [21, Theorem 2].) We mention this example since (4.1) resembles the hypothetical (1.2) with $r = -1 = \mathrm{ord}_{s=1}\,\zeta_K(s)$ and $C = \rho_K/e^{r\gamma}$. However, $s = 1$ is at the edge of the critical strip for $\zeta_K(s)$ while it is on the critical line for $L(E, s)$. Also, the e^{γ} in Goldfeld's theorem will not arise from Theorem 4.1, but rather from Lemma 5.1.

Theorem 4.3 *For $\mathrm{Re}(s) > 1$, let*

$$\widetilde{L}(s) = \prod_{\mathfrak{p}} \frac{1}{(1-\beta_{\mathfrak{p},1}\,\mathrm{N}\mathfrak{p}^{-s})\cdots(1-\beta_{\mathfrak{p},d}\,\mathrm{N}\mathfrak{p}^{-s})}$$

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be a normalized Euler product. Assume $\widetilde{L}(s)$ *extends to a holomorphic nonvanishing function on* $\mathrm{Re}(s) = 1$ *except for a zero of order* $R_0 \in \mathbf{Z}$ *at, say,* $1 + it_0$*. Then*

$$\lim_{x\to\infty} \sum_{\sqrt{x}<\mathrm{N}\mathfrak{p}\le x} \frac{\beta_{\mathfrak{p},1} + \cdots + \beta_{\mathfrak{p},d}}{\mathrm{N}\mathfrak{p}^{1+it}} = \begin{cases} 0, & \text{if } t \neq t_0, \\ -R_0 \log 2, & \text{if } t = t_0. \end{cases}$$

When $R_0 < 0$, the zero at $1 + it_0$ is a pole, and when $R_0 = 0$ there is no zero or pole.

Proof Replacing $\widetilde{L}(s)$ with $\widetilde{L}(s + it_0)$, we may assume $t_0 = 0$. For $\mathrm{Re}(s) > 1$, set

$$(\log \widetilde{L})(s) = \sum_{\mathfrak{p}} \sum_{k\ge 1} \frac{\beta^k_{\mathfrak{p},1} + \cdots + \beta^k_{\mathfrak{p},d}}{k\,\mathrm{N}\mathfrak{p}^{ks}} = \sum_{n\ge 1} \frac{b_n}{n^s},$$

where $b_n = \sum_{\mathrm{N}\mathfrak{p}^k=n} (\beta^k_{\mathfrak{p},1} + \cdots + \beta^k_{\mathfrak{p},d})/k$. Write

$$(\log \widetilde{L})(s) = \sum_{n\ge 1} \sum_{\mathrm{N}\mathfrak{p}=n} \frac{\beta_{\mathfrak{p},1} + \cdots + \beta_{\mathfrak{p},d}}{\mathrm{N}\mathfrak{p}^s} + \sum_{n\ge 1} \sum_{\substack{\mathrm{N}\mathfrak{p}^k=n \\ k\ge 2}} \frac{\beta^k_{\mathfrak{p},1} + \cdots + \beta^k_{\mathfrak{p},d}}{k\,\mathrm{N}\mathfrak{p}^{ks}},$$

where the second series is absolutely convergent for $\mathrm{Re}(s) > 1/2$, so for any real t,

$$\sum_{n\le x} \frac{b_n}{n^{1+it}} = \sum_{\mathrm{N}\mathfrak{p}\le x} \frac{\beta_{\mathfrak{p},1} + \cdots + \beta_{\mathfrak{p},d}}{\mathrm{N}\mathfrak{p}^{1+it}} + c_t + o(1)$$

for some constant c_t. Therefore

$$\sum_{\sqrt{x}<\mathrm{N}\mathfrak{p}\le x} \frac{\beta_{\mathfrak{p},1} + \cdots + \beta_{\mathfrak{p},d}}{\mathrm{N}\mathfrak{p}^{1+it}} = \sum_{\sqrt{x}<n\le x} \frac{b_n}{n^{1+it}} + o(1). \tag{4.2}$$

By hypothesis, $\widetilde{L}'(s)/\widetilde{L}(s)$ is holomorphic on $\mathrm{Re}(s) \ge 1$ except for a simple pole at $s = 1$ with residue R_0. Thus for $\mathrm{Re}(s) \ge 1$ with $s \neq 1$,

$$(\log \widetilde{L})(s) = R_0 \log(s - 1) + H(s), \tag{4.3}$$

where H is holomorphic on $\mathrm{Re}(s) \ge 1$. By Theorem 4.1, with $f(s) = \zeta_K(s)$ and $g(s) = (\log \widetilde{L})(s)$,

$$\sum_{n\ge 1} \frac{b_n}{n^{1+it}} = (\log \widetilde{L})(1 + it) \tag{4.4}$$

for $t \neq 0$ and

$$\sum_{n\le x} \frac{b_n}{n} = -R_0 \log\log x + c + o(1)$$

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for some c. Thus when $t \neq 0$,

$$\sum_{\sqrt{x}<\mathrm{N}\mathfrak{p}\leq x} \frac{\beta_{\mathfrak{p},1} + \cdots + \beta_{\mathfrak{p},d}}{\mathrm{N}\mathfrak{p}^{1+it}} \to 0,$$

since this is a tail-end piece of a convergent series by (4.2) and (4.4). When $t = 0$,

$$\begin{aligned}\sum_{\sqrt{x}<\mathrm{N}\mathfrak{p}\leq x} \frac{\beta_{\mathfrak{p},1} + \cdots + \beta_{\mathfrak{p},d}}{\mathrm{N}\mathfrak{p}} &= \sum_{\sqrt{x}<n\leq x} \frac{b_n}{n} + o(1) \\ &= -R_0 \log\log x + R_0 \log\log \sqrt{x} + o(1) \\ &= R_0 \log(1/2) + o(1). \qquad \blacksquare\end{aligned}$$

We now introduce a condition on $L_2(s)$ in (2.2) to extend the last part of Lemma 3.1 to the line $\mathrm{Re}(s) = 1/2$.

Definition 4.4 Let $L(s)$ be a normalized Euler product over K. We say it satisfies the *second moment hypothesis* when its second moment over K, $L_2(s)$, extends from $\mathrm{Re}(s) > 1$ to a holomorphic nonvanishing function on $\mathrm{Re}(s) = 1$, except for a zero or pole at perhaps one point on $\mathrm{Re}(s) = 1$.

We label the exceptional point in Definition 4.4 as $1+it_0$, and its order of vanishing is R_0. For ease of exposition, when there is no such point we let $1+it_0$ be any number on the line $\mathrm{Re}(s) = 1$, with $R_0 = 0$.

The second moment hypothesis concerns behavior only on the line $\mathrm{Re}(s) = 1$. In practice, the exceptional point in the second moment hypothesis should arise when the symmetric or exterior square of $L(s)$ over K has a factor that is essentially the Riemann zeta function.

Example 4.5 The second moment of $\zeta_K(s)$ over K is $\zeta_K(s)$, so the second moment hypothesis is satisfied with a simple pole at $s = 1$ ($R_0 = -1$).

Example 4.6 For a nontrivial Dirichlet character χ, the second moment of $L(\chi, s)$ over $\mathbf{Q}$ is $L(\chi^2, s)$. The second moment hypothesis is satisfied, with a simple pole at $s = 1$ when χ is quadratic ($R_0 = -1$) and with no zeros or poles when χ is nonquadratic ($R_0 = 0$).

Example 4.7 Let $E_{/\mathbf{Q}}$ be an elliptic curve. Using a theorem of Shimura [27], the symmetric square of $L(E, s)$ is holomorphic and nonvanishing on $\mathrm{Re}(s) = 2$. The exterior square is $\zeta(s-1)$ up to a finite number of Euler factors. Using $L(E, s + 1/2)$ instead of $L(E, s)$, the second moment hypothesis is satisfied for $L(E, s + 1/2)$, with a simple zero at $s = 1$ ($R_0 = 1$).

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Remark 4.8 Under the second moment hypothesis, applying the Wiener-Ikehara Tauberian theorem to $L_2'(s+it_0)/L_2(s+it_0)$ implies

$$\frac{\sum_{\mathrm{N}\mathfrak{p}\le x}(\alpha_{\mathfrak{p},1}^2+\cdots+\alpha_{\mathfrak{p},d}^2)\,\mathrm{N}\mathfrak{p}^{-it_0}}{\#\{\mathfrak{p}:\mathrm{N}\mathfrak{p}\le x\}}\to -R_0. \tag{4.5}$$

Therefore $|R_0|\le d$.

When $t_0=0$, the average on the left side of (4.5) is analogous to integrals of the type $\int_I x^2\,d\mu$ which classically define the second moment of a measure μ. Therefore, it is reasonable to call $-R_0$ a second moment. This moment is 1 for quadratic characters, 0 for linear characters with finite order greater than two, and -1 for elliptic curves.

When $L(s)$ is the Artin L-function of a character χ, d equals $\chi(1)$ and the Chebotarev density theorem implies $-R_0$ is the average value of $\chi(g^2)$ over the Galois group for χ. This average is the Frobenius-Schur indicator, and is known to be either 1, 0, or -1 when χ is irreducible. Therefore when χ is possibly reducible, the condition $|R_0|=d$ only occurs when χ is a sum of linear characters, so $R_0=d$ does not occur and $R_0=-d$ occurs precisely when χ is a sum of trivial and quadratic characters.

Theorem 4.9 *Let $L(s)$ be as in* (2.1)*, and satisfy the second moment hypothesis. Then, as $x\to\infty$,*

$$\sum_{\mathrm{N}\mathfrak{p}\le x}\sum_{k\ge1}\frac{\alpha_{\mathfrak{p},1}^k+\cdots+\alpha_{\mathfrak{p},d}^k}{k\,\mathrm{N}\mathfrak{p}^{k(1/2+it)}}=\sum_{\mathrm{N}\mathfrak{p}^k\le x}\frac{\alpha_{\mathfrak{p},1}^k+\cdots+\alpha_{\mathfrak{p},d}^k}{k\,\mathrm{N}\mathfrak{p}^{k(1/2+it)}}
+\begin{cases}o(1), & \text{if } t\ne t_0/2,\\ -R_0\log\sqrt{2}+o(1), & \text{if } t=t_0/2.\end{cases}$$

When $R_0\ne0$, note the exceptional behavior for $L(s)$ happens at $1/2+it_0/2$, not at $1/2+it_0$.

Proof By the argument in the proof of Lemma 3.1, at $s=1/2+it$ we look at

$$\sum_{\sqrt{x}<\mathrm{N}\mathfrak{p}\le x}\frac{\alpha_{\mathfrak{p},1}^2+\cdots+\alpha_{\mathfrak{p},d}^2}{2\,\mathrm{N}\mathfrak{p}^{2s}}=\sum_{\sqrt{x}<\mathrm{N}\mathfrak{p}\le x}\frac{\alpha_{\mathfrak{p},1}^2+\cdots+\alpha_{\mathfrak{p},d}^2}{2\,\mathrm{N}\mathfrak{p}^{1+2it}}$$

as $x\to\infty$. By Theorem 4.3 with $\widetilde{L}(s)=L_2(s)$, this sum tends to $-(1/2)R_0\log 2$ as $x\to\infty$ when $2t=t_0$, and it tends to 0 when $2t\ne t_0$. ∎

Example 4.10 Applying Theorem 4.9 to $L(s)=\zeta_K(s)$, for $\mathrm{Re}(s)\ge 1/2$

$$\sum_{\mathrm{N}\mathfrak{p}\le x}\sum_{k\ge1}\frac{1}{k\,\mathrm{N}\mathfrak{p}^{ks}}=\sum_{\mathrm{N}\mathfrak{p}^k\le x}\frac{1}{k\,\mathrm{N}\mathfrak{p}^{ks}}+\begin{cases}o(1), & \text{if } s\ne 1/2,\\ \log\sqrt{2}+o(1), & \text{if } s=1/2.\end{cases}$$

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Example 4.10 illustrates the $\sqrt{2}$ phenomenon in a very basic form. The appearance of $\sqrt{2}$ in Theorem 4.9 comes from the terms in the Dirichlet series for $\log L(s)$ corresponding to squares of primes. This is also the source of irregularities in the distribution of primes among quadratic residue and nonresidue classes [23] and an extra term of $1/2$ in a heuristic formula of Nagao [20, p. 213] for the rank of an elliptic curve. Viewing this $\sqrt{2}$ as an effect of symmetric and exterior squares, there is a relation to the quadratic excess of Brock and Granville [5], as explained by Katz [14].

We now show a result in the direction of the $\sqrt{2}$ factor in Theorem 1.1.

Corollary 4.11 *Let $L(s)$ be as in (2.1), and satisfy the second moment hypothesis. The following conditions are equivalent:*

(1) *As $x \to \infty$,*

$$\prod_{\mathrm{N}\mathfrak{p} \le x} \frac{1}{(1 - \alpha_{\mathfrak{p},1}/\sqrt{\mathrm{N}\mathfrak{p}}) \cdots (1 - \alpha_{\mathfrak{p},d}/\sqrt{\mathrm{N}\mathfrak{p}})} \sim \frac{C}{(\log x)^r}$$

for some $C \in \mathbf{C}^\times$ and $r \in \mathbf{C}$.

(2) *As $x \to \infty$,*

$$\sum_{\mathrm{N}\mathfrak{p}^k \le x} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\,\mathrm{N}\mathfrak{p}^{k/2}} = -r' \log\log x + C' + o(1)$$

for some C' and r' in $\mathbf{C}$.

In this case $r' = r$ and either $C = e^{C'}$ or $C = e^{C'}/\sqrt{2}^{R_0}$.

Proof By Lemma 3.2, we have such an equivalence when the sum $\sum_{\mathrm{N}\mathfrak{p}^k \le x}$ in the second part is replaced with $\sum_{\mathrm{N}\mathfrak{p} \le x} \sum_{k \ge 1}$. To pass from the latter to the former will, by Theorem 4.9 at $t = 0$, have no effect if $t_0 \ne 0$ or introduce a new term $-R_0 \log \sqrt{2}$ if $t_0 = 0$. ∎

Unlike Theorem 3.3, in Corollary 4.11 the partial Euler products are allowed to tend to 0 (in a specific manner).

5 Extending Goldfeld's Theorem to More Euler Products

Having linked a peculiar role for $\sqrt{2}$ to second moments in Corollary 4.11, we are ready to describe a version of Goldfeld's theorem for fairly general Euler products.

Lemma 5.1 *As $s \to 0^+$,*

$$s \int_e^\infty \frac{\log\log x}{x^{s+1}}\, dx = -\log s - \gamma + o(1).$$

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Proof Let $u = \log x$ and then $y = su$. Euler's constant arises since

$$\int_s^\infty e^{-y}(\log y)\,dy \to \int_0^\infty e^{-y}(\log y)\,dy = -\gamma. \qquad \blacksquare$$

Lemma 5.1 is the source of e^γ in Goldfeld's theorem.

Lemma 5.2 *Let $h\colon [1,\infty) \to \mathbf{C}$ be piecewise continuous with $h(x) \to 0$ as $x \to \infty$. Then as $s \to 0^+$,*

$$s\int_1^\infty \frac{h(x)}{x^{s+1}}\,dx \to 0.$$

Proof Pick $\varepsilon > 0$, break up the integral at a point beyond which $|h(x)| \le \varepsilon$, estimate, take $\overline{\lim}_{s\to 0^+}$ and then let $\varepsilon \to 0$. $\blacksquare$

Theorem 5.3 *Let $L(s)$ be as in (2.1). If*

$$\sum_{\mathrm{N}\mathfrak{p}^k \le x} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\,\mathrm{N}\mathfrak{p}^{k/2}} = -r\log\log x + C' + o(1) \tag{5.1}$$

as $x \to \infty$, for some C' and r in $\mathbf{C}$, then $L(s)$ extends to a holomorphic nonvanishing function on $\mathrm{Re}(s) > 1/2$, and

$$L(s) \sim e^{C'}e^{r\gamma}(s-1/2)^r \tag{5.2}$$

as $s \to 1/2^+$. If $L(s)$ is holomorphic at $s = 1/2$, then the order of vanishing there is r and its leading Taylor coefficient is $e^{C'}e^{r\gamma}$.

Proof Set

$$b_n = \sum_{\mathrm{N}\mathfrak{p}^k = n} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k},$$

so $\log L(s) = \sum b_n n^{-s}$ for $\mathrm{Re}(s) > 1$. The condition (5.1) says $\sum_{n\le x} b_n/\sqrt{n} = -r\log\log x + C' + o(1)$ as $x \to \infty$. From this slow growth, $\sum b_n n^{-s}$ converges for $\mathrm{Re}(s) > 1/2$, so $L(s)$ is holomorphic and nonvanishing on $\mathrm{Re}(s) > 1/2$. For $\mathrm{Re}(s) > 0$,

$$(\log L)(s+1/2) = \sum_{n\ge 1} \frac{b_n}{\sqrt{n}}\frac{1}{n^s} = s\int_1^\infty \frac{A(x)}{x^{s+1}}\,dx,$$

where $A(x) = \sum_{n\le x} b_n/\sqrt{n} = -r\log\log x + C' + o(1)$ as $x \to \infty$. As $s \to 0^+$,

$$\begin{aligned}(\log L)(s+1/2) &= s\int_1^e \frac{A(x)}{x^{s+1}}\,dx + s\int_e^\infty \frac{A(x)}{x^{s+1}}\,dx\\ &= s\int_e^\infty \frac{-r\log\log x + C' + o_x(1)}{x^{s+1}}\,dx + o_s(1)\\ &= r\log s + r\gamma + C' + o_s(1) \quad \text{by Lemmas 5.1 and 5.2,}\end{aligned}$$

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where $o_x(1)$ (resp., $o_s(1)$) is a term tending to 0 as $x \to \infty$ (resp., as $s \to 0^+$). Therefore $L(s + 1/2) \sim e^{C'} e^{r\gamma} s^r$ as $s \to 0^+$. ■

Corollary 5.4 *Let $L(s)$ be as in (2.1). For some real t, assume*

$$\sum_{\mathrm{N}\mathfrak{p}^k \le x} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k\,\mathrm{N}\mathfrak{p}^{k(1/2+it)}} = -r_t \log\log x + C_t' + o(1)$$

as $x \to \infty$, for some C_t' and r_t in $\mathbf{C}$. Then $L(s)$ extends to a holomorphic nonvanishing function on $\mathrm{Re}(s) > 1/2$. If $L(s)$ is holomorphic at $1/2 + it$, then its order of vanishing there is r_t and its leading Taylor coefficient is $e^{C_t'} e^{r_t\gamma}$.

Proof We can assume $t = 0$ by replacing $\alpha_{\mathfrak{p},j}$ with $\alpha_{\mathfrak{p},j}\,\mathrm{N}\mathfrak{p}^{-it}$. Now apply Theorem 5.3. ■

Corollary 5.5 *Let χ be a unitary Hecke character on a number field K, and assume it is not of the form $(\mathrm{N}(\cdot))^{it}$ for any real t. If*

$$\prod_{\mathrm{N}\mathfrak{p} \le x} \frac{1}{1 - \chi(\mathfrak{p})/\sqrt{\mathrm{N}\mathfrak{p}}} \sim \frac{C}{(\log x)^r} \tag{5.3}$$

as $x \to \infty$, where $C \neq 0$, then $L(\chi, s)$ is holomorphic and nonvanishing for $\mathrm{Re}(s) > 1/2$, $r = \mathrm{ord}_{s=1/2} L(\chi, s)$, and

$$C = \begin{cases} \sqrt{2}Be^{-r\gamma}, & \text{if } \chi \text{ is quadratic,} \\ Be^{-r\gamma}, & \text{otherwise,} \end{cases}$$

where B is the leading Taylor coefficient of $L(\chi, s)$ at $s = 1/2$.

Proof Apply Corollary 4.11 and Theorem 5.3. Whether or not the second moment of $L(\chi, s)$ has a pole on $\mathrm{Re}(s) = 1$ depends on whether or not χ is quadratic. ■

In the statement of Corollary 5.5, we excluded the possibility that χ is $(\mathrm{N}(\cdot))^{it}$ simply because part of the conclusion (no pole along $\mathrm{Re}(s) = 1$) shows such χ do not satisfy (5.3).

The case of Dirichlet L-functions is worth recording separately (with $r = 0$).

Corollary 5.6 *Let χ be a nontrivial Dirichlet character. If $\prod_{p \le x}(1 - \chi(p)/\sqrt{p})^{-1}$ has a nonzero limit as $x \to \infty$, then $L(\chi, s)$ satisfies the Riemann hypothesis and*

$$\prod_p \frac{1}{1 - \chi(p)/\sqrt{p}} = \begin{cases} L(\chi, 1/2), & \text{if } \chi \text{ is nonquadratic,} \\ \sqrt{2}L(\chi, 1/2), & \text{if } \chi \text{ is quadratic.} \end{cases}$$

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Corollary 5.7 (Goldfeld) *Let $E_{/\mathbf{Q}}$ be an elliptic curve. If* $\operatorname{Prod}(E,x) \sim C/(\log x)^r$ *as* $x \to \infty$, *where* $C \in \mathbf{C}^\times$ *and* $r \in \mathbf{C}$, *then* $L(E,s) \neq 0$ *for* $\operatorname{Re}(s) > 1$, $r = \operatorname{ord}_{s=1} L(E,s)$, *and* $C = B/\sqrt{2}e^{r\gamma}$, *where* B *is the leading Taylor coefficient of* $L(E,s)$ *at* $s = 1$.

Proof Apply Corollary 4.11 and Theorem 5.3 to $L(s) = L(E, s + 1/2)$. ■

This proof of Corollary 5.7 simplifies Goldfeld's proof in [12]. We used his derivation of (5.2) more fully and avoided some delicate arguments from [12]. In particular, we did not need explicit formulas. However, explicit formulas will be used in Section 6. Our use of the last part of Theorem 5.3 tacitly appeals to the elliptic modularity theorem to get holomorphy of $L(E,s)$ at $s = 1$. While holomorphy at this point is logically weaker than the full elliptic modularity theorem, it seems unlikely that this particular consequence will ever be (generally) proved in a simpler way on its own.

Example 5.8 Let χ_4 be the quadratic character mod 4. We can not prove the partial Euler products for $L(\chi_4, s)$ at $s = 1/2$ converge, but if they do, we know the limit is not $L(\chi_4, 1/2) \approx .67$, but rather $\sqrt{2}L(\chi_4, 1/2) \approx .94$. Table 1 gives crude numerical evidence.

Example 5.9 The curve $E_3 : y^2 + y = x^3 - 7x + 6$ has analytic rank 3, by [6]. If $(\log x)^3 \operatorname{Prod}(E_3, x)$ has a limit, it is not $L^{(3)}(E_3, 1)/3! \approx 1.73$, but

$$L^{(3)}(E_3, 1)/3!\sqrt{2}e^{3\gamma} \approx .22.$$

This is consistent with Table 2.

x	$\prod_{p\le x}\left(1 - \chi_4(p)/\sqrt{p}\right)^{-1}$
10^2	.94
10^3	.89
10^4	.98
10^5	.97

Table 1

x	$(\log x)^3 \operatorname{Prod}(E_3, x)$
10^2	.22
10^3	.18
10^4	.25
10^5	.22

Table 2

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Example 5.10 The L-function of a CM elliptic curve $E_{/\mathbf{Q}}$ fits both Corollaries 5.5 and 5.7, which leads to an interesting comparison. Write $L(E, s+1/2) = L(\chi, s)$ for a unitary Hecke character χ on the appropriate imaginary quadratic field K. Taking partial Euler products at $s = 1/2$, we consider hypothetical asymptotic relations for $L(E, s)$ at $s = 1$ and $L(\chi, s)$ at $s = 1/2$:

$$\prod_{\substack{p \leq x \\ p \mid N}} \frac{1}{1 - a_p/p} \cdot \prod_{\substack{p \leq x \\ (p,N)=1}} \frac{1}{1 - a_p/p + 1/p} \sim \frac{C_1}{(\log x)^{r_1}} \tag{5.4}$$

and

$$\prod_{\mathrm{N}\mathfrak{p} \leq x} \frac{1}{1 - \chi(\mathfrak{p})/\sqrt{\mathrm{N}\mathfrak{p}}} \sim \frac{C_2}{(\log x)^{r_2}}. \tag{5.5}$$

Corollaries 5.5 and 5.7 tell us that if (5.4) and (5.5) hold, then $r_1 = r_2$ but $C_1 \neq C_2$. Instead, $C_1 = C_2/\sqrt{2}$. That implies the ratio of the first partial Euler product to the second tends not to 1, but to $1/\sqrt{2}$. This can be proved unconditionally, and boils down to the fact that

$$\prod_{\substack{\sqrt{x} < p \leq x \\ p \text{ inert in } \mathcal{O}_K}} \frac{1}{1 + 1/p} \to \frac{1}{\sqrt{2}}.$$

Taking logarithms, this is equivalent to

$$\sum_{\substack{\sqrt{x} < p \leq x \\ p \text{ inert in } \mathcal{O}_K}} \frac{1}{p} \to \frac{1}{2} \log 2, \tag{5.6}$$

and (5.6) can be verified by writing the sum over both split and inert primes using the quadratic character η of $\mathrm{Gal}(K/\mathbf{Q})$. Alternatively, (5.6) follows from Theorem 4.3 with $\beta_{p,1} = 1$, $\beta_{p,2} = -\eta(p)$, and $R_0 = -1$.

Here is a generalization of Goldfeld's theorem to any point on the critical line for a reasonable Euler product.

Theorem 5.11 *Let $L(s)$ be as in (2.1). Assume $L(s)$ extends to a holomorphic function on $\mathrm{Re}(s) \geq 1/2$ and satisfies the second moment hypothesis, with $1 + it_0$ being the exceptional point for $L_2(s)$. For a complex number with real part $1/2$, say $1/2 + it$, assume*

$$\prod_{\mathrm{N}\mathfrak{p} \leq x} \frac{1}{(1 - \alpha_{\mathfrak{p},1}\, \mathrm{N}\mathfrak{p}^{-(1/2+it)}) \cdots (1 - \alpha_{\mathfrak{p},d}\, \mathrm{N}\mathfrak{p}^{-(1/2+it)})} \sim \frac{C_t}{(\log x)^{r_t}} \tag{5.7}$$

as $x \to \infty$, where $C_t \in \mathbf{C}^\times$ and $r_t \in \mathbf{C}$. Then, with B_t denoting the leading Taylor coefficient of $L(s)$ at $s = 1/2 + it$,

(1) *$L(s) \neq 0$ for $\mathrm{Re}(s) > 1/2$,*
(2) *$r_t = \mathrm{ord}_{s=1/2+it} L(s)$,*

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(3) *if* $t \neq t_0/2$, *then* $C_t = B_t/e^{r_t\gamma}$, *and if* $t = t_0/2$, *then* $C_t = B_t/\sqrt{2}^{R_0} e^{r_t\gamma}$.

Proof We do not shift to the point $1/2$, as in the proof of Corollary 5.4, but work directly at $1/2 + it$.

By Lemma 3.2, (5.7) is equivalent to

$$\sum_{\mathrm{N}\mathfrak{p}\leq x}\sum_{k\geq 1}\frac{\alpha_{\mathfrak{p},1}^k+\cdots+\alpha_{\mathfrak{p},d}^k}{k\,\mathrm{N}\mathfrak{p}^{k(1/2+it)}} = -r_t\log\log x + \log C_t + o(1)$$

as $x \to \infty$, where $\log C_t$ is a particular logarithm of C_t. Replacing $\sum_{\mathrm{N}\mathfrak{p}\leq x}\sum_k$ with $\sum_{\mathrm{N}\mathfrak{p}^k\leq x}$ by Theorem 4.9 (here we use the second moment hypothesis) yields

$$\sum_{\mathrm{N}\mathfrak{p}^k\leq x}\frac{\alpha_{\mathfrak{p},1}^k+\cdots+\alpha_{\mathfrak{p},d}^k}{k\,\mathrm{N}\mathfrak{p}^{k(1/2+it)}} = \begin{cases} -r_t\log\log x + \log C_t + o(1), & \text{if } t \neq t_0/2,\\ -r_t\log\log x + \log C_t + R_0\log\sqrt{2} + o(1), & \text{if } t = t_0/2.\end{cases}$$

Now use Corollary 5.4. ■

Remark 5.12 If the formal Euler product of $L(s)$ at $s = 1/2 + it$ is nonzero, it is $L(1/2+it)$ when $t \neq t_0/2$ and is $L(1/2+it)/\sqrt{2}^{R_0}$ when $t = t_0/2$. This value depends on $L(1/2+it)$ and (surprisingly) on the order of vanishing of $L_2(s)$ at $s = 1 + 2it$.

Example 5.13 In Example 3.5, we referred to an example of Hardy's where

$$\prod(1 + a_n x^n) \to 2\prod(1 + a_n)$$

as $x \to 1^-$. A close reading of Hardy's example leads to the following general phenomenon.

Suppose $b_n \in \mathbf{C}$ satisfies the following conditions: $\sum b_n$ and $\sum b_n^2$ converge, and $\sum b_n^3$ is absolutely convergent. (These are "moment" hypotheses, up to order 3.) If, for $0 \leq x < 1$,

$$\sum |b_n|^2 x^n = r\log(1 - x) + h(x), \tag{5.8}$$

where $r \in \mathbf{R}$ and h is continuous on $[0, 1]$, and $b_n \neq -1$ for each n, then the sequence $a_n := 2\operatorname{Re}(b_n) + |b_n|^2$ satisfies

$$\prod_{n\geq 1}(1 + a_n x^n) \to \frac{1}{2^r}\prod_{n\geq 1}(1 + a_n) \tag{5.9}$$

as $x \to 1^-$. The assumption (5.8) resembles (4.3) when we take $\widetilde{L}(s) = L_2(s)$ for an Euler product $L(s)$ satisfying the second moment hypothesis.

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6 A Sharper Version of Theorem 5.11

To understand more clearly what (5.7) means, we will show under plausible hypotheses extending those in Theorem 5.11 that (5.7), for any choice of t, is equivalent to an estimate that seems deeper than the Riemann hypothesis for $L(s)$.

We recall the notation:

$$L(s) = \prod_{\mathfrak{p}} \frac{1}{(1 - \alpha_{\mathfrak{p},1} \, \mathrm{N}\mathfrak{p}^{-s}) \cdots (1 - \alpha_{\mathfrak{p},d} \, \mathrm{N}\mathfrak{p}^{-s})}$$

with $|\alpha_{\mathfrak{p},j}| \leq 1$. Now let

$$b_n = \sum_{\mathrm{N}\mathfrak{p}^k = n} \frac{\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k}{k}, \tag{6.1}$$

so when $\mathrm{Re}(s) > 1$,

$$\log L(s) = \sum_{n \geq 2} \frac{b_n}{n^s}, \quad -\frac{L'(s)}{L(s)} = \sum_{n \geq 1} \frac{b_n \log n}{n^s}. \tag{6.2}$$

When we say $L(s)$ has an analytic continuation and functional equation, we mean: for some integer $m \geq 1$ the function

$$\Lambda(s) := A^s \prod_{i=1}^{m} \Gamma(\lambda_i s + \mu_i) \cdot L(s), \tag{6.3}$$

where $A > 0$, $\lambda_i > 0$, and $\mathrm{Re}(\mu_i) \geq 0$, satisfies the three conditions

- $\Lambda(s)$ is entire,
- $\Lambda(s)$ is bounded in vertical strips,
- for some nonzero complex number w, $\Lambda(s)$ satisfies

$$\Lambda(1 - s) = w\overline{\Lambda(\bar{s})}. \tag{6.4}$$

Necessarily, $|w| = 1$, and in practice $\lambda_i = 1$ or $1/2$. Boundedness of $\Lambda(s)$ in strips is needed to justify shifting lines of integration in the derivation of explicit formulas.

The *Riemann hypothesis* for $L(s)$ is the claim that all zeros of $\Lambda(s)$ are on the line $\mathrm{Re}(s) = 1/2$. (We are only concerned with this condition for individual L-functions, not in families.) Equivalently, all zeros of $L(s)$ should be on the line $\mathrm{Re}(s) = 1/2$ or at the poles of $\prod \Gamma(\lambda_i s + \mu_i)$, and in the latter case the multiplicity of the zero should equal the multiplicity of the Γ-pole. Any zeros of $L(s)$ whose locations and multiplicities are explained by a Γ-pole are called *trivial*. They have real part ≤ 0, by the constraints on λ_i and μ_i. The *nontrivial* zeros of $L(s)$ are the remaining zeros. Since $L(s)$ is nonvanishing on $\mathrm{Re}(s) > 1$, the functional equation (6.4) implies the nontrivial zeros have real part in $[0, 1]$, and the Riemann hypothesis for $L(s)$ is equivalent to $L(s) \neq 0$ for $\mathrm{Re}(s) > 1/2$.

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The formalism of (6.3) and (6.4) makes sense if $m = 0$, but this is possible only when $L(s)$ is identically 1 [8, Theorem 3.1]. (For earlier results in this direction, see [3, 29].) So there is no harm in taking $m \geq 1$ when considering Euler products over number fields. However, Euler products over function fields have $m = 0$. We will discuss the function field case in Section 8.

Using the notation of (6.1), the basic hypothesis in Corollary 5.4 is that

$$\sum_{n \leq x} \frac{b_n}{n^{1/2+it}} + r_t \log\log x \text{ converges} \tag{6.5}$$

as $x \to \infty$, for some r_t, and one of the conclusions is that r_t is the order of vanishing of $L(s)$ at $1/2 + it$. (We are assuming $L(s)$ is entire.) Our next theorem shows that the validity of (6.5), using the proper value of r_t, is independent of t. We need the following lemma concerning a growth condition that is the same for all t.

Lemma 6.1 *For a sequence of complex numbers c_n, suppose $\sum_{n \leq x} c_n = o(\sqrt{x} \log x)$. Then, for each real t,*

$$\sum_{n \leq x} \frac{c_n}{n^{it}} = o(\sqrt{x} \log x).$$

Proof Use partial summation. ■

A similar result holds if we replace $\sqrt{x} \log x$ with $x^a (\log x)^b$ for $0 < a < 1$ and $b > 0$.

Returning to the notation of (6.1), set

$$\psi_L(x) = \sum_{n \leq x} b_n \log n = \sum_{\mathrm{N}\mathfrak{p}^k \leq x} (\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k) \log \mathrm{N}\mathfrak{p}.$$

Then for $\mathrm{Re}(s) > 1$,

$$-\frac{L'(s)}{L(s)} = s \int_1^\infty \frac{\psi_L(x)}{x^{s+1}}\, dx \tag{6.6}$$

and the Riemann hypothesis holds for $L(s)$ if and only if $\psi_L(x) = O(\sqrt{x} (\log x)^2)$.

Theorem 6.2 *Let $L(s)$ be as in* (2.1)*. Assume $L(s)$ admits an analytic continuation and functional equation. Fix a real number t, and let m_t be the order of $L(s)$ at $s = 1/2 + it$. Then the following conditions are equivalent:*

(1) *As $x \to \infty$,*

$$\sum_{n \leq x} \frac{b_n}{n^{1/2+it}} + m_t \log\log x$$

converges.

(2) $\psi_L(x) = o(\sqrt{x} \log x)$.

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That is, when $L(s)$ is entire with a functional equation, the hypothesis of Corollary 5.4*, using the correct coefficient of* $\log\log x$*, is equivalent to* $\psi_L(x) = o(\sqrt{x}\log x)$*.*

Theorem 1.3 follows from this with $L(s) = L(E, s + 1/2)$.

Proof (The reader may want to read the application of this result in Theorem 6.3 before beginning the proof.)

The function $\widetilde{L}(s) = L(s + it)$ (here t is from the statement of the theorem, and is unrelated to s) is a normalized Euler product, with b_n replaced by $b_n n^{-it}$ and $\Lambda(s)$ replaced by $\widetilde{\Lambda}(s) = A^{-it}\Lambda(s+it)$. Lemma 6.1 says the conditions $\psi_L(x) = o(\sqrt{x}\log x)$ and $\psi_{\widetilde{L}}(x) = o(\sqrt{x}\log x)$ are equivalent, so we may safely assume $t = 0$ and prove

$$\sum_{n\leq x}\frac{b_n}{\sqrt{n}} + m_0\log\log x \text{ converges as } x\to\infty \iff \psi_L(x) = o(\sqrt{x}\log x). \tag{6.7}$$

Each side of (6.7) implies $L(s) \neq 0$ on the region $\operatorname{Re}(s) > 1/2$ (the left side shows $\log L(s)$ is holomorphic here, and the right side shows $L'(s)/L(s)$ is holomorphic here by (6.6)), so we may—and will—use the Riemann hypothesis for $L(s)$ to prove (6.7).

We will approach (6.7) in two stages. First we will use an explicit formula to find a convenient expression for

$$\sum_{n\leq x}\frac{b_n\log n}{\sqrt{n}} + m_0\log x,$$

which is (6.12) below. Then, using two other explicit formulas, we will obtain (6.7).

Recall the Dirichlet series for $-L'(s)/L(s)$ in (6.2). For $c > 1$ and a real, Perron's formula gives for $x > 1$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L'(s)}{L(s)}\frac{x^s}{s-a}\,ds = x^a{\sum_{n\leq x}}'\frac{b_n\log n}{n^a},$$

where the $'$ means the last term in the sum is weighted by $1/2$ if x is an integer. Applying this at $a = 1/2$ and $a = 0$ and subtracting,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L'(s)}{L(s)}\frac{x^s}{2s(s-1/2)}\,ds = \sqrt{x}\sum_{n\leq x}\frac{b_n\log n}{\sqrt{n}} - \sum_{n\leq x}b_n\log n. \tag{6.8}$$

When x is integral, the last terms in the two sums cancel, so we removed the $'$ on each Σ.

We now write the integral in (6.8) as a sum of residues in the half-plane $\operatorname{Re}(s) \leq 1$, yielding an explicit formula. The justification for this is tedious, but not delicate since the $s(s-1/2)$ in the denominator makes the residue sum absolutely convergent. We omit the justification (thus hiding our use of the functional equation for $L(s)$ and the boundedness of $\Lambda(s)$ in vertical strips), and simply report the results.

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Writing the local expansions of $L'(s)/L(s)$ at $s = 0$ and $s = 1/2$ as

$$\frac{L'(s)}{L(s)} = \frac{c}{s} + d + \cdots, \quad \frac{L'(s)}{L(s)} = \frac{c'}{s-1/2} + d' + \cdots,$$

we have

$$\text{(6.9)} \quad \mathrm{Res}_{s=\rho}\left(-\frac{L'(s)}{L(s)}\frac{x^s}{2s(s-1/2)}\right) = \begin{cases} c(\log x + 2) + d, & \text{if } \rho = 0, \\ -c'\sqrt{x}(\log x - 2) - d'\sqrt{x}, & \text{if } \rho = 1/2. \end{cases}$$

As ρ runs over the nontrivial zeros of $L(s)$, $\sum_\rho 1/|\rho|^2$ converges by the Riemann hypothesis for $L(s)$. (In the sum, ρ is repeated according to its multiplicity.) Therefore, excluding $1/2$,

$$\text{(6.10)} \qquad \sum_{\substack{\rho \text{ nontr.} \\ \rho \neq 1/2}} \mathrm{Res}_{s=\rho}\left(-\frac{L'(s)}{L(s)}\frac{x^s}{2s(s-1/2)}\right) = O(\sqrt{x}).$$

Since the Γ-poles of $L(s)$ run in finitely many arithmetic progressions to the left, the residue sum over these poles is absolutely convergent, and tends to 0 or is oscillatory as $x \to \infty$. (Oscillations occur only when there is a Γ-pole other than 0 on the imaginary axis).

Adding up the residues in (6.9), (6.10), and at the Γ-poles, the explicit formula version of (6.8) implies

$$\text{(6.11)} \qquad -c'\sqrt{x}\log x + O(\sqrt{x}) = \sqrt{x}\sum_{n \leq x}\frac{b_n \log n}{\sqrt{n}} - \sum_{n \leq x} b_n \log n$$

as $x \to \infty$. Dividing by $\sqrt{x}$ and noting $c' = m_0$, we arrive at the key equation

$$\text{(6.12)} \qquad \sum_{n \leq x}\frac{b_n \log n}{\sqrt{n}} = -m_0 \log x + \frac{\sum_{n \leq x} b_n \log n}{\sqrt{x}} + O(1).$$

This completes the first stage of our proof.

Now we derive (6.7) from (6.12). We want to divide the nth term on the left side of (6.12) by $\log n$, which reminds us of the elementary implication (for $c_n \in \mathbf{C}$)

$$\text{(6.13)} \quad \sum_{n \leq x} c_n = A \log x + O(1) \Longrightarrow \sum_{2 \leq n \leq x} \frac{c_n}{\log n} = A \log\log x + \text{const.} + o(1).$$

If we could use (6.13) in (6.12), we would have the left side of (6.7) directly. Alas, (6.13) can not be applied since $(\sum_{n \leq x} b_n \log n)/\sqrt{x} = \psi_L(x)/\sqrt{x}$ is surely not bounded. (This is plausible by comparison to known oscillations for $(\psi(x) - x)/\sqrt{x}$, and serves simply as motivation for our next step.) We consider instead a condition which models (6.12) but is weaker than the hypothesis in (6.13):

$$\sum_{n \leq x} c_n = A \log x + g(x) + O(1).$$

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We carry out the argument for (6.13) in this setting. (To simply absorb $A\log x$ into $g(x)$ on account of the generality of the notation would not help.) Writing $C(n) = c_1 + \cdots + c_n = A\log n + g(n) + O(1)$, we find by partial summation

$$\begin{aligned}
\sum_{2\le n\le N} \frac{c_n}{\log n} &= \frac{C(N)}{\log N} + \sum_{n=2}^{N-1} C(n)\Big(\frac{1}{\log n} - \frac{1}{\log(n+1)}\Big) \\
&= A + \frac{g(N)}{\log N} + O\Big(\frac{1}{\log N}\Big) + \sum_{n=2}^{N-1} C(n)\Big(\frac{1}{n(\log n)^2} + O\Big(\frac{1}{n^2(\log n)^2}\Big)\Big) \\
&= \frac{g(N)}{\log N} + \sum_{n=2}^{N-1}\Big(\frac{A}{n\log n} + \frac{g(n)}{n(\log n)^2} + O\Big(\frac{C(n)}{n^2(\log n)^2}\Big)\Big) + A' + o(1) \\
&= A\log\log N + \frac{g(N)}{\log N} + \sum_{n=2}^{N-1}\Big(\frac{g(n)}{n(\log n)^2} + O\Big(\frac{C(n)}{n^2(\log n)^2}\Big)\Big) \\
&\qquad + A'' + o(1),
\end{aligned}$$

where A' and A'' are new constants. We apply this to (6.12): $c_n = b_n(\log n)/\sqrt{n}$, $A = -m_0$, and $g(x) = \psi_L(x)/\sqrt{x}$. Since $g(x) = O(\sqrt{x})$ (this is another way of saying $\psi_L(x) = O(x)$, not $\psi_L(x) = O(\sqrt{x})$), we get $C(n) = O(\sqrt{n})$. Then $\sum_{n\ge 2} C(n)/n^2(\log n)^2$ is absolutely convergent, so as $N \to \infty$, our partial summation computation shows (6.12) implies

(6.14)
$$\sum_{n\le x} \frac{b_n}{\sqrt{n}} + m_0\log\log x \text{ converges} \Longleftrightarrow \frac{\psi_L(x)}{\sqrt{x}\log x} + \sum_{n\le x}\frac{\psi_L(n)}{n\sqrt{n}(\log n)^2} \text{ converges}$$

as $x \to \infty$. We now show the sum on the right side of (6.14) converges as $x \to \infty$. This follows from the following two facts: for c_n in $\mathbf{C}$,

$$\frac{1}{x}\sum_{n\le x} c_n = O(\sqrt{x}) \Longrightarrow \sum_{n\ge 2}\frac{c_n}{n\sqrt{n}(\log n)^2} \text{ converges absolutely} \tag{6.15}$$

(this is left to the reader to check by partial summation), and

$$\frac{1}{x}\sum_{n\le x}\psi_L(n) = O(\sqrt{x}). \tag{6.16}$$

The proof of (6.16) (analogous to a property of $\psi(x) - x$), uses a modification of (6.8):

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L'(s)}{L(s)}\frac{x^s}{s(s+1)}\,ds = \sum_{n\le x} b_n\log n\left(1 - \frac{n}{x}\right). \tag{6.17}$$

By partial summation, the right side is $(1/x)\sum_{n\le x}\psi_L(n) + O(1)$. Evaluating the left side as an absolutely convergent sum of residues gives a second explicit formula:

$$\frac{1}{x}\sum_{n\le x}\psi_L(n) = \mathrm{Res}_0 + \mathrm{Res}_{-1} + \sum_{\substack{L(\rho)=0\\ \rho\ne 0,-1}}\frac{x^\rho}{\rho(\rho+1)} + O(1),$$

where each ρ is repeated according to its multiplicity as a zero, and Res_0 and Res_{-1} are residues of the integrand in (6.17). By computation, the residue at 0 is $O(\log x)$, the residue at -1 is $O((\log x)/x)$, the residues at the trivial zeros of $L(s)$ (excluding 0 and -1) have a bounded sum (as $x \to \infty$) and the residues at the nontrivial zeros are $O(\sqrt{x})$ by the Riemann hypothesis for $L(s)$. Therefore we obtain (6.16). By (6.15) and (6.16), (6.14) becomes, as $x \to \infty$,

$$\sum_{n \leq x} \frac{b_n}{\sqrt{n}} + m_0 \log\log x \text{ converges} \iff \frac{\psi_L(x)}{\sqrt{x}\log x} \text{ converges.}$$

Our final step is based on a paper of Gallagher [11]. Assuming the Riemann hypothesis for $\zeta(s)$, Gallagher used an explicit formula with remainder term for $\psi(x)$ to refine the standard estimate $\psi(x) - x = O(\sqrt{x}(\log x)^2)$ coming from the Riemann hypothesis to

$$\psi(x) - x = O(\sqrt{x}(\log\log x)^2) \tag{6.18}$$

off a closed subset of $[2, \infty)$ with finite logarithmic measure. (That is, (6.18) is satisfied for $x \geq 2$ outside a closed set E with $\int_E dx/x < \infty$.) Gallagher's argument carries over to $\psi_L(x)$, but we omit the details related to this third explicit formula. In short, then, the Riemann hypothesis for $L(s)$ implies $\psi_L(x) = O(\sqrt{x}(\log\log x)^2)$ off a set of x with finite logarithmic measure, so

$$\underline{\lim}\left|\frac{\psi_L(x)}{\sqrt{x}\log x}\right| = 0.$$

Therefore $\psi_L(x)/\sqrt{x}\log x$ converges if and only if it tends to 0. We are done. ■

Theorem 6.3 *Let $L(s)$ be a normalized Euler product, as in* (2.1)*. Assume $L(s)$ admits an analytic continuation and functional equation, and satisfies the second moment hypothesis. Then the following conditions are equivalent:*

(1) *For some real t, there are $C_t \in \mathbf{C}^\times$ and $r_t \in \mathbf{C}$ such that*

$$\prod_{\mathrm{N}\mathfrak{p} \leq x} \frac{1}{(1 - \alpha_{\mathfrak{p},1}\,\mathrm{N}\mathfrak{p}^{-(1/2+it)}) \cdots (1 - \alpha_{\mathfrak{p},d}\,\mathrm{N}\mathfrak{p}^{-(1/2+it)})} \sim \frac{C_t}{(\log x)^{r_t}}$$

as $x \to \infty$.

(2) *For all real t, there are $C_t \in \mathbf{C}^\times$ and $r_t \in \mathbf{C}$ such that*

$$\prod_{\mathrm{N}\mathfrak{p} \leq x} \frac{1}{(1 - \alpha_{\mathfrak{p},1}\,\mathrm{N}\mathfrak{p}^{-(1/2+it)}) \cdots (1 - \alpha_{\mathfrak{p},d}\,\mathrm{N}\mathfrak{p}^{-(1/2+it)})} \sim \frac{C_t}{(\log x)^{r_t}}$$

as $x \to \infty$.

(3) *As $x \to \infty$, $\psi_L(x) = o(\sqrt{x}\log x)$.*

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Proof When the first condition holds, the proof of Theorem 5.11 tells us that r_t is the order of $L(s)$ at $s = 1/2 + it$ and that the first condition in Theorem 6.2 holds. Therefore we deduce $\psi_L(x) = o(\sqrt{x}\log x)$. Now apply Lemma 3.2 and the equivalence in Theorem 6.2 with any other choice of t. We are using the second moment hypothesis in the application of Theorem 5.11. ■

We noted in the introduction that Montgomery conjectured that the true order of magnitude of $\psi(x) - x$ is at most $\sqrt{x}(\log\log\log x)^2$. Specifically, Montgomery conjectures

$$\underline{\lim}\,\frac{\psi(x)-x}{\sqrt{x}(\log\log\log x)^2} = -\frac{1}{2\pi},\quad \overline{\lim}\,\frac{\psi(x)-x}{\sqrt{x}(\log\log\log x)^2} = \frac{1}{2\pi}.$$

He arrived at this conjecture from considerations related to the hypothesis that the nontrivial zeros of $\zeta(s)$ are all simple and (under the Riemann hypothesis for $\zeta(s)$) the imaginary parts of the nontrivial zeros are linearly independent over $\mathbf{Q}$. These simplicity and linear independence hypotheses can fail for L-functions, *e.g.,* the L-function of an elliptic curve might have a multiple zero at the real point on its critical line. It is reasonable to believe that when the L-function is primitive (in the sense of [24], say), the only exceptions to these hypotheses should be because of one multiple zero (with an algebraic or geometric cause), so it still seems reasonable to conjecture that $\psi_L(x) = O(\sqrt{x}(\log\log\log x)^2)$, which would imply $\psi_L(x) = o(\sqrt{x}\log x)$. It would be interesting if these estimates on $\psi_L(x)$ can be related to statements about the vertical distribution of zeros on the line $\mathrm{Re}(s) = 1/2$.

In [15], W. Kuo and R. Murty prove Theorem 1.3, and implicitly also Theorem 6.2, by a simpler method than ours. They require neither functional equations nor explicit formulas. (They do need holomorphy on the critical line, which in practice is nearly always proved at the same time as a functional equation.) Our proof of Theorem 6.2 nevertheless remains interesting, for the following reason. Nagao [20] has used a heuristic limit formula for the (analytic) rank of an elliptic curve to search for curves with high (algebraic) rank. His formula does not converge, but the proof of Theorem 6.2 suggests an alternate limit formula. Under the plausible conditions in Theorem 6.2, this alternate formula provably converges to the analytic rank, and it does not seem that the method of [15] can be directly applied here. This will be the subject of a future paper with R. Murty.

7 Dirichlet Series on the Critical Line

Theorems 5.11 and 6.3 give information about Euler products along the critical line. A topic that naturally arises when thinking about these theorems is Dirichlet series along the critical line, for $L(s)$, $\log L(s)$, and $L'(s)/L(s)$. Should the ordinary Dirichlet series for these functions converge here? (If so, away from singularities the limits must be the expected values by Abel's theorem for Dirichlet series.) Throughout this section, we assume $L(s)$ has an analytic continuation and functional equation, and satisfies the Riemann hypothesis.

Using these assumptions, a contour integration argument shows $\log L(s)$ and $L'(s)/L(s)$ are represented by their ordinary Dirichlet series and $L(s)$ is represented

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by its Euler product (2.1) for $\mathrm{Re}(s) > 1/2$. I thank Rohrlich for pointing out to me that for any entire L-function over a number field (with a functional equation connecting s and $1-s$, and so on), its ordinary Dirichlet series should converge for $\mathrm{Re}(s) > 1/2 - 1/2D$, where D is the number of real Gamma factors plus twice the number of complex Gamma factors. (In practice, all but finitely many Euler factors in $L(s)$ have degree d and $D = d$. Alternatively, using the axiomatic notation of (6.3), one expects $D = 2\sum \lambda_i$, and this sum has an intrinsic characterization through the asymptotic formula counting nontrivial zeros of $L(s)$ with imaginary part up to a given height [25].) Applying this to $L(E, s+1/2)$, the Dirichlet series for $L(E, s)$ should converge for $\mathrm{Re}(s) > (1/2 - 1/2\cdot 2) + 1/2 = 3/4$. It is known to converge for $\mathrm{Re}(s) > 5/6$ [19, pp. 15–18], so $L(E, 1) = \sum a_n/n$ (and similarly for higher derivatives of $L(E, s)$ at $s = 1$), but $\sum a_n/n$ only converges conditionally since $\#\{n \le x : a_n \ne 0\}$ grows like cx or $cx/\sqrt{\log x}$ for some $c > 0$, depending on whether or not E has complex multiplication [26].

By Corollary 5.4, the ordinary Dirichlet series for $\log L(s)$ does not converge at logarithmic singularities on the line $\mathrm{Re}(s) = 1/2$. (These singularities exist since $L(s)$ has nontrivial zeros.) At any other point on this line, the ordinary Dirichlet series for $\log L(s)$ converges if and only if $\psi_L(x) = o(\sqrt{x}\log x)$, by Theorem 6.2. Therefore convergence is plausible, but a proof is not possible at present.

On the other hand, the ordinary Dirichlet series for $L'(s)/L(s)$ does not converge anywhere on the critical line, by the next result. I am grateful to Stark for the statement and the proof.

Theorem 7.1 *Let $f(s) = \sum a_n n^{-s}$ for $\mathrm{Re}(s) \ge \sigma_0 > \sigma_1$. Assume f has a meromorphic continuation to $\mathrm{Re}(s) \ge \sigma_1$ and a simple pole somewhere along the line $\mathrm{Re}(s) = \sigma_1$. Then $\sum a_n n^{-s}$ converges at no point on this line.*

Proof Replacing s with $s+\sigma_1$, we can replace the line $\mathrm{Re}(s) = \sigma_1$ with the imaginary axis. Shifting vertically, it suffices to show $\sum a_n$ does not converge.

Let $A(x) = \sum_{n\le x} a_n$. Assume $A(x)$ converges as $x \to \infty$, say to c. Then $f(s)$ equals $\sum a_n n^{-s}$ for $\mathrm{Re}(s) > 0$ and $f(s) \to c$ as $s \to 0^+$ by Abel's theorem. Since f is meromorphic at $s = 0$ and bounded as $s \to 0^+$, it must be holomorphic at $s = 0$ with $f(0) = c$.

When $\mathrm{Re}(s) > 0$,

$$\int_1^\infty \frac{A(x) - c}{x^{s+1}}\,dx = \frac{f(s) - c}{s}.$$

Let the simple pole of f on the imaginary axis be at $i\gamma_0$. Necessarily $\gamma_0 \ne 0$. We consider $(f(s) - c)/s$ at $s = \sigma + i\gamma_0$ and then let $\sigma \to 0^+$.

Fixing $\varepsilon > 0$, $|A(x) - c| \le \varepsilon$ for $x \ge N > 1$. Then

$$\begin{aligned}\left|\frac{f(\sigma+i\gamma_0) - c}{\sigma + i\gamma_0}\right| &\le \int_1^\infty |A(x) - c|\frac{dx}{x^{\sigma+1}}\\ &\le \int_1^N |A(x) - c|\frac{dx}{x^{\sigma+1}} + \frac{\varepsilon}{\sigma}.\end{aligned}$$

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Multiply through by σ and let $\sigma \to 0^+$ to get

$$\left|\frac{\mathrm{Res}_{i\gamma_0} f}{\gamma_0}\right| \leq \varepsilon.$$

Now let $\varepsilon \to 0$ to get a contradiction of the simple pole at $i\gamma_0$. ■

Example 7.2 For a nontrivial Dirichlet character χ, the Dirichlet series for

$$L'(\chi, s)/L(\chi, s)$$

will converge on $\mathrm{Re}(s) > 1/2$ under the Riemann hypothesis for $L(\chi, s)$, but converges nowhere on the line $\mathrm{Re}(s) = 1/2$.

Example 7.3 While $\log \zeta(s) = \sum 1/kp^{ks}$ converges on $\mathrm{Re}(s) = 1$ with $s \neq 1$, there is no contradiction of Theorem 7.1 since this series has a logarithmic singularity at $s = 1$ rather than a (simple) pole.

8 Function Fields

We now consider partial Euler products over a function field with finite constant field. Proofs similar to the number field case will be omitted.

In Section 6, we found (5.7) seems to lie deeper than the Riemann hypothesis over number fields. In function fields, where the Riemann hypothesis is already a theorem, the analogue of (5.7) turns out to be provable in some examples.

Let K be a function field (in one variable) with constant field of size q. Normalized Euler products over K, by definition, have the form

$$L(z) = \prod_{v} \frac{1}{(1 - \alpha_{v,1} z^{\deg v}) \cdots (1 - \alpha_{v,d} z^{\deg v})}, \tag{8.1}$$

where v runs over all the places of K and $|\alpha_{v,j}| \leq 1$. The product is absolutely convergent for $|z| < 1/q$, and we are interested in partial Euler products on the circle $|z| = 1/\sqrt{q}$. In particular, we are interested in asymptotic relations of the form

$$\prod_{\deg v \leq n} \frac{1}{(1 - \alpha_{v,1} z^{\deg v}) \cdots (1 - \alpha_{v,d} z^{\deg v})} \sim \frac{C}{n^r}$$

as $n \to \infty$. The second moment $L_2(z)$ is defined just as in the number field case, and we say $L(z)$ satisfies the second moment hypothesis when $L_2(z)$ extends from $|z| < 1/q$ to a holomorphic nonvanishing function on $|z| = 1/q$, except for a zero or pole at perhaps one point z_0, with $|z_0| = 1/q$ and order of vanishing $R_0 \in \mathbf{Z}$. (We allow $R_0 < 0$.) We will retain this meaning for z_0, R_0 throughout this section.

Since Dirichlet series over function fields are power series in $z = q^{-s}$, their analytic treatment is simpler than in the number field case. We leave the function field analogues of Theorems 4.1 and 4.3 to the reader, and pass to the statement of the analogue of Theorem 4.9, which brings in $\sqrt{2}$.

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Theorem 8.1 *Let $L(s)$ be as in* (8.1), *and satisfy the second moment hypothesis. Pick z with $|z| = 1/\sqrt{q}$. As $n \to \infty$,*

$$\sum_{\deg v \le n} \sum_{k \ge 1} \frac{\alpha_{v,1}^k + \cdots + \alpha_{v,d}^k}{k} z^{k \deg v} = \sum_{k \deg v \le n} \frac{\alpha_{v,1}^k + \cdots + \alpha_{v,d}^k}{k} z^{k \deg v} + \begin{cases} o(1) & \text{if } z^2 \ne z_0, \\ -R_0 \log \sqrt{2} + o(1) & \text{if } z^2 = z_0. \end{cases}$$

Since the additive condition $2t = t_0$ in number fields becomes the multiplicative condition $z^2 = z_0$, there are two exceptional points on the circle $|z| = 1/\sqrt{q}$ (if $R_0 \ne 0$), rather than one point as over number fields.

Corollary 5.4 has a function field analogue, as follows.

Theorem 8.2 *Let $L(z)$ be as in* (8.1). *For some u with $|u| = 1/\sqrt{q}$, assume*

$$\sum_{k \deg v \le n} \frac{\alpha_{v,1}^k + \cdots + \alpha_{v,d}^k}{k} u^{k \deg v} = -r_u \log n + C_u' + o(1)$$

as $n \to \infty$, for some C_u' and r_u in $\mathbf{C}$. *Then $L(z)$ extends to a holomorphic nonvanishing function on $|z| < 1/\sqrt{q}$. If $L(z)$ is holomorphic at u, then its order of vanishing there is r_u and its leading term is $e^{C_u'} e^{r_u \gamma} (1 - z/u)^{r_u}$.*

We write the first term of $L(z)$ at u with a power of $1 - z/u$ instead of with a power of $z - u$, for ease of comparison with Corollary 5.4. Owing to this, the coefficient $e^{C_u'} e^{r_u \gamma}$ is not $L^{(r_u)}(u)/r_u!$, but $(-u)^{r_u} L^{(r_u)}(u)/r_u!$.

There is no function field analogue of the technique of proof of Theorem 5.11, since Lemma 3.2 is false in the function field case. That is, if $\gamma_{v,1}, \ldots, \gamma_{v,d}$ lie in the open unit disk and $\max_i |\gamma_{v,i}| \to 0$ as $\deg v \to \infty$, the condition

$$\prod_{\deg v \le n} \frac{1}{(1 - \gamma_{v,1}) \cdots (1 - \gamma_{v,d})} \sim \frac{C}{n^r} \tag{8.2}$$

as $n \to \infty$, for some $C \in \mathbf{C}^\times$ and $r \in \mathbf{C}$, need not imply

$$-\sum_{\deg v \le n} (\log(1 - \gamma_{v,1}) + \cdots + \log(1 - \gamma_{v,d})) = -r \log n + C' + o(1) \tag{8.3}$$

for some C' as $n \to \infty$. (The reverse implication, however, is trivially true.) This failure is not a surprise, since $\#\{v : \deg v = n\}$ is unbounded as n grows.

Example 8.3 We give a counterexample of (8.2) $\Rightarrow$ (8.3) with $d = 1$, so we write γ_v instead of $\gamma_{v,1}$. Let N_n be the number of degree n places on K. For all v with a common degree n, set $\gamma_v = 1 - e^{2\pi i/N_n}$. Then $|\gamma_v| \to 0$ as $\deg v \to \infty$, and

$$\prod_{\deg v \le n} \frac{1}{1 - \gamma_v} = 1.$$

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Thus (8.2) holds with $r = 0$. However, (8.3) fails since

$$-\sum_{\deg v \le n} \log(1 - \gamma_v) = -2\pi i n.$$

For readers concerned about the logic of carrying over our results from number fields to function fields, in light of the failure of Lemma 3.2, we note that this lemma was not used in any results which we are extending to function fields with analogous proofs. For the record, Lemma 3.2 is either explicitly or implicity used in the proofs of Theorem 3.3, Corollaries 4.11, 5.5, 5.6, and 5.7, and Theorems 5.11 and 6.3.

Since (8.2) and (8.3) are not generally equivalent, we give up trying to prove function field equivalences like those in Corollary 4.11 or Theorem 6.3. Instead we exploit the different structure of function field L-functions to prove better equivalences more easily.

From now on, we assume $L(z)$ as in (8.1) is entire and satisfies a functional equation

$$L(1/qz) = cz^{-D}\overline{L(\overline{z})}, \tag{8.4}$$

where $c \neq 0$ and $D \in \mathbf{Z}$. Then $D \ge 0$ and $L(z)$ is a polynomial of degree D:

$$L(z) = (1 - \lambda_1 z) \cdots (1 - \lambda_D z). \tag{8.5}$$

Note for any u that

$$\operatorname{ord}_{z=u} L(z) = \#\{i : \lambda_i u = 1\}. \tag{8.6}$$

Theorem 8.4 *When $L(z)$ is entire and satisfies* (8.4)*, the following are equivalent:*

(1) *For some u with $|u| = 1/\sqrt{q}$,*

$$\sum_{k \deg v \le n} \frac{\alpha_{v,1}^k + \cdots + \alpha_{v,d}^k}{k} u^{k \deg v} + m_u \log n$$

converges as $n \to \infty$, where $m_u = \operatorname{ord}_{z=u} L(z)$.

(2) *For all i, $|\lambda_i| = \sqrt{q}$.*

Proof The first condition implies $L(z) \neq 0$ for $|z| < 1/\sqrt{q}$ by Theorem 8.2. Therefore by the functional equation, $L(z)$ satisfies the Riemann hypothesis.

For the converse direction, suppose all $|\lambda_i|$ equal $\sqrt{q}$. Pick any u with $|u| = 1/\sqrt{q}$, so $|\lambda_i u| = 1$ for all i. Taking logarithms in (8.1) and (8.5), when $|z| < 1/q$, and comparing coefficients,

$$\sum_{k \deg v = n} \frac{\alpha_{v,1}^k + \cdots + \alpha_{v,d}^k}{k} = -\sum_{i=1}^{D} \frac{\lambda_i^n}{n}.$$

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Therefore

$$\begin{aligned}\sum_{k \deg v \le n} \frac{\alpha_{v,1}^k + \cdots + \alpha_{v,d}^k}{k} u^{k \deg v} &= \sum_{1 \le j \le n} \left(- \sum_{i=1}^{D} \frac{\lambda_i^j}{j} \right) u^j \\ &= - \sum_{i=1}^{D} \sum_{j=1}^{n} \frac{(\lambda_i u)^j}{j} \\ &= -\#\{i : \lambda_i u = 1\}(\log n) + \text{ const.} + o(1),\end{aligned}$$

since $\sum_j z^j/j$ converges when $|z| = 1$ except at $z = 1$. Using (8.6), we are done. ∎

Recall the notation z_0, R_0 when $L(z)$ satisfies the second moment hypothesis.

Corollary 8.5 *Let $L(z)$ be entire and satisfy* (8.4), *the Riemann hypothesis, and the second moment hypothesis. For u with $|u| = 1/\sqrt{q}$, let $m_u = \operatorname{ord}_{z=u} L(z)$ and let the leading term of $L(z)$ as a series in $1 - z/u$ be $B_u(1 - z/u)^{m_u}$. Then*

$$\prod_{\deg v \le n} \frac{1}{(1 - \alpha_{v,1} u^{\deg v}) \cdots (1 - \alpha_{v,d} u^{\deg v})} \sim \frac{C_u}{n^{m_u}}$$

as $n \to \infty$, with

$$C_u = \begin{cases} B_u / e^{m_u \gamma}, & \text{if } u^2 \neq z_0, \\ B_u / \sqrt{2}^{R_0} e^{m_u \gamma}, & \text{if } u^2 = z_0. \end{cases}$$

Proof By the Riemann hypothesis and Theorem 8.4,

$$\sum_{k \deg v \le n} \frac{\alpha_{v,1}^k + \cdots + \alpha_{v,d}^k}{k} u^{k \deg v} = -m_u \log n + C_u' + o(1)$$

for some C_u'. By Theorem 8.2, $B_u = e^{C_u'} e^{m_u \gamma}$. Now use Theorem 8.1 and exponentiate. ∎

Example 8.6 Let $E_{/\mathbf{F}_2(T)}$ be the elliptic curve defined by

$$y^2 + xy = x^3 + T^3.$$

This curve lies in a family recently studied by Ulmer [28]. By Tate's algorithm, the conductor is $\mathfrak{n} = (0) + 5(\infty)$, so $L(E, z)$ has degree $\deg \mathfrak{n} - 4 = 2$. A calculation shows $L(E, z) = 1 - 4z^2$, so there is a simple zero at $z = 1/2$ (corresponding to $s = 1$). The second moment hypothesis for $L(E, z/\sqrt{2})$ follows from holomorphy and nonvanishing of the symmetric square on $|z| = 1/2$, which is a special case of Deligne's work [9], and is also discussed in [18].

The hypotheses of Corollary 8.5 are satisfied for $L(z) = L(E, z/\sqrt{2})$ with $m_{1/\sqrt{2}} = 1$, so the partial Euler products for $L(E, z)$ at $z = 1/2$ tend to 0. For comparison

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with the hypothetical (1.1), which motivated this whole chain of ideas in the first place, we write the decay of the partial Euler products in terms of the growth of their reciprocals:

$$\prod_{\deg v \le n} \frac{\#E_{\mathrm{ns}}(\mathbf{F}_v)}{\mathrm{N}v} \sim \frac{\sqrt{2}^{R_0} e^{\gamma} n}{2} = \frac{e^{\gamma} n}{\sqrt{2}}, \tag{8.7}$$

where $\mathbf{F}_v$ is the residue field at v. This estimate, unlike (1.1) over $\mathbf{Q}$, is unconditionally true. We illustrate it in Table 3, where the right column has a limit of $e^{\gamma}/\sqrt{2} \approx 1.26$.

n	$(1/n) \prod_{\deg v \le n} \#E_{\mathrm{ns}}(\mathbf{F}_v)/\,\mathrm{N}v$
5	1.22
6	1.36
7	1.27
8	1.28
9	1.22
10	1.29

Table 3

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