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MULTILINEAR FUNCTIONS OF ROW STOCHASTIC MATRICES

STEPHEN PIERCE

1. Introduction. In the study of inequalities, the cases of equality are often the most difficult and interesting part. The case of equality is, in some sense, a measure of the tightness of the inequality. In this paper, we generalize two inequalities of Brualdi and Newman [**1**, Theorems 3, 4], but the instances of equality are probably more interesting because of the variety of cases which can occur.

Let $A = (a_{ij})$ be an $n \times n$ matrix. Define the permanent of A by

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{t=1}^{n} a_{t,\sigma(t)}.$$

We say that A is row stochastic if all entries are non-negative and all row sums are 1. In [**1**], several inequalities involving permanents of row stochastic matrices were proved. In two of these results, the case of equality was not determined. We will generalize both of these results to a class of functions which includes the permanent, and determine all cases of equality. All proofs are purely combinatorial. We assume familiarity with [**1**].

2. Results. Let $1 \leqq r \leqq n$. Define $Q_{r,n}$ to be the $\binom{n}{r}$ strictly increasing sequences $\alpha = (\alpha_1, \ldots, \alpha_r)$ of length r chosen from $\{1, \ldots, n\}$. If $\alpha, \beta \in Q_{r,n}$, let $A[\alpha|\beta]$ be the $r \times r$ submatrix whose (i, j) entry is a_{α_i, β_j}. If $\beta = \alpha$, we write $A[\alpha]$. Let H be a subgroup of S_n and set

$$d_H(A) = \sum_{\sigma \in H} \prod_{t=1}^{n} a_{t,\sigma(t)}.$$

If $\alpha \in Q_{r,n}$, let $H(\alpha)$ be the subgroup of H consisting of all $\sigma \in H$ which fix each integer not in α. Define

$$d^r(A[\alpha]) = \sum_{\sigma \in H(\alpha)} \prod_{t=1}^{r} a_{\alpha_t, \sigma(\alpha_t)}.$$

Let $r + s = n$. If $\alpha \in Q_{r,n}$, let α' be the sequence in $Q_{s,n}$ complimentary to α. For convenience, let Ω be the set of all permutation matrices.

THEOREM 1. *Let $1 \leqq r \leqq n - 1$. If A is row stochastic, then*

$$(1) \qquad \sum_{\alpha \in Q_{r,n}} d^r(A[\alpha])(1 - d^s(A[\alpha'])) \leqq \binom{n-1}{r}(1 - d_H(A)).$$

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Note that, if $H = S_n$, then d_H, d^r, and d^s are all permanents and (1) becomes [**1**, Theorem 3].

THEOREM 2. *Equality holds in* (1) *if and only if* $d_H(A) = \mathrm{per}(A)$ *and* A *is permutation isomorphic to a matrix* B *having one of the following forms:*

(a) *A is a permutation matrix;*

(b) $$B = \left[\begin{array}{c|c} x_1 & x_2 \ldots x_n \\ \hline 0 & P \end{array}\right],$$

where $P \in \Omega$ and P is the identity if $r \leqq n - 2$;

(c) $$B = \left[\begin{array}{cc|c} x & 1-x & 0 \\ y & 1-y & \\ \hline \multicolumn{2}{c|}{0} & P \end{array}\right],$$

where $P \in \Omega$ and P is the identity if $r \leqq n - 2$;

(d) $$B = \left[\begin{array}{ccc|c} 0 & x & 1-x & \\ 1-y & 0 & y & 0 \\ z & 1-z & 0 & \\ \hline \multicolumn{3}{c|}{0} & P \end{array}\right],$$

where $r = n - 1$ and $P \in \Omega$.

It is of interest to note that the cases of equality are independent of the group H. Also note that the four cases can overlap.

From Theorem 2, we easily obtain

THEOREM 3. *Let* $1 \leqq r \leqq n - 1$. *Let A be $n \times n$ row stochastic. Then*

$$\sum_{\alpha \in Q_{r,n}} d^r(A[\alpha]) \leqq \binom{n-1}{r} + \binom{n-1}{r-1} d_H(A). \tag{2}$$

Equality holds in (2) *if and only if* $d_H(A) = \mathrm{per}(A)$ *and A is permutation isomorphic to a matrix B of the form* (b) *in Theorem* 2, *or $r = n - 1$ and A is permutation isomorphic to a matrix B of the form*

$$B = \left[\begin{array}{ccc|c} 0 & x & 1-x & \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ \hline \multicolumn{3}{c|}{0} & P \end{array}\right],$$

where $P \in \Omega$.

Brualdi and Newman determined the case of equality in (2) when A is doubly stochastic. From Theorem 3, the only possibility is the identity matrix.

3. Proofs. For a fixed $\alpha \in Q_{r,n}$, consider the expression

$$d^r(A[\alpha])(1 - d^s(A[\alpha'])). \tag{3}$$

Replace the 1 in (3) by

$$\prod_{j \in \alpha'} (a_{j1} + \ldots + a_{jn}).$$

Then rewrite (3) as

$$\sum_{\rho} \sum_{\sigma} a_{\alpha_1 \rho(\alpha_1)} \ldots a_{\alpha_r \rho(\alpha_r)} a_{\alpha_1' \sigma(\alpha_1')} \ldots a_{\alpha_s' \sigma(\alpha_s')}, \tag{4}$$

where ρ runs over $H(\alpha)$ and σ runs over all functions of $\{\alpha_1', \ldots, \alpha_s'\}$ into $\{1, \ldots, n\}$ such that $\sigma \notin H(\alpha')$. Every term in (4) appears as a term in

$$\sum_{\tau} a_{1\tau(1)} \ldots a_{n\tau(n)} = \sum_{\tau} a_{\tau}, \tag{5}$$

where τ runs over all functions of $\{1, \ldots, n\}$ into itself except for those which are permutations in H. All terms in (4) are formally distinct, but a term in the sum (5) may occur in (4) for more than one $\alpha \in Q_{r,n}$.

Thus far, we have imitated the procedure in [**1**], and we also call on [**1**] to conclude that if $\tau \notin S_n$, then a_τ can appear in (4) for at most $\binom{n-1}{r}$ distinct $\alpha \in Q_{r,n}$. Suppose that $\tau \in S_n$, but $\tau \notin H$. We must show that a_τ occurs in (4) for at most $\binom{n-1}{r}$ $\alpha \in Q_{r,n}$.

If $\tau \notin H$, then, in particular, τ is not the identity. If a_τ occurs in (4) for some α, then the action of τ on α corresponds to a member of $H(\alpha)$. Since $\tau \notin H$, it cannot fix every member of α'. Thus we pick $i_1, j_1 \in \alpha'$ such that $\tau(i_1) = j_1$. If β is another member of $Q_{r,n}$ for which a_τ occurs in (4), then both i_1, j_1 are in β or both are in β'. The number of elements in $Q_{r,n}$ satisfying this condition for i_1, j_1 is

$$\binom{n-2}{r} + \binom{n-2}{r-2}.$$

If there is a β in $Q_{r,n}$ such that a_τ occurs in (4) and $i_1, j_1 \in \beta$, then we can pick $i_2, j_2 \in \beta'$ such that $\tau(i_2) = j_2$. For all $\alpha \in Q_{r,n}$ such that a_τ appears in (4), i_2, j_2 are both in α or both in α'. The number of α in $Q_{r,n}$ satisfying the above conditions on the pairs (i_1, j_1) and (i_2, j_2) is

$$\binom{n-4}{r} + 2\binom{n-4}{r-2} + \binom{n-4}{r-4}.$$

Continue this procedure until we obtain k pairs (i_1, j_1), $(i_2, j_2), \ldots, (i_k, j_k)$ such that if a_τ appears in (4) for some $\alpha \in Q_{r,n}$. Then each pair is in α or in α'. Moreover, not all k pairs are in α; otherwise, we could make an additional step. The total number of $\alpha \in Q_{r,n}$ satisfying these conditions on the k pairs is

$$\sum_{t=0}^{k-1} \binom{k}{t} \binom{n-2k}{r-2t}. \tag{6}$$

For small values of n, we easily verify that (6) is strictly less than $\binom{n-1}{r}$. Thus, if

$$\sum_{t=0}^{k-1} \binom{k}{t}\binom{n-2k}{r-2t} < \binom{n-1}{r},$$

$$\sum_{t=0}^{k-1} \binom{k}{t}\binom{n-2k}{r-1-2t} < \binom{n-1}{r-1},$$

addition yields the inequality for the case $n+1$. Thus, if $\tau \in S_n \backslash H$, a_τ occurs in (4) for fewer than $\binom{n-1}{r}$ $\alpha \in Q_{r,n}$. Since (5) is $1 - d_H(A)$, Theorem 1 is proved.

We now assume that equality holds in (1). We will apply a lemma of Marcus and Pierce [**2**, Theorem 2], which we state here in a form suitable to us.

LEMMA 1. *Let $A = (a_{ij})$ be $n \times n$. Let Γ_m be the set of integer sequences $\gamma = (\gamma_1, \ldots, \gamma_n)$, $1 \leqq \gamma_i \leqq n$, such that some integer occurs in γ with multiplicity at least m. Suppose that for all $\gamma \in \Gamma_m$, $a_{1\gamma_1} \ldots a_{n\gamma_n} = 0$. Then every column of A has fewer than m non-zero entries.*

LEMMA 2. *If equality holds in* (1), *then every column of A has at most* 2 *non-zero entries.*

Proof. From the proof of (1), we see that for each τ which is not in H, either a_τ occurs in (4) for exactly $\binom{n-1}{r}$ $\alpha \in Q_{r,n}$, or $a_\tau = 0$. But if $\tau \in \Gamma_3$, there cannot be $\binom{n-1}{r}$ distinct α in $Q_{r,n}$ for which a_τ appears in (4). Thus, $a_\tau = 0$ if $\tau \in \Gamma_3$ and we apply Lemma 1.

In addition, if two distinct integers both occur at least twice in τ, then $a_\tau = 0$. Thus, we cannot find rows i_1, i_2, i_3, i_4 and columns j_1, j_2 in A such that $a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_2} a_{i_4 j_2} \neq 0$. Let us say that a matrix satisfying this condition and that of Lemma 2 satisfies (*).

For convenience, we now eliminate one case of equality in (1).

LEMMA 3. *If equality holds in* (1) *and A has a zero column, then A falls into class* (b) *of Theorem* 2.

Proof. Our problem is invariant under permutation isomorphism if we use a suitable group conjugate to H, so assume that column 1 of A is zero. Then (1) becomes

$$\sum_{\substack{\alpha \in Q_{r,n} \\ 1 \notin \alpha}} d^r(A[\alpha]) \leqq \binom{n-1}{r}.$$

Thus, for equality in (1), we must have $d^r(A[\alpha]) = 1$ for all $\alpha \in Q_{r,n}$, $1 \notin \alpha$, and this is possible only if $A[2, \ldots, n] \in \Omega$ if $r = n-1$ and $A[2, \ldots, n]$ is the identity if $r \leqq n-2$. Lemma 3 is proved, and henceforth we will assume that A has no zero column.

LEMMA 4. *Let A be a row stochastic matrix satisfying* (*). *Then the number of rows in A which have a* 1 *is at least* $n - 3$.

Proof. We use induction on n. Verification for $n = 4, 5$ is easy, so assume that $n \geqq 6$. First, assume that there are 1's in A, so we may take $a_{nn} = 1$. If $a_{1n} = \ldots = a_{n-1,n} = 0$, induction applies to $A[1, \ldots, n-1]$ and the lemma is proved. Otherwise, let $a_{n-1,n} = x \neq 0$. Let B be the matrix obtained from A by adding x to $a_{n-1,n-1}$. By (*), $a_{1n} = \ldots = a_{n-2,n} = 0$, so $B[1, \ldots, n-1]$ is row stochastic. It is easily verified that $B[1, \ldots, n-1]$ satisfies (*), so induction applies. If x were 1, the lemma would be proved. If $0 < x < 1$, we may conclude that A has at most four rows which have no 1 and that every column of A which has a 1 also has exactly one other non-zero entry which is less than 1. Thus, if there are exactly $n - 4$ 1's in A, A is permutation equivalent to a matrix B of the form

$$B = \left[\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & I_{n-4} \end{array}\right],$$

where B_1 is 4×4 and every column of B_2 has a non-zero entry less than 1. But $n \geqq 6$, so in order to avoid violation of (*), we must have all non-zero entries of B_2 in the same row, say the first. Then rows 2, 3, 4 of B_1 each have at least 2 non-zero entries. Thus, $B_1\,[2, 3, 4|1, 2, 3, 4]$ has a column with two non-zero entries and B cannot satisfy (*).

Finally, assume that A has no 1's. Let the non-zero elements in column n be among a_{nn} and $a_{n-1,n}$. Again replace $a_{n-1,n-1}$ with $a_{n-1,n-1} + a_{n-1,n}$. Then $A[1, \ldots, n-1]$ has at least $n - 5$ 1's, so $n \leqq 5$. This proves Lemma 4.

LEMMA 5. *If equality holds in* (1), *then for all* $\alpha \in Q_{r,n}$,

$$d_H(A) = \operatorname{per}(A); \tag{7}$$

$$d^r(A[\alpha]) = \operatorname{per}(A[\alpha]); \tag{8}$$

$$d^r(A[\alpha])(1 - d^s(A[\alpha'])) = \operatorname{per}(A[\alpha])(1 - \operatorname{per}(A[\alpha'])). \tag{9}$$

Proof. Let $\tau \in S_n \backslash H$. If $a_\tau \neq 0$, there are $\binom{n-1}{r}$ α's in $Q_{r,n}$ for which a_τ appears in (4). But (6) is strictly less than $\binom{n-1}{r}$ so $a_\tau = 0$. This proves (7). Now suppose that $A[\alpha]$ and $A[\alpha']$ both contain diagonals with no zeros. The union of these two diagonals, by (7), corresponds to a member of H. Using the Laplace expansion theorem of Marcus and Soules [**3**], expand $d_H(A)$ by rows $\alpha_1, \ldots, \alpha_r$. One of the terms in the expansion will be $d^r(A[\alpha])(d^s(A[\alpha']))$ and the product of the two diagonals mentioned must appear in this term. Since $d_H(A) = \operatorname{per}(A)$, $d^r(A[\alpha]) = \operatorname{per}(A[\alpha])$ and $d^s(A[\alpha']) = \operatorname{per}(A[\alpha'])$, so (8) and (9) are satisfied. If all diagonals of $A[\alpha]$ contain a zero, then (8) and (9) are satisfied. Finally, suppose that $A[\alpha]$ has a diagonal with no zeros, but $\operatorname{per}(A[\alpha']) = 0$. If $d^r(A[\alpha]) < \operatorname{per}(A[\alpha])$, we would have

$$\begin{aligned} d^r(A[\alpha])(1 - d^s(A[\alpha'])) &= d^r(A[\alpha]) \\ &< \operatorname{per}(A[\alpha]), \end{aligned}$$

and thus (1) would be strict inequality. Lemma 5 is proved.

We will use Lemma 5 repeatedly for specific computations. We now have A permutation equivalent to a matrix B of the form

$$B = \left[\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & C \end{array}\right],$$

where B_1 is of degree 3 or less, no 1's occur in B_1 or B_2, and C is a $(0, 1)$ matrix.

LEMMA 6. *If B_1 is 2×2 or 3×3, then $C \in \Omega$.*

Proof. We do the 2×2 case; the 3×3 case is the same. Since B satisfies (*), if $C \notin \Omega$, assume that C has the form

$$C = \begin{bmatrix} 1 & & & & 0 \\ 1 & & & 0 & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \\ 0 & & & 1 & 0 \end{bmatrix}.$$

Then B_2 has all zeros, except for exactly one non-zero element in column n. (Recall that A has no zero columns.) Then B_1 has at least 3 non-zero entries, and hence one column of B_1 has 2 non-zero entries. This violates (*).

LEMMA 7. *If B_1 is 2×2, then B_2 has at least one zero row. If B_1 is 3×3, then B_2 is zero.*

Proof. This is an easy consequence of (*).

LEMMA 8. *If B_1 is 2×2 or 3×3, then B_1 is a principal submatrix of A.*

Proof. As in Lemma 6, we give the proof for the 2×2 case; the 3×3 case is similar, but more tedious. Suppose that B_1 does not intersect the main diagonal of A. Assume, then, that $B_1 = A[1, 2|3, 4]$, with $a_{11} = a_{12} = 0$, $a_{13} = x$, $a_{14} = 1 - x$, by Lemma 7. By Lemmas 5 and 6, the right side of (1) is

$$\binom{n-1}{r}(1 - \operatorname{per}(B_1)).$$

Consider the possibilities for $\{1, 2, 3, 4\} \cap \alpha$, as α runs through $Q_{r,n}$. If all of $1, 2, 3, 4$ are in α, and $d^r(A[\alpha]) \neq 0$, then by Lemma 5, $d^r(A[\alpha]) = \operatorname{per}(A[\alpha]) = \operatorname{per}(B_1)$. Thus, by Lemma 6, $A[\alpha'] \in \Omega$, so we get a zero term on the left of (1). Thus, the left side of (1) is at most

$$(11)\quad \binom{n-4}{r}(1-d_H(A))+\binom{n-4}{r-1}a_{22}$$
$$+\binom{n-4}{r-2}\,(x(1-a_{24})+a_{24}(1-x)+(1-x)(a_{23})+a_{23}(1-x))$$
$$+\binom{n-4}{r-3}\left(x\left(\max_{j\neq 3,4} a_{2j}\right)+(1-x)\left(\max_{j\neq 3,4} a_{2j}\right)\right).$$

The coefficient of $\binom{n-4}{r-k}$ comes from considering those $\alpha \in Q_{r,n}$ such that the cardinality of $\{1, 2, 3, 4\} \cap \alpha$ is k. Now (11) is strictly less than

$$\left(\binom{n-4}{r}+\binom{n-4}{r-1}+2\binom{n-4}{r-2}+\binom{n-4}{r-3}\right)(1-d_H(A))$$
$$\leqq \binom{n-1}{r}(1-d_H(A)).$$

The strictness follows because $1 \leqq r \leqq n-1$,

so $$\binom{n-4}{r-3} > 0,$$

and because $0 < x < 1$, so

$$\max_{j\neq 3,4} a_{2j} \leqq 1-a_{23}-a_{24} < 1-a_{23}(1-x)-a_{24}x = 1-d_H(A).$$

Now suppose that B_1 intersects the main diagonal of A at exactly one point. It suffices to assume that $B_1 = A[1, 2|2, 3]$ and that $a_{12} = x$, $a_{13} = 1-x$. The left side of (1) is at most

$$\binom{n-3}{r}(1-d_H(A))+\binom{n-3}{r-1}(a_{22})+\binom{n-3}{r-2}\left(\max_{j\neq 2,3} a_{2j}x+(1-x)(1-a_{22})\right)$$
$$<\left(\binom{n-3}{r}+\binom{n-3}{r-1}+\binom{n-3}{r-2}\right)(1-d_H(A)) \leqq \binom{n-1}{r}(1-d_H(A)).$$

We remark that from Lemma 7, if B_1 is 3×3, B_1 is permutation equivalent to a matrix C_1 of the form

$$C_1 = \begin{bmatrix} x & 1-x & 0 \\ 0 & y & 1-y \\ 1-z & 0 & z \end{bmatrix},$$

$0 < x, y, z < 1$. When attempting to prove that B_1 is principal in A, we must consider how many zeros there are in the intersection of B_1 with the main diagonal of A.

We now finish Theorem 2. If B_1 is 3×3, A is the direct sum of B_1 and an $(n-3) \times (n-3)$ matrix in Ω. As stated in the previous remark, there are three possibilities for B_1:

$$B_1 = \begin{bmatrix} x & 1-x & 0 \\ 0 & y & 1-y \\ 1-z & 0 & z \end{bmatrix}, \quad \begin{bmatrix} x & 0 & 1-x \\ 0 & 1-y & y \\ 1-z & z & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & x & 1-x \\ 1-y & 0 & y \\ z & 1-z & 0 \end{bmatrix},$$

$0 < x, y, z < 1$. In all cases, $1 - d_H(A) = x + y + z - xy - xz - yz$. In the first case, the left of (1) is at most

$$\binom{n-3}{r}(1 - d_H(A)) + \binom{n-3}{r-1}(x(1-yz) + y(1-xz) + z(1-xy))$$
$$+ \binom{n-3}{r-2}(xy(1-z) + xz(1-y) + yz(1-x))$$
$$< \left(\binom{n-3}{r} + 2\binom{n-3}{r-1} + \binom{n-3}{r-2}\right)(1 - d_H(A)) = \binom{n-1}{r}(1 - d_H(A)).$$

In the second case, we similarly verify strict inequality in (1). In the third case, the left side of (1) is at most

$$(12)\quad \binom{n-3}{r}(1 - d_H(A)) + \binom{n-3}{r-1}\cdot 0$$
$$+ \binom{n-3}{r-2}(x(1-y) + y(1-z) + z(1-x))$$
$$= \left(\binom{n-3}{r} + \binom{n-3}{r-2}\right)(1 - d_H(A)) \leqq \binom{n-1}{r}(1 - d_H(A)).$$

Equality holds in (12) if and only if

$$\binom{n-3}{r-1} = 0,$$

i.e., $r = n - 1$. By Lemma 6, $A[4, \ldots, n] \in \Omega$, so A satisfies (d) in Theorem 2.

If B_1 is 2×2, let $B_1 = A[1, 2]$ and set

$$B_1 = \begin{pmatrix} x & 1-x \\ y & z \end{pmatrix}.$$

Thus $d_H(A) = xz + y(1-x)$, so the left side of (1) is at most

$$(13)\quad \binom{n-2}{r}(1 - d_H(A)) + \binom{n-2}{r-1}(x(1-z) + z(1-x))$$
$$\leqq \left(\binom{n-2}{r} + \binom{n-2}{r-1}\right)(1 - d_H(A))$$
$$= \binom{n-1}{r}(1 - d_H(A)).$$

For equality in (13), we need $y + z = 1$, and $d^r(A[\alpha]) = z$ for all $\alpha \in Q_{r,n}$ such that $1 \notin \alpha$. Thus $A[3, \ldots, n]$ is the identity unless $r = n - 1$, and hence A is in class (c) of Theorem 2.

If B_1 is 1×1 with entry x, $0 < x < 1$, x may not be on the main diagonal of A. We may assume that $x = a_{11}$ or a_{12}. Suppose that $x = a_{11}$ and that $\operatorname{per}(A) \neq 0$. Then $\operatorname{per}(A) = x$ and $A[2, \ldots., n] \in \Omega$. It is now easy to verify, using Lemma 5, that A is in class (b) of Theorem 2. If $\operatorname{per}(A) = 0$, then $A[2, \ldots., n] \notin \Omega$, so assume that the last column of $A[2, \ldots n]$ is zero. The left side of (1) is at most

$$\binom{n-2}{r} + \binom{n-2}{r-1}x < \binom{n-1}{r} = \binom{n-1}{r}(1 - d_H(A)).$$

Now assume that $x = a_{12}$. Then A has the form

$$A = \left[\begin{array}{cc|c} a_{11} & x & a_{13} \ldots\ldots a_{1n} \\ a_{21} & 0 & a_{23} \ldots\ldots a_{2n} \\ \hline & 0 & \\ & \vdots & C \\ & 0 & \end{array}\right],$$

where C is a $(0, 1)$ matrix.

Suppose that $\operatorname{per}(A) = x$. Then the left side of (1) is at most

$$\binom{n-2}{r}(1 - x) + \binom{n-2}{r-1}\left(\max_{j\neq 2} a_{1j}\right) \leqq \binom{n-1}{r}(1 - d_H(A)).$$

For equality to hold it is necessary that $\max_{j\neq 2} a_{1j} = 1 - x$. If $a_{11} = 1 - x$, we must have $C \in \Omega$. Thus, $a_{21} = 1$ and A is in class (c), so we may assume that $a_{13} = 1 - x$. Then

$$A = \left[\begin{array}{ccc|c} 0 & x & 1-x & 0\ldots\ldots 0 \\ a_{21} & 0 & a_{23} & \\ a_{31} & 0 & a_{23} & \\ \hline & 0 & & C \end{array}\right], \tag{14}$$

where C is $(0, 1)$. Look at the possibilities for $\{1, 2, 3\} \cap \alpha$. The left side of (1) is at most

$$\binom{n-3}{r}(1 - d_H(A)) + \binom{n-3}{r-1}(a_{33}(1 - x)) + \binom{n-3}{r-2}(a_{21}x(1 - a_{33}) + a_{31}(1 - x)).$$

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If $a_{21} = 1$ and $a_{33} = 0$, $\operatorname{per}(A) = 0$, which is a contradiction. So the left side of (1) is at most

$$\left(\binom{n-3}{r} + \binom{n-3}{r-1} + \binom{n-3}{r-2}\right)(1-x) \leqq \binom{n-1}{r}(1 - d_H(A)).$$

Thus, for equality in (1), we need $r = n - 1$, $a_{31} = 1$, $C \in \Omega$. Since $\operatorname{per}(A) = x$, $a_{23} = 1$ and A satisfies (d). Now let $\operatorname{per}(A) = 0$. Let $x = a_{12}$. If $A[3, \ldots n] \in \Omega$, $a_{21} = 0$. Then the left of (1) is at most

$$\binom{n-2}{r} + \binom{n-2}{r-1}(\max a_{1j}) < \binom{n-1}{r}.$$

So $A[3, \ldots., n] \notin \Omega$. Then the left of (1) is at most

$$\binom{n-2}{r} + \binom{n-2}{r-1}\left(x + \max_{j \neq 2} a_{1j}\right) \leqq \binom{n-1}{r}.$$

Since $A[3, \ldots. n] \notin \Omega$, equality cannot hold unless

$$\binom{n-2}{r} = 0,$$

i.e., $r = n - 1$, and $\max_{j \neq 2} a_{1j} = 1 - x$. If $a_{11} = 1 - x$, we force $A[3, \ldots. n] \in \Omega$, which is a contradiction. So let $a_{13} = 1 - x$. Then A has the form (14). We can verify that $C \in \Omega$ and $a_{21} = a_{31} = 1$. Thus A satisfies (d) of Theorem 2. Finally, if A is a (0, 1) matrix and has no zero columns, $A \in \Omega$, and using Lemma 5, we complete the proof of Theorem 2.

As in the Brualdi and Newman paper [**1**], Theorem 3 follows from (1) and the obvious fact that

$$d^r(A[\alpha])d^s(A[\alpha']) \leqq d_H(A). \tag{15}$$

Thus, equality holds in (2) if and only if equality holds in (1) and (15). By Lemma 5, if equality holds in (1), it holds when $d_H = \operatorname{per}$, so it suffices to do the problem for the pernament. Brualdi and Newman proved [**1**, Lemma 1] that if $\operatorname{per}(A) \neq 0$, and equality holds in (15), then A is permutation isomorphic to a lower triangular matrix. Thus, if $\operatorname{per}(A) \neq 0$, A is of the form (b) in Theorem 2.

If $\operatorname{per}(A) = 0$, and A has a zero column, A is in (b), by Lemma 3. The only other possibility for $\operatorname{per}(A) = 0$ and equality in (1) is $r = n - 1$ and

$$A = \left[\begin{array}{ccc|c} 0 & x & 1-x & \\ 1 & 0 & 0 & \\ 1 & 0 & 0 & 0 \\ \hline & & & \\ & & 0 & B \end{array}\right],$$

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$B \in \Omega$. Then

$$\begin{aligned}\sum_{\alpha \in Q_{r,n}} \operatorname{per}(A[\alpha]) &= x\binom{n-3}{r-2} + (1-x)\binom{n-3}{r-2} \\ &= 1 \\ &= \binom{n-1}{r} + \binom{n-1}{r-1} \operatorname{per}(A),\end{aligned}$$

and Theorem 3 is proved.

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University of Toronto,
Toronto, Ontario