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A NON-REFLEXIVE BANACH SPACE HAS NON-SMOOTH THIRD CONJUGATE SPACE

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J. Dixmier has observed [3, p. 1070] that a non-reflexive Banach space has non-rotund fourth conjugate space. It is the aim of this paper to improve Dixmier's result by showing that a non-reflexive Banach space already has non-smooth third conjugate space in that the images under natural embedding of the continuous linear functionals which do not attain their norm on the unit sphere are non-smooth points of the third conjugate space.

For a normed linear space X over the real or complex numbers, we will use the asterisk notation for conjugate spaces, X^*, etc. and the circumflex notation for natural embedding $\hat{x}$, $\hat{X}$, etc. We will denote the unit sphere $\{x \in X: \|x\|=1\}$ by $S(X)$. For the space X we will consider the set-valued mapping $x \to D(x)$ of $S(X)$ into $S(X^*)$ where for each $x \in S(X)$, $D(x) \equiv \{f \in S(X^*): f(x)=1\}$, (by the Hahn-Banach Theorem $D(x)$ is non-empty for each $x \in S(X)$), and we will consider a *support mapping* $x \to f_x$ of X into X^* where for each $x \in S(X)$ and real $\lambda \geq 0$, $f_x \in D(x)$ and $f_{\lambda x} = \lambda f_x$. We say that X is *smooth* at $x \in S(X)$ if $D(x)$ contains only one point and that X is *rotund* at $x \in S(X)$ if $D(x) \cap D(y) = \phi$ for all $y \in S(X) \backslash \{x\}$. It is well known that the space X is smooth (rotund) on $S(X)$ if X^* is rotund (smooth) on $S(X^*)$ and that smoothness and rotundity are dual properties if X is reflexive. Our proofs depend essentially on the *subreflexivity* property established by Errett Bishop and R. R. Phelps [1] that, for a Banach space X, $D(X)$ is norm-dense in $S(X^*)$.

We will need the following characterization of smoothness.

LEMMA 1. *For a normed linear space X when X has the norm topology and X^* has the $\sigma(X^*, X)$-topology,*

(*i*) *if X is smooth at $x \in S(X)$ then every support mapping of X into X^* is continuous on $S(X)$ at x,*

(*ii*) *if there exists a support mapping of X into X^* which is continuous on $S(X)$ at x then X is smooth at x,* [4, *p.* 107 *and p.* 109].

Our Theorem is a consequence of the following simple lemma.

LEMMA 2. *Let X be a Banach space and $f \in S(X^*)$. If there exist nets $\{x_n\}$ and $\{f_n\}$ where $x_n \in S(X)$ and $f_n \in D(x_n)$ for all n such that $\{f_n\}$ is norm-convergent to f and $\{\hat{x}_n\}$ is $\sigma(X^{**}, X^{***})$-convergent, then $f \in D(X)$.*

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Proof. Since $\hat{X}$ is a norm-closed linear subspace it is $\sigma(X^{**}, X^{***})$-closed so $\{\hat{x}_n\}$ is $\sigma(X^{**}, X^{***})$-convergent to some $\hat{x} \in \hat{X}$. But then

$$|1-f(x)| \leq \|f_n-f\| \, \|x_n\|+|f(x_n)-f(x)|,$$

and it follows that $f \in D(x)$.

It is known that a Banach space is reflexive if, when X^* and X^{**} have the norm topologies, there exists a support mapping of X^* into X^{**} which is continuous on $S(X^*)$, [4, p. 112]. The following corollary improves this result.

COROLLARY. *A Banach space is reflexive and rotund on $S(X)$ if and only if, when X^* has the norm-topology and X^{**} the $\sigma(X^{**}, X^{***})$-topology, there exists a support mapping of X^* into X^{**} which is continuous on $S(X^*)$.*

Proof. If X is reflexive and rotund on $S(X)$ then X^* is smooth on $S(X^*)$ and the continuity property of the support mapping follows from Lemma 1(i). Conversely, from the continuity property of the support mapping we deduce from Lemma 1(ii) that X^* is smooth on $S(X^*)$ which implies that X is rotund on $S(X)$, and we deduce from the subreflexivity of X and from Lemma 2 that $D(X)= S(X^*)$. So then $D(X^*)=S(\hat{X})$ and since X^* is subreflexive, $S(\hat{X})$ is norm-dense in $S(X^{**})$. However, since $S(\hat{X})$ is norm-closed $X=X^{**}$; i.e. X is reflexive.

A counter-example exists to show that a Banach space X which has a support mapping of X into X^* continuous on $S(X)$ when X has the norm-topology and X^* has the $\sigma(X^*, X^{**})$-topology, is not necessarily reflexive, [5].

It should be noted that Lemma 2 provides a limit to possible extensions of the subreflexivity property of a Banach space such as that recently given by Béla Bollobás [2]. Let X be a Banach space and $f \in S(X^*)\backslash D(X)$; Lemma 2 implies that for every net $\{f_n\}$ where $f_n \in D(X)$ for all n such that $\{f_n\}$ is norm-convergent to f, the net $\{\hat{x}_n\}$ where $\hat{x}_n \in D(f_n)$ for all n cannot be $\sigma(X^{**}, X^{***})$-convergent.

By the R. C. James' Theorem [6] that a Banach space X is reflexive if and only if $D(X)=S(X^*)$, we have that, for a non-reflexive Banach space X, $S(X^*)\backslash D(X)$ is non-empty.

THEOREM. *For a non-reflexive Banach space X, X^{***} is not smooth at $\hat{f} \in S(X^{***})$ for every $f \in S(X^*)\backslash D(X)$.*

Proof. Given any $f \in S(X^*)\backslash D(X)$, it follows from the subreflexivity of X and Lemma 2 that there exist nets $\{x_n\}$ and $\{f_n\}$ where $x_n \in S(X)$ and $f_n \in D(x_n)$ for all n such that $\{f_n\}$ is norm-convergent to f and $\{\hat{x}_n\}$ is not $\sigma(X^{**}, X^{***})$-convergent. By the Hahn-Banach Theorem there exists an $F_f \in D(f)$ but $\{\hat{x}_n\}$ is not $\sigma(X^{**}, X^{***})$-convergent to F_f. We have then that $\{\hat{f}_n\}$ is norm-convergent to $\hat{f}$ but $\{\hat{\hat{x}}_n\}$ is not $\sigma(X^{**\wedge}, X^{***\wedge})$-convergent to $\hat{F}_f$; i.e. $\{\hat{\hat{x}}_n\}$ is not $\sigma(X^{****}, X^{***})$-convergent to $\hat{F}_f$. But then by Lemma 1(i), X^{***} is not smooth at $\hat{f}$.

Dixmier has shown that for a non-reflexive Banach space X, X^{****} is not rotund at $\hat{F} \in S(X^{****})$ for every $F \in S(X^{**})\backslash S(\hat{X})$, [7, p. 105]. Consequently we can deduce the following corollary from our Theorem.

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COROLLARY. *For a non-reflexive Banach space, X, X^{****} is both not smooth and not rotund at $\hat{F} \in S(X^{****})$ for every $F \in S(X^{**})\backslash D(X^{*})$.*

Added in proof

From correspondence I gather that the result was first proved by R. R. Phelps in 1960, though it has never till now been published. The proof given by Phelps uses the fact that in a non-reflexive Banach space the natural embedding of the second into the fourth conjugate space, differs at points $D(X^{*})\backslash S(\hat{X})$ from the second adjoint of the natural embedding of the space into its second conjugate space; (see [8, p 70]).

I am indebted to Dr. Phelps for the following example of a non-reflexive Banach space with rotund second conjugate space. Consider the sequence space l^{∞} renormed by $\|x\| = \sup\{|\lambda_n|\} + (\sum |\lambda_n|^2/2^n)^{1/2}$, where $x = \{\lambda_n\}$. It is known [9, p. 981] that $(l^{\infty}, \|.\|)$ is rotund, and that $\|.\|$ is equivalent to the supremum norm and is lower semi-continuous in the $\sigma(l^{\infty}, l^1)$-topology, hence $(l^{\infty}, \|.\|)$ is conjugate to the space l^1 with norm $\|y\| = \sup\{\sum \lambda_n\mu_n : \|x\| \leq 1\}$, where $y = \{\mu_n\}$. Now this norm is equivalent to the usual norm on l^1. Furthermore, $(l^1, \|.\|)$ is conjugate to the subspace $(c_0, \|.\|)$ of $(l^{\infty}, \|.\|)$. We have then a non-reflexive Banach space $(c_0, \|.\|)$ with rotund second conjugate space $(l^{\infty}, \|.\|)$.

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