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Abstract

Let F be a p-adic field. Consider a dual pair $(\mathrm{SO}(2n+1)_+, \widetilde{\mathrm{Sp}}(2n))$, where $\mathrm{SO}(2n+1)_+$ is the split orthogonal group and $\widetilde{\mathrm{Sp}}(2n)$ is the metaplectic cover of the symplectic group $\mathrm{Sp}(2n)$ over F. We study lifting of characters between orthogonal and metaplectic groups. We say that a representation of $\mathrm{SO}(2n+1)_+$ lifts to a representation of $\widetilde{\mathrm{Sp}}(2n)$ if their characters on corresponding conjugacy classes are equal up to a transfer factor. We study properties of this transfer factor, which is essentially the character of the difference of the two halves of the oscillator representation. We show that the lifting commutes with parabolic induction. These results were motivated by the paper 'Lifting of characters on orthogonal and metaplectic groups' by Adams who considered the case $F=\mathbb{R}$.

1. Introduction

Investigating lifting of representations is a very important part of the theory of representations and automorphic forms. For example, the local Langlands conjecture states that representations of a *linear* algebraic group G are parameterized by data related to the 'dual' group ${}^LG$. Assuming this, a map between dual groups $\phi\colon {}^LH \to {}^LG$ should be related to a 'lifting' of representations between G and H. Conversely, a natural relationship between representations of H and G might be explained in terms of such a homomorphism.

Another situation in which there is a natural relationship of representations is the Howe theta correspondence. This correspondence matches representations of G and G', for any dual pair of subgroups (G, G') of the metaplectic cover $\widetilde{\mathrm{Sp}}(2N)$ of the symplectic group $\mathrm{Sp}(2N)$. It is important that in this case the groups G and G' need not be linear.

A particularly interesting example of a dual pair is $(\mathrm{SO}(2n+1)_+, \widetilde{\mathrm{Sp}}(2n)) \subset \widetilde{\mathrm{Sp}}(2N)$, where $N = 2n(2n+1)$ and $\mathrm{SO}(2n+1)_+$ denotes the split orthogonal group. The properties of such pairs in the real case were investigated by Adams and Barbasch in [AB98]. Their main result is that there is a natural bijection between genuine irreducible representations of the metaplectic group $\widetilde{\mathrm{Sp}}(2n)$ and the irreducible representations of the groups $\mathrm{SO}(p,q)$, where $p+q=2n+1$ and $(-1)^q$ is fixed. (A representation of $\widetilde{\mathrm{Sp}}(2n)$ is called genuine if it does not factor to a representation of $\mathrm{Sp}(2n)$.)

A natural question is if the dual pair correspondence can be interpreted on the level of characters. In the real case this problem was solved by Adams [Ada98]. He defined a lifting

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of stable characters between orthogonal and metaplectic groups. A virtual character of a linear group $\mathbf{G}(F)$ is stable, roughly speaking, if it is invariant by conjugation by $\mathbf{G}(\overline{F})$, where $\overline{F}$ denotes the algebraic closure of F. Stable characters arise naturally in the study of characters for linear groups; see § 8 for a discussion of stability. For tempered representations the lifting of stable characters agrees with the stabilized dual pair correspondence.

The purpose of this paper is to study lifting of characters in the case of p-adic fields. Recall that two strongly regular and semisimple elements $g \in \mathrm{Sp}(2n)$ and $g' \in \mathrm{SO}(2n+1)_+$ stably correspond if they have the same non-trivial eigenvalues. The stable correspondence is a bijection between strongly regular semisimple stable conjugacy classes in $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)_+$ (see Proposition 2.3). Therefore, we can define the lifting of representations directly on the character level. We say that a character $\Theta_{\rho'}$ of $\mathrm{SO}(2n+1)_+$ lifts to a character Θ_ρ of $\widetilde{\mathrm{Sp}}(2n)$ if

$$\Theta_\rho(\widetilde{g}) = \Phi(\widetilde{g})\Theta_{\rho'}(g'), \tag{1}$$

for any pair of elements $\widetilde{g}$ and g' such that $p(\widetilde{g})$ and g' stably correspond. Here $p: \widetilde{\mathrm{Sp}}(2n) \longrightarrow \mathrm{Sp}(2n)$ is the projection and Φ is the 'transfer factor' which is necessary for technical reasons.

We would like to show that if $\Theta_{\rho'}$ is a stable virtual character of $\mathrm{SO}(2n+1)_+$ then Θ_ρ defined by (1) is a genuine stable virtual character of $\widetilde{\mathrm{Sp}}(2n)$ and vice versa. The case when $n=1$ was studied by Schultz [Sch98]. He established a bijection between stable virtual characters of $\widetilde{\mathrm{SL}}(2)$ and irreducible representations of $\mathrm{SO}(3)_+$ via the character theory. In the real case, Adams used knowledge of discrete series characters to prove this first for discrete series. He then showed that the formula 'commutes' with parabolic induction. In the p-adic case much less is known about characters of discrete series (in particular supercuspidal) representations.

We study properties of the transfer factor Φ (see § 4). Just as in the real case, the transfer factor is essentially the character of the *difference* of the two halves of the oscillator representation of $\widetilde{\mathrm{Sp}}(2n)$. In particular, the absolute value of the transfer factor is the quotient of Weyl denominators for $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(2n)$. We use it to define stability in the metaplectic group (§ 8). We show that in the metaplectic case a parabolically induced stable representation remains stable and that the characters that satisfy (1) are necessarily stable. We conclude the paper with showing that the lifting of characters commutes with parabolic induction. A similar statement in the case when $\mathbf{G}$ is a reductive linear algebraic group and $\mathbf{H}$ is its endoscopy group was proven by Shelstad [She82]. We use different methods than Shelstad, i.e. we use the formula for the character of an induced representation given by van Dijk [Van72]. This approach leads to further study of the Weyl groups and stable Weyl groups of Cartan subgroups in symplectic and orthogonal groups (see §§ 6 and 7 for details).

2. Stability in symplectic and orthogonal groups

We review some well-known facts concerning stable conjugacy. Let F be a p-adic field. Let G denote the group $\mathrm{Sp}(2n)$ or $\mathrm{SO}(2n+1)_+$ over F. Let $\overline{F}$ be the algebraic closure of F and let $\mathbf{G}$ be an algebraic group such that $\mathbf{G}(F) = G$. We will identify $\mathbf{G}$ with $\mathbf{G}(\overline{F})$. Recall that a semisimple element $g \in G$ is strongly regular if its centralizer is a Cartan subgroup. Let $g, h \in G$ be strongly regular semisimple elements. We say that g and h are stably conjugate if they are conjugate in $\mathbf{G}(\overline{F})$. We will write $g \sim_{\mathrm{st}} h$. Equivalently, g and h are stably conjugate if they have the same eigenvalues. Now consider two Cartan subgroups $\mathbf{T}$ and $\mathbf{T}'$ in $\mathbf{G}$ that are defined over F. We say that $\mathbf{T}(F)$ and $\mathbf{T}'(F)$ are stably conjugate if there exists $g \in \mathbf{G}$ such that $g\mathbf{T}(F)g^{-1} = \mathbf{T}'(F)$.

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We introduce the following notation:

$$\mathcal{C}_{\mathrm{st}}(G) = \{T \mid T \text{ is a Cartan subgroup in } G\}/\text{stable conjugacy}.$$

Let $\mathbf{T}$ be a Cartan subgroup defined over F and let W denote the Weyl group of $\mathbf{T}$ in $\mathbf{G}$. Let Γ be the Galois group of $\overline{F}/F$. The following lemma is well known.

LEMMA 2.1. *The sets $\mathcal{C}_{\mathrm{st}}(G)$ and $H^1(\Gamma, W)$ are in bijection.*

Proof. The proof of an existence of an injective map $\Phi : \mathcal{C}_{\mathrm{st}}(G) \longrightarrow H^1(\Gamma, W)$ is standard. The surjectivity of Φ follows from [Rag04, Theorem 1.1]. □

DEFINITION 2.2. Let $g \in \mathrm{Sp}(2n)$, $g' \in \mathrm{SO}(2n+1)_+$ be strongly regular semisimple elements. We say that g and g' stably correspond if and only if g and g' have the same non-trivial (i.e. $\neq 1$) eigenvalues. We will write $g \overset{\text{stably}}{\longleftrightarrow} g'$.

PROPOSITION 2.3. *There is a bijection between $\mathcal{C}_{\mathrm{st}}(\mathrm{Sp}(2n))$ and $\mathcal{C}_{\mathrm{st}}(\mathrm{SO}(2n+1)_+)$ that matches isomorphic Cartan subgroups. This bijection induces a bijection between strongly regular, semisimple stable conjugacy classes of $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)_+$. Moreover, if $x \in \mathrm{Sp}(2n)$ and $x' \in \mathrm{SO}(2n+1)_+$ are strongly regular semisimple elements such that $x \overset{\text{stably}}{\longleftrightarrow} x'$, then there exists an isomorphism $\psi : \mathrm{Cent}_{\mathrm{Sp}(2n)}(x) \longrightarrow \mathrm{Cent}_{\mathrm{SO}(2n+1)_+}(x')$ such that $\psi(x) = x'$.*

Proof. Choose Cartan subgroups $\mathbf{T}_s \subset \mathbf{Sp(2n)}$ and $\mathbf{T}'_s \subset \mathbf{SO(2n{+}1)}$ defined over F and such that their rational points are split. Let $\psi_s : \mathbf{T}_s \to \mathbf{T}'_s$ be an isomorphism that commutes with the Galois action and such that the elements t and $\psi_s(t)$ have the same eigenvalues for all $t \in \mathbf{T}_s$. Let $\phi_s : W(C_n) \to W(B_n)$ be an isomorphism that satisfies the condition

$$\psi_s(w \cdot t) = \phi_s(w) \cdot \psi_s(t), \quad w \in W(C_n), t \in \mathbf{T}_s.$$

This induces an isomorphism between $H^1(\Gamma, W(C_n))$ and $H^1(\Gamma, W(B_n))$ and hence (by Lemma 2.1) a bijection between the sets $\mathcal{C}_{\mathrm{st}}(\mathrm{Sp}(2n))$ and $\mathcal{C}_{\mathrm{st}}(\mathrm{SO}(2n+1)_+)$. Let $\mathbf{T}$ be any Cartan subgroup in $\mathbf{Sp(2n)}$ that is defined over F. Choose $g \in \mathbf{Sp(2n)}$ such that $\mathbf{T} = g\mathbf{T}_s g^{-1}$. We have that the class of the cocycle $(g^{-1}\sigma(g))_{\sigma\in\Gamma}$ is an element of $H^1(\Gamma, W(C_n))$. By [Rag04, Theorem 1.1], its image in $H^1(\Gamma, W(B_n))$ contains a cocycle of the form $(h^{-1}\sigma(h))_{\sigma\in\Gamma}$ for some $h \in \mathbf{SO(2n{+}1)}$. Let $\mathbf{T}' = h\mathbf{T}'_s h^{-1}$. Then $\psi = \mathrm{int}(h) \circ \psi_s \circ \mathrm{int}(g^{-1})$ is an isomorphism between $\mathbf{T}$ and $\mathbf{T}'$ and let $\phi = \mathrm{int}(h) \circ \phi_s \circ \mathrm{int}(g^{-1})$ be the isomorphism between the Weyl groups of $\mathbf{T}$ and $\mathbf{T}'$. We have that $\psi(w \cdot t) = \phi(w) \cdot \psi(t)$, for all $t \in \mathbf{T}$ and all $w \in W(\mathbf{Sp(2n)}, \mathbf{T})$. We will show that $\psi(\mathbf{T}(F)) = \mathbf{T}'(F)$. Let $t \in \mathbf{T}(F)$ and $\sigma \in \Gamma$. If n is an element of the normalizer of a Cartan subgroup, we will denote by $\overline{n}$ its image in the corresponding Weyl group. We have that $\sigma(\psi(t)) = \sigma(h)\psi_s(\sigma(g^{-1})t\sigma(g))\sigma(h^{-1})$. Hence, $\sigma(\psi(t)) = \psi(t) = h(\psi_s(g^{-1}tg))h^{-1}$ if and only if $\psi_s(g^{-1}hg) = \overline{h^{-1}\sigma(h)} \cdot \psi_s(\sigma(g^{-1})t\sigma(g))$. This is true, since $\overline{h^{-1}\sigma(h)} = \phi_s(\overline{g^{-1}\sigma(g)}) \in W(B_n)$ and by the choice of the isomorphisms ψ_s and ϕ_s we have that

$$\phi_s(\overline{g^{-1}\sigma(g)}) \cdot \psi_s(\sigma(g^{-1})t\sigma(g)) = \psi_s(\overline{g^{-1}\sigma(g)} \cdot \sigma(g^{-1})t\sigma(g)) = \psi_s(g^{-1}tg).$$

The second statement of the proposition follows from the fact that $x \overset{\text{stably}}{\longleftrightarrow} \psi(x)$ for all strongly regular semisimple elements $x \in \mathbf{T}(F)$.

Finally, let $x \in \mathrm{Sp}(2n)$ and $x' \in \mathrm{SO}(2n+1)_+$ be strongly regular semisimple elements such that $x \overset{\text{stably}}{\longleftrightarrow} x'$. Without loss of generality, we can assume that $x \in \mathbf{T}(F)$. By the construction of ψ, we have that $x \overset{\text{stably}}{\longleftrightarrow} \psi(x)$ and hence $x' \sim_{\mathrm{st}} \psi(x)$. Therefore, the Cartan subgroups $\psi(\mathrm{Cent}_{\mathrm{Sp}(2n)}(x))$ and $\mathrm{Cent}_{\mathrm{SO}(2n+1)_+}(x')$ are also stably conjugate. We compose the map ψ with a conjugation by an appropriate element to obtain the required isomorphism (by an abuse of notation, we will also denote it by ψ). □

3. The character of the oscillator representation

First we recall after Maktouf [Mak99] the construction of the metaplectic double cover of the symplectic group $\mathrm{Sp}(2n)$. We fix an additive character η. Recall that an orientation of a vector space W is a non-zero element e of $\bigwedge^{\dim W} W$. We will write W^e for an oriented vector space. Let V^e be a $2n$-dimensional oriented vector space equipped with a symplectic form $\langle\ ,\ \rangle$. For any two Lagrangian subspaces l_1, l_2, we define a map $g_{l_1,l_2} : l_1 \to (l_2)^*$ by

$$g_{l_1,l_2}(v)(w) = \langle v, w\rangle, \quad v \in l_1, w \in l_2.$$

The map g_{l_1,l_2} induces an isomorphism between $l_1/l_1 \cap l_2$ and $(l_2/l_1 \cap l_2)^*$. We choose orientations e_1 and e_2 on l_1 and l_2 respectively and we consider $\det g_{l_1^{e_1},l_2^{e_2}} \bmod F^{*2}$ (it is independent of the choice of orientation on $l_1 \cap l_2$). We define

$$m(l_1^{e_1}, l_2^{e_2}) = \gamma(1)^{2(1-(n-\dim(l_1\cap l_2)))}\gamma(\det g_{l_1^{e_1},l_2^{e_2}})^{-2},$$

where $\gamma(a) =$ Weil index of $x \to \eta(ax^2)$, $a \in F$. (see [Rao93, Appendix] for the definition and properties of the Weil index). For the quadratic form $Q_a(x) = ax^2$, $a \neq 0$, we define $\gamma(Q_a) = \gamma(a)$. If $Q = 0$, then $\gamma(Q) = 1$. If $Q = Q_1 \oplus Q_2$, then $\gamma(Q) = \gamma(Q_1)\gamma(Q_2)$. For any triple of Lagrangian subspaces l_1, l_2, l_3, denote by Q_{l_1,l_2,l_3} a quadratic form defined on $l_1 \times l_2 \times l_3$ as follows:

$$Q(x, y, z) = \langle x, y\rangle + \langle y, z\rangle + \langle z, x\rangle.$$

Let $c(l_1, l_2, l_3) = \gamma(Q_{l_1,l_2,l_3})$. The following lemma will be useful.

Lemma 3.1. *Let l, l_1 and l_2 be Lagrangian subspaces. If $l = (l \cap l_1) + (l \cap l_2)$, then $c(l_1, l, l_2) = 1$.*

Proof. See [LVe80, Lemma 1.5.11, p. 44]. □

We define $\widetilde{\mathrm{Sp}}(2n)$ to be the set of pairs (g, ψ), where $g \in \mathrm{Sp}(2n)$ and ψ is a function on the set of Lagrangian subspaces of the space V satisfying the conditions

$$\psi^2(l) = m(l^e, gl^e)^{-1},$$
$$\psi(l') = \psi(l)c(l', l, gl)c(l', l, g^{-1}l')^{-1}.$$

The value of $m(l^e, gl^e)$ is independent of the choice of an orientation e. The multiplication is defined as follows:

$$(g, \psi)(h, \psi') = (gh, \psi''),$$
$$\psi''(l) = \psi(l)\psi'(l)c(l, gl, ghl).$$

Remark. Note that this definition of the metaplectic group $\widetilde{\mathrm{Sp}}(2n)$ depends on the fixed additive character η. Let $\omega(\eta) = \omega_+(\eta) \oplus \omega_-(\eta)$ be the oscillator representation of $\widetilde{\mathrm{Sp}}(2n)$ attached to η. For simplicity, we will drop η from the notation.

We will denote by $\Theta_{\omega_\pm}$ the character of $\omega_\pm$ and by $\Theta_{\omega_+ \pm \omega_-}$ the character of the formal sum (difference) of ω_+ and ω_-. Let $(g, \psi) \in \widetilde{\mathrm{Sp}}(2n)$, where g is a regular semisimple element. We decompose V into a direct sum of subspaces W_1 and W_2 such that W_1 has a g invariant

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Lagrangian subspace l_1 and W_2 has a Lagrangian subspace l_2 such that $l_2 \cap (gl_2) = 0$. On $(1-g^{-1})^{-1}l_2$, we define the quadratic form $Q_{g,l_2}(v) = \langle (g^{-1}-1)v, v\rangle$.

PROPOSITION 3.2. *We have* $\Theta_{\omega_+ \pm \omega_-}((g,\psi)) = \psi(l_1+l_2)\overline{\gamma(Q_{\pm g,l_2})}|\det(1 \mp g)|^{-1/2}$.

Proof. It follows directly from the formula

$$\Theta_{\omega_\pm}(g,\psi) = \tfrac{1}{2}\psi(l_1+l_2)(\overline{\gamma(Q_{g,l_2})}|\det(1-g)|^{-\frac{1}{2}} \pm \overline{\gamma(Q_{-g,l_2})}|\det(1+g)|^{-\frac{1}{2}})$$

given by [Mak99, § 31, p. 296]. □

COROLLARY 3.3. *Let* $(-I,\psi') \in \widetilde{\mathrm{Sp}}(2n)$. *There exists a constant* λ *(depending only on the choice of* ψ'*) such that* $\Theta_{\omega_+ + \omega_-}((g,\psi)(-I,\psi')) = \lambda\, \Theta_{\omega_+ - \omega_-}((g,\psi))$.

Proof. First note that $(g,\psi)(-I,\psi') = (-g,\psi\psi')$. Indeed, by Lemma 3.1, we have that $c(l,gl,-gl) = 1$ for any Lagrangian subspace l. Therefore,

$$\begin{aligned}\Theta_{\omega_+ + \omega_-}((-g,\psi\psi')) &= \psi'(l_1+l_2)\psi(l_1+l_2)\overline{\gamma(Q_{-g,l_2})}|\det(1+g)|^{-1/2}\\ &= \psi'(l_1+l_2)\Theta_{\omega_+ - \omega_-}((g,\psi)).\end{aligned}$$

Now we claim that the function ψ' is constant. Recall that for any pair of Lagrangian subspaces l and l' we have that $\psi'(l') = \psi'(l)c(l',l,-l)c(l',l,-l')^{-1}$. By Lemma 3.1, we get that $c(l',l,-l) = 1$. Now consider the form $Q_{l',l,-l'}$. Note that $Q_{l',l,-l'} = -Q_{l,l',-l'}$ and therefore $c(l',l,-l') = \gamma(Q_{l',l,-l'}) = \gamma(-Q_{l,l',-l'}) = \gamma(Q_{l,l',-l'})^{-1} = c(l,l',-l')^{-1} = 1$ (the middle equality follows from the properties of the Weil index and the last one follows again from Lemma 3.1). We define $\lambda = \psi'(l') = \psi'(l)$. □

4. Transfer factor

Let $G = \mathrm{Sp}(2n)$ or $\mathrm{SO}(2n+1)_+$. Let T be a Cartan subgroup in G and let $g \in T$. We define $D_G(g) = \prod_{\alpha\in R}(1-\alpha(g))$, where R is the set of roots of T in G. We will denote the set of regular elements of G (i.e. elements $g \in G$ such that $D_G(g) \neq 0$) by G_{reg}.

DEFINITION 4.1. The transfer factor on $\widetilde{\mathrm{Sp}}(2n)$ is equal to the difference of the two halves of the oscillator representation, i.e.

$$\Phi = \Theta_{\omega_+ - \omega_-}.$$

LEMMA 4.2. *Let* $(g,\psi) \in \widetilde{\mathrm{Sp}}(2n)$, *where* $g \in \mathrm{Sp}(2n)$ *is a strongly regular semisimple element. Let* $g \overset{\text{stably}}{\longleftrightarrow} g' \in \mathrm{SO}(2n+1)_+$. *Then*

$$|\Phi(\widetilde{g})| = \frac{|D_{\mathrm{SO}(2n+1)_+}(g')|^{\frac{1}{2}}}{|D_{\mathrm{Sp}(2n)}(g)|^{\frac{1}{2}}} = |\det(1+g)|^{-\frac{1}{2}}.$$

Proof. First note that by Corollary 3.2 we have $|\Phi(\widetilde{g})| = |\det(1+g)|^{-\frac{1}{2}}$. Now we will evaluate the quotient of Weyl denominators for $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(2n)$. Assume that g has eigenvalues $g_1,\ldots,g_{2n}$. Therefore, g' has the same eigenvalues together with 1. We have that

$$\frac{D_{\mathrm{SO}(2n+1)_+}(g')}{D_{\mathrm{Sp}(2n)}(g)} = \frac{\prod(1-g_i)\prod(1-g_i/g_j)}{\prod(1-g_i^2)\prod(1-g_i/g_j)} = \prod(1+g_i)^{-1} = \det(1+g)^{-1}.$$ □

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Let $A \subset \mathrm{Sp}(2n)$ and $A' \subset \mathrm{SO}(2n+1)_+$ be isomorphic split tori. Let M be the centralizer of A in $\mathrm{Sp}(2n)$ and M' the centralizer of A' in $\mathrm{SO}(2n+1)_+$. Assume that $M \cong \mathrm{Sp}(2n_0) \times \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k)$ and $M' \cong \mathrm{SO}(2n_0+1)_+ \times \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k)$.

Definition 4.3. Let $g = (g_0, \dots, g_k) \in M$ and $g' = (g'_0, \dots, g'_k) \in M'$ be strongly regular semisimple elements. We say that g stably corresponds to g' with respect to M and M' if $g_i \in \mathrm{GL}(n_i)$ is conjugate to $g'_i \in \mathrm{GL}(n_i)$ for $i \in \{1, \dots, k\}$ and if g_0 stably corresponds to g'_0 in $\mathrm{Sp}(2n_0)$. We will write $g \overset{M,M'}{\longleftrightarrow} g'$.

Now consider $\widetilde{A} = p^{-1}(A)$, $\widetilde{M} = p^{-1}(M)$, where $p : \widetilde{\mathrm{Sp}}(2n) \to \mathrm{Sp}(2n)$ is the projection map. We also consider the cover $p' : \widetilde{\mathrm{Sp}}(2n_0) \times \widetilde{\mathrm{GL}}(n_1) \times \cdots \times \widetilde{\mathrm{GL}}(n_k) \longrightarrow \widetilde{M}$. Let $(\widetilde{g}_0, \dots, \widetilde{g}_k) \in \widetilde{\mathrm{Sp}}(2n_0) \times \widetilde{\mathrm{GL}}(n_1) \times \cdots \times \widetilde{\mathrm{GL}}(n_k)$ be such that its image in M is strongly regular and semisimple. Let $\widetilde{g} = p'((\widetilde{g}_0, \dots, \widetilde{g}_k)) = \widetilde{g}_0 \cdots \widetilde{g}_k \in \widetilde{M}$.

Definition 4.4. We define the transfer factor on $\widetilde{M}$ as follows:

$$\Phi_{\widetilde{M}}(\widetilde{g}) = \frac{\Phi(\widetilde{g})}{|\Phi(\widetilde{g})|} |\Phi_{\widetilde{\mathrm{Sp}}(2n_0)}(\widetilde{g}_0)|.$$

Suppose now that $p(\widetilde{g}) = g \overset{M,M'}{\longleftrightarrow} g'$. By Lemma 4.2, we have that

$$\Phi_{\widetilde{M}}(\widetilde{g}) = \frac{|D_{\mathrm{Sp}(2n)}(g)|^{\frac{1}{2}}}{|D_M(g)|^{\frac{1}{2}}} \frac{|D_{M'}(g')|^{\frac{1}{2}}}{|D_{\mathrm{SO}(2n+1)_+}(g')|^{\frac{1}{2}}} \Theta_{\omega_+ - \omega_-}(\widetilde{g}) = \frac{|\det(1+g)|^{\frac{1}{2}}}{|\det(1+g_0)|^{\frac{1}{2}}} \Theta_{\omega_+ - \omega_-}(\widetilde{g}).$$

We will treat $\widetilde{\mathrm{Sp}}(n_0)$ and each of the $\widetilde{\mathrm{GL}}(n_i)$ as a subgroup of $\widetilde{\mathrm{Sp}}(2n)$. In particular, if $(g, \psi) \in \widetilde{\mathrm{GL}}(n_i)$, then ψ is a constant function and it is equal to $\pm\gamma(\det(g))\gamma(1)^{-1}$. Therefore, we will write $(g, \psi) = (g, \psi_\epsilon)$, where $\psi_\epsilon = \epsilon\gamma(\det(g))\gamma(1)^{-1}$ for $\epsilon = \pm 1$. The map $(g, \psi_\epsilon) \longmapsto \epsilon\gamma(\det(g))\gamma(1)^{-1}$ is a genuine character on $\widetilde{\mathrm{GL}}(n_i)$ and we will denote it by χ_i.

Lemma 4.5. *We have* $\Phi_{\widetilde{M}}(\widetilde{g}) = \chi_1(\widetilde{g}_1) \cdots \chi_k(\widetilde{g}_k) \Theta^{\widetilde{\mathrm{Sp}}(2n_0)}_{\omega_+ - \omega_-}(\widetilde{g}_0)$.

Proof. Write $\widetilde{g} = (g, \psi), \widetilde{g}_0 = (g_0, \psi_0), \widetilde{g}_i = (g_i, \psi_{\epsilon_i})$, $\epsilon_i = \pm 1$ for $i = 1, \dots, k$. Choose $g'_0 \in \mathrm{SO}(2n_0+1)_+$ such that $g_0 \overset{M,M'}{\longleftrightarrow} g'_0$ and let $g' = (g'_0, g_1, \dots, g_k) \in \mathrm{SO}(2n+1)_+$. Hence, $g \overset{M,M'}{\longleftrightarrow} g'$ and

$$\Phi_{\widetilde{M}}(\widetilde{g}) = \frac{|\det(1+g)|^{\frac{1}{2}}}{|\det(1+g_0)|^{\frac{1}{2}}} \Theta_{\omega_+ - \omega_-}(\widetilde{g}).$$

We evaluate $\Theta_{\omega_+ - \omega_-}(\widetilde{g})$. We decompose the underlying symplectic space into a direct sum of subspaces W_1 and W_2, where W_1 has a Lagrangian subspace l_1 such that $g_i l_1 = l_1$ for $i = 0, \dots, k$ and W_2 has a Lagrangian subspace l_2 such that $l_2 \cap (g_0 l_2) = 0$ and $g_i l_2 = l_2$ for $i = 1, \dots, k$. Let $l = l_1 + l_2$. By Proposition 3.2, we have that

$$\Theta_{\omega_+ - \omega_-}((g, \psi)) = \psi(l)\overline{\gamma(Q_{-g,l_2})}|\det(1+g)|^{-1/2}.$$

Since $g_i l = l$ for $i \neq 0$ by Lemma 3.1, we have that $\psi(l) = \psi_0(l)\psi_{\epsilon_1}(l) \cdots \psi_{\epsilon_k}(l)$, where each $\psi_{\epsilon_i}(l)$ equals $\chi_i(\widetilde{g}_i)$. Note also that $\gamma(Q_{-g,l_2}) = \gamma(Q_{-g_0,l_2})$, since $(1+g^{-1})^{-1} l_2 = (1+g_0^{-1})^{-1} l_2 \subset V_2$ and $Q_{-g,l_2} = Q_{-g_0,l_2}$. Combining all of this together, we get that

$$\Theta_{\omega_+ - \omega_-}(\widetilde{g}) = \chi_1(\widetilde{g}_1) \cdots \chi_k(\widetilde{g}_k)\psi_{\epsilon_0}(l)\overline{\gamma(Q_{-g_0,l_2})}|\det(1+g)|^{-1/2}.$$

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The final conclusion follows from the fact that

$$\Theta^{\widetilde{\mathrm{Sp}}(2n_0)}_{\omega_+ - \omega_-}(\widetilde{g}_0) = \psi_{\epsilon_0}(l)\overline{\gamma(Q_{-g_0,l_2})}|\det(1+g_0)|^{-1/2}. \qquad \square$$

5. The induced character formula

Let $G = \mathrm{Sp}(2n)$ or $\mathrm{SO}(2n+1)_+$. Let A be a split torus in G and let M be its centralizer in G. For any Cartan subgroup T of G, we denote by A_T the split component of T and by $W(A,T)$ the set of all injections $s : A \to A_T$ for which there exists $y \in G$ such that $s(a) = yay^{-1}$ for all $a \in A$. Alternatively (see [Van72]), $W(A,T) \cong \bigcup_{\{H \subset M : H \sim_G T\}/\sim_M} W(G,H)/W(M,H)$.

For $s = s_y \in W(A,T)$ and for any representation ρ of M, let $s\rho$ be the representation of $M^s = yMy^{-1}$ defined by $s\rho(m) = \rho(y^{-1}my)$. The following theorem is due to van Dijk [Van72, Theorem 3].

PROPOSITION 5.1. *Consider the parabolic subgroup $P = MN$. Let ρ be any admissible representation of M with a character Θ_ρ. We extend ρ to a representation of P by putting $\rho(mn) = \rho(m), m \in M, n \in N$. If T is a Cartan subgroup that is not conjugate to a Cartan subgroup in M, then Θ_π vanishes on $T \cap G_{\mathrm{reg}}$. If T is conjugate to a Cartan subgroup in M, then the character Θ_π of the representation $\pi = \mathrm{Ind}_P^G(\rho)$ has the formula*

$$\Theta_\pi(g) = \sum_{s \in W(A,T)} \Theta_{s\rho}(g)\frac{|D_{M^s}(g)|^{\frac{1}{2}}}{|D_G(g)|^{\frac{1}{2}}}, \quad g \in T \cap G_{\mathrm{reg}}.$$

We will assume the following two facts (well known to experts) about the metaplectic group. First we assume that a character of an admissible representation π of $\widetilde{\mathrm{Sp}}(2n)$ defined as a distribution $f \mapsto \mathrm{tr}\int f(x)\pi(x)\,dx$, $f \in C_c^\infty(G)$ is given by a locally integrable function.

The second fact is that van Dijk's result holds for the metaplectic group. Let ρ be an admissible representation of $\widetilde{M} = p^{-1}(M)$. We inflate ρ to a representation of a parabolic subgroup $\widetilde{P} = p^{-1}(P)$, and we denote its character by Θ_ρ. Let π be the induced representation $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(2n)}(\rho)$ with a character Θ_π. Let $\widetilde{T} = p^{-1}(T)$ be any Cartan subgroup of $\widetilde{\mathrm{Sp}}(2n)$ and choose $\widetilde{g} \in \widetilde{T}$ such that $p(\widetilde{g})$ is regular. If $\widetilde{T}$ is conjugate to a Cartan subgroup in $\widetilde{M}$, then

$$\Theta_\pi(\widetilde{g}) = \sum_{s \in W(\widetilde{A},\widetilde{T})} \Theta_{s\rho}(\widetilde{g})\frac{|D_{M^s}(g)|^{\frac{1}{2}}}{|D_{\mathrm{Sp}(2n)}(g)|^{\frac{1}{2}}}.$$

Otherwise, $\Theta_\pi(\widetilde{g}) = 0$. The set $W(\widetilde{A},\widetilde{T})$ is defined similarly as in the linear case, i.e. if $\widetilde{A} = p^{-1}(A)$ and $A_{\widetilde{T}}$ is the split component of the Cartan subgroup $\widetilde{T}$, then we denote by $W(\widetilde{A},\widetilde{T})$ the set of all injections $s : \widetilde{A} \to A_{\widetilde{T}}$ for which there exists $\widetilde{y} \in \widetilde{\mathrm{Sp}}(2n)$ such that $s(\widetilde{a}) = \widetilde{y}\,\widetilde{a}\,\widetilde{y}^{-1}$ for all $\widetilde{a} \in \widetilde{A}$.

LEMMA 5.2. *The sets $W(A,T)$ and $W(\widetilde{A},\widetilde{T})$ are in bijection.*

Proof. Let $s_y \in W(A,T)$. We choose $\widetilde{y} \in \widetilde{\mathrm{Sp}}(2n)$ such that $p(\widetilde{y}) = y$. We have that $\widetilde{y}\,\widetilde{a}\,\widetilde{y}^{-1} \subset A_{\widetilde{T}}$ for all $\widetilde{a} \in \widetilde{A}$. If $\widetilde{x} \in \widetilde{\mathrm{Sp}}(2n)$ is another element that maps to y, then (since our extension is central) we have that $\widetilde{y}\,\widetilde{a}\,\widetilde{y}^{-1} = \widetilde{x}\,\widetilde{a}\,\widetilde{x}^{-1}$ for $\widetilde{a} \in \widetilde{A}$. Therefore, $s_{\widetilde{y}} = s_{\widetilde{x}} \in W(\widetilde{A},\widetilde{T})$. On the other hand, if $s_{\widetilde{y}} \in W(\widetilde{A},\widetilde{T})$, then clearly $s_{p(\widetilde{y})}$ belongs to $W(A,T)$. $\square$

6. Weyl groups

The goal of this section is to show that there exists a bijection between the sets $W(A,T)$ and $W(A',T')$, where $A\subset \mathrm{Sp}(2n)$ and $A'\subset \mathrm{SO}(2n+1)_+$ are isomorphic split tori and $T\subset M=\mathrm{Cent}_{\mathrm{Sp}(2n)}(A)$ and $T'\subset M'=\mathrm{Cent}_{\mathrm{SO}(2n+1)_+}(A')$ are isomorphic Cartan subgroups.

By [How73, Lemma, p. 296], any Cartan subgroup in $\mathrm{Sp}(2n)$ (and thus also in $\mathrm{SO}(2n+1)_+$) can be decomposed into a direct product $(F^*)^a\times K_1^*\times\cdots\times K_l^*\times E_1^1\times\cdots\times E_k^1$, where each K_i^* is the group of units of some non-trivial field extension K_i of F and each E_j^1 is the group of norm units of E_j over L_j for some tower of field extensions $E_j \overset{2}{-} L_j - F$.

From now on, if H is a Cartan subgroup in $\mathrm{Sp}(2n)$ (or in $\mathrm{SO}(2n+1)_+$), then we will denote H_F, H_K, and H_N as the subgroups of H, where

$$H\cong H_F\times H_K\times H_N$$

and $H_F\cong (F^*)^a$ is a split torus, $H_K\cong K_1^*\times\cdots\times K_l^*\subset \mathrm{GL}(b)$ as in the decomposition above, $H_N\subset \mathrm{Sp}(2c)$ (or $\mathrm{SO}(2c+1)_+$) is a product of norm-one tori, and $a+b+c=n$.

First we will show that the quotients $W(\mathrm{Sp}(2n),T)/W(M,T)$ and $W(\mathrm{SO}(2n+1)_+,T')/W(M',T')$ are isomorphic. We start with describing the group $W(\mathrm{Sp}(2b),T_K)$.

Lemma 6.1. *The group $W(\mathrm{Sp}(2b),T_K)$ is isomorphic to $W(\mathrm{GL}(b),T_K)\ltimes(\mathbb{Z}_2)^l$.*

Proof. Assume first that T_K is a Cartan subgroup in $\mathrm{Sp}(2b)$ coming from a single field extension, i.e. $T_K=K^*$ for some field extension $K\overset{b}{-}F$ of degree b. We choose a symplectic basis and we consider the following embedding of K^* into $\mathrm{Sp}(2b)$:

$$\iota(z)=\begin{pmatrix} M_z & 0\\ 0 & {}^tM_{z^{-1}}\end{pmatrix},$$

where $M_z\in \mathrm{GL}(b)$ denotes the matrix of multiplication by $z\in K^*$. Let $w\in K^*$ be a regular element and let $Z=Z_K\in\mathrm{GL}(b)$ be such that $ZM_wZ^{-1}={}^tM_w$. It follows that $ZM_zZ^{-1}={}^tM_z$ for all $z\in K^*$. Let

$$\delta_K=\begin{pmatrix} 0 & {}^tZ^{-1}\\ -Z & 0\end{pmatrix}\in \mathrm{Sp}(2b).$$

The element δ_K acts on the Cartan subgroup $K^*\subset\mathrm{Sp}(2b)$ as follows:

$$\delta_K\begin{pmatrix} M_z & 0\\ 0 & {}^tM_{z^{-1}}\end{pmatrix}\delta_K^{-1}=\begin{pmatrix} M_{z^{-1}} & 0\\ 0 & {}^tM_{z}\end{pmatrix}.$$

Let $\mathbf{T}_K$ be a Cartan subgroup in $\mathbf{Sp(2b)}$ such that $\mathbf{T}_K(F)=K^*$. Then $W(\mathrm{Sp}(2b),K^*)$ is a subgroup of $W(\mathbf{Sp(2b)},\mathbf{T}_K)$. The latter is of type C_b and consists of permutations and 'sign changes'. From these, the only operations that preserve $\mathbf{T}_K(F)\subset\mathrm{Sp}(2b)$ are those coming from the action of $\mathrm{GL}(b)$ on K^* and the action of the element δ_K that corresponds to the simultaneous sign change of all the eigenvalues. Note also that δ_K commutes with the subgroup $W(\mathrm{GL}(b),K^*)$ and hence in this particular case we have $W(\mathrm{Sp}(2b),T_K)\cong W(\mathrm{GL}(b),T_K)\times\mathbb{Z}_2$.

The proof of the general case is analogous: the subgroup $(\mathbb{Z}_2)^l$ is generated by the elements $\{\delta_{K_i}, i=1,\ldots,l\}$. Each δ_{K_i} acts on K_i^* as described earlier, i.e. it replaces $\iota(z)$ with $\iota(z^{-1})$ for $z\in K_i^*$ and leaves all the other K_j^* fixed. The subgroup $(\mathbb{Z}_2)^l$ is normal in $W(\mathrm{Sp}(2b),T_K)$. ☐

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LEMMA 6.2. *We have the following isomorphisms of groups*

(1) $W(\mathrm{Sp}(2n), T) \cong W(C_a) \times (W(\mathrm{GL}(b), T_K) \ltimes (\mathbb{Z}_2)^l) \times W(\mathrm{Sp}(2c), T_N)$.
(2) $W(\mathrm{SO}(2n+1)_+, T') \cong W(B_a) \times (W(\mathrm{GL}(b), T_K) \ltimes (\mathbb{Z}_2)^l) \times W(\mathrm{SO}(2c+1)_+, T_N)$.

Proof. First will show that $N_{\mathrm{Sp}(2n)}(T) \cong N_{\mathrm{Sp}(2a)}(T_F) \times N_{\mathrm{Sp}(2b)}(T_K) \times N_{\mathrm{Sp}(2c)}(T_N)$. If $n \in N_{\mathrm{Sp}(2n)}(T)$, then n has to also normalize every individual component of T, i.e T_F, T_K, and T_N. That is because the eigenvalues of elements of each of these components are of different nature: every element of T_F is diagonalizable over F, while the eigenvalues of a generic element in T_K belong to some non-trivial field extension $K_i - F$. Furthermore, the eigenvalues of a generic element in T_N are norm-one elements in some non-trivial field extension $E_j - L_j - F$.

We claim that $n \in \mathrm{Sp}(2a) \times \mathrm{Sp}(2b) \times \mathrm{Sp}(2c)$. Indeed, there exist elements $n_F \in N_{\mathrm{Sp}(2a)}(T_F)$ and $n_K \in N_{\mathrm{Sp}(2b)}(T_K)$ whose actions on T_F and T_K coincide with the action of n. That is because n normalizes T_F and T_K and the fact that all operations allowed on T_F and T_K are realized by $W(\mathrm{Sp}(2a), T_F)$ and $W(\mathrm{Sp}(2b), T_K)$ (see the proof of Lemma 6.1). Therefore, the element $nn_F^{-1}n_K^{-1}$ fixes $T_F \times T_K$ and hence it belongs to the centralizer of $T_F \times T_K$; in particular, it is contained in $\mathrm{Sp}(2a) \times \mathrm{Sp}(2b) \times \mathrm{Sp}(2c)$. Now (1) follows from the above and from Lemma 6.1.

The proof of (2) is analogous. Assume that $\mathrm{SO}(2n+1)_+$ preserves the bilinear form

$$\begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the Weyl group of type D_a acts on the torus T_F. Let w_1 be an element of $O(2a)$ that acts on $\mathrm{diag}(x_1, \ldots, x_a, x_1^{-1}, \ldots, x_a^{-1})$ by interchanging x_1 and x_1^{-1}. Now $\overline{w}_1 = w_1 \times -I \in \mathrm{SO}(2n+1)_+$ together with $W(D_a)$ generate $W(B_a)$.

The proof of the statement concerning the torus T_K is similar to the proof of Lemma 6.1. The only difference is that we replace the elements δ_{K_i} with δ'_{K_i}, where

$$\delta'_{K_i} = \begin{pmatrix} 0 & {}^tZ_i^{-1} & 0 \\ Z_i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the matrices Z_i are the same ones that were used to construct $\delta_{K_i}s$. □

Next we study the Weyl groups $W(M, T)$ and $W(M', T')$, where $M \cong \mathrm{Sp}(2n_0) \times \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k)$ and $M' \cong \mathrm{SO}(2n_0+1)_+ \times \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k)$, for some $n_0 + n_1 + \cdots + n_k = n$. Accordingly, we decompose

$$T \cong T' \cong T_0 \times T_1 \times \cdots \times T_k,$$

where T_0 is a Cartan subgroup in $\mathrm{Sp}(2n_0)$ (or $\mathrm{SO}(2n_0+1)_+$) and T_i is a Cartan subgroup in $\mathrm{GL}(n_i)$ for $i = 1, \ldots, k$. We write $T_0 \cong F^{*a_0} \times (T_0)_K \times T_N$, where $(T_0)_K \subset \mathrm{GL}(b_0)$ is a product of l_0 groups of units of some field extensions of F. Applying Lemma 6.2 to T_0, we get the following.

LEMMA 6.3. *We have the following isomorphisms of groups*

$$\begin{aligned} W(M, T) \cong\ & W(C_{a_0}) \times W(\mathrm{GL}(b_0), (T_0)_K) \ltimes (\mathbb{Z}_2)^{l_0} \\ & \times W(\mathrm{Sp}(2c), T_N) \times \prod W(\mathrm{GL}(n_i), T_i), \\ W(M', T') \cong\ & W(B_{a_0}) \times W(\mathrm{GL}(b_0), (T_0)_K) \ltimes (\mathbb{Z}_2)^{l_0} \\ & \times W(\mathrm{SO}(2c+1)_+, T_N) \times \prod W(\mathrm{GL}(n_i), T_i). \end{aligned}$$

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As a consequence of Lemmas 6.2 and 6.3, we get the following lemma.

Lemma 6.4. *We have* $W(\mathrm{Sp}(2n), T)/W(M, T) \cong W(\mathrm{SO}(2n+1)_+, T')/W(M', T')$.

To show that $W(A, T) \cong W(A', T')$, we will need one more result.

Lemma 6.5. *Suppose that* T (T') *is conjugate to a Cartan subgroup* $H \subset \mathrm{GL}(a+b) \times \mathrm{Sp}(2c)$ $(\mathrm{GL}(a+b) \times \mathrm{SO}(2c+1)_+)$. *Then:*

(1) $H \cong H_{FK} \times H_N$, *where* H_{FK} *is conjugate in* $\mathrm{GL}(a+b)$ *to* $T_F \times T_K$ *and* H_N *is stably conjugate in* $\mathrm{Sp}(2c)$ $(\mathrm{SO}(2c+1)_+)$ *to* T_N;

(2) *If* $xTx^{-1} = H$ $(xT'x^{-1} = H)$, *then* $x \in \mathrm{Sp}(2(a+b)) \times \mathrm{Sp}(2c)$ $(x \in O(2(a+b)) \times O(2c+1))$;

(3) H *is conjugate in* M (M') *to* $H_{FK} \times T_N$.

Proof. (1) follows from the fact that conjugate elements of T_N and H_N (and hence of $T_F \times T_K$ and H_{FK}) have the same eigenvalues.

To show (2), consider a $2n$-dimensional vector space V with a non-degenerate symplectic form $\langle\ ,\ \rangle$ that is preserved by $\mathrm{Sp}(2n)$. Let $g = (d_1, e_1) \in (T_F \times T_K) \times T_N$ be a regular element. Decompose V into a direct product of subspaces W_1 and W_2, where $d_1 \in \mathrm{Sp}(W_1)$ and $e_1 \in \mathrm{Sp}(W_2)$. By (1), $x(d_1, 1)x^{-1} = (d_2, 1)$ for some $d_2 \in H_{FK}$. Let $w_2 \in W_2$. Since $(d_1, 1)w_2 = w_2$, we have that $(x(d_1, 1)x^{-1})xw_2 = xw_2$. Since d_2 is regular (i.e. has no trivial eigenvalues) and since $(d_2, 1)$ fixes W_2, we get that $xw_2 \in W_2$. Note also that x takes W_1 into W_1, since $0 = \langle W_1, W_2\rangle = \langle xW_1, xW_2\rangle = \langle xW_1, W_2\rangle$ and x preserves $\langle,\rangle|_{W_i}$; hence, the assertion follows. The proof of the 'orthogonal' version is similar.

To show (3), write $x = (x_1, x_2) \in \mathrm{Sp}(2(a+b)) \times \mathrm{Sp}(2c)$. Then $(1, x_2^{-1})H(1, x_2) = H_{FK} \times T_N$ and $(1, x_2^{-1}) \in I \times \mathrm{Sp}(2c) \subset M$. Now suppose that $x = (x_1, x_2) \in O(2(a+b)) \times O(2c+1)$. If $\det x_2 = 1$, then we proceed as in the symplectic case. If $\det x_2 = -1$, then we replace $(1, x_2)$ with $(1, -x_2) \in I \times \mathrm{SO}(2c+1)_+ \subset M'$. □

Lemma 6.6. *The sets* $W(A, T)$ *and* $W(A', T')$ *are in bijection.*

Proof. It is enough to show (cf. Lemma 6.4) that we can choose the representatives of the sets $\{H \subset M : H \sim_{\mathrm{Sp}(2n)} T\}$ and $\{H' \subset M' : H' \sim_{\mathrm{SO}(2n+1)_+} T'\}$ to be isomorphic Cartan subgroups. Let $H_1, \ldots, H_m$ be the representatives of the set $\{H \subset M : H \sim_{\mathrm{Sp}(2n)} T\}/{\sim_M}$. By Lemma 6.5, we can decompose each H_i into a product $(H_i)_{FK} \times (H_i)_N$, where $(H_i)_{FK}$ is a Cartan subgroup in $\mathrm{GL}(a+b)$ that is conjugate to $T_F \times T_K$ in $\mathrm{GL}(a+b)$ and $(H_i)_N$ is a Cartan subgroup in $\mathrm{Sp}(2c)$ that is stably conjugate in $\mathrm{Sp}(2c)$ to T_N. Also, by Lemma 6.5, each H_i is conjugate in M to $(H_i)_{FK} \times T_N$. Without loss of generality, we will assume then that $(H_i)_N = T_N$ for all i. Therefore, $(H_1)_{FK} \times T_N, \ldots, (H_m)_{FK} \times T_N$ is a complete list of representatives of the indexing set for $\mathrm{Sp}(2n)$.

We form a list $H'_1, \ldots, H'_m$ for $\mathrm{SO}(2n+1)$ as follows. Define $H'_i \cong (H_i)_{FK} \times T'_N \subset \mathrm{SO}(2n+1)_+$. We need to show the following: (1) $H'_i \sim_{\mathrm{SO}(2n+1)_+} T'$, (2) $H'_i \nsim_{M'} H'_j$ for $i \neq j$, (3) the list is complete.

Assertion (1) follows from the fact that $(H_i)_{FK}$ and $T_F \times T_K$ are conjugate in $\mathrm{GL}(a+b)$; hence also $(H_i)_{FK} \times T'_N$ and $T' \cong T_F \times T_K \times T'_N$ are conjugate in $\mathrm{SO}(2n+1)_+$. Part (2) is true because $T_i \nsim_M T_i$ for $i \neq j$.

Now we will show that $H'_1, \ldots, H'_m$ exhaust the list of the representatives of the indexing set for $\mathrm{SO}(2n+1)_+$. Let $H' \subset M'$ be a Cartan subgroup that is conjugate in $\mathrm{SO}(2n+1)_+$

to T'. We decompose H' into a product $H'_{FK} \times H'_N$, where H'_{FK} is conjugate to $T_F \times T_K$ in $\mathrm{GL}(a+b)$ and H'_N is stably conjugate in $\mathrm{SO}(2c+1)_+$ to T'_N. There exists $j \in \{1, \dots, m\}$ such that $H'_{FK} \cong (H_j)_{FK}$. Therefore, $H' \cong (H_j)_{FK} \times H'_N \sim (H_j)_{FK} \times T'_N \cong H'_j$. By Lemma 6.5(3), we get that $H' \sim_{M'} H'_j$. This completes the proof. □

7. Stable Weyl groups

The bijection between the sets $W(A,T)$ and $W(A',T')$ obtained in the previous section does not have sufficiently nice properties. The reason is that it is just a bijection between sets; it does not come from a Weyl group isomorphism (in general, the groups $W(\mathrm{Sp}(2n),T)$ and $W(\mathrm{SO}(2n+1)_+,T')$ are not isomorphic). In order to obtain the required properties, we need to consider the stable Weyl groups. The same technique was also used by Adams [Ada98].

Let $\mathbf{G} = \mathbf{Sp(2n)}$ or $\mathbf{SO(2n{+}1)}$ and let $\mathbf{H} \subset \mathbf{G}$ be a Cartan subgroup that is defined over F. We denote by $W(\mathbf{G},\mathbf{H})^\Gamma$ the subgroup of $W(\mathbf{G},\mathbf{H})$ that is fixed by the action of Γ.

LEMMA 7.1. *The subgroup $W(\mathbf{G},\mathbf{H})^\Gamma$ of $W(\mathbf{G},\mathbf{H})$ consists of those elements w in $W(\mathbf{G},\mathbf{H})$ which act on $\mathbf{H}(F)$. $W(\mathbf{G},\mathbf{H})^\Gamma = \{w \in W(\mathbf{G},\mathbf{H}) \mid w \text{ acts on } \mathbf{H}(F)\}$.*

Proof. First note that $W(\mathbf{G},\mathbf{H})^\Gamma$ acts on $\mathbf{H}(F)$. Indeed, if $w \in W(\mathbf{G},\mathbf{H})^\Gamma$ and $t \in \mathbf{H}(F)$ then $\sigma(w \cdot t) = \sigma(w) \cdot \sigma(t) = w \cdot t$ for all $\sigma \in \Gamma$. Assume now that $w \in W(\mathbf{G},\mathbf{H})$ acts on $\mathbf{H}(F)$. Let $t \in \mathbf{H}(F)$ be a strongly regular element. We have that $\sigma(w \cdot t) = w \cdot t$, i.e. $w^{-1}\sigma(w)$ commutes with t. Therefore, $w^{-1}\sigma(w) \in \mathbf{H}$ and hence $\sigma(w) = w$. □

DEFINITION 7.2. We define the stable Weyl group of H in G to be $W(\mathbf{G},\mathbf{H})^\Gamma$. We will denote it by $W_{\mathrm{st}}(G,H)$.

Let $T = \mathbf{T}(F) \subset \mathrm{Sp}(2n)$ and $T' = \mathbf{T}'(F) \subset \mathrm{SO}(2n+1)_+$ be isomorphic Cartan subgroups. Let $x \in T$ and $x' \in T'$ be strongly regular elements that stably correspond. Recall that in § 2 we constructed isomorphisms $\phi : W(\mathbf{Sp(2n)},\mathbf{T}) \longrightarrow W(\mathbf{SO(2n{+}1)},\mathbf{T}')$ and $\psi : \mathbf{T} \longrightarrow \mathbf{T}'$ such that $\psi(x) = x'$, $\psi(\mathbf{T}(F)) = \mathbf{T}'(F)$, and $\psi(w \cdot t) = \phi(w) \cdot \psi(t)$, for all $t \in \mathbf{T}$ and all $w \in W(\mathbf{Sp(2n)},\mathbf{T})$.

LEMMA 7.3. *The image of $W_{\mathrm{st}}(\mathrm{Sp}(2n),T)$ under the isomorphism ϕ is equal to*

$$W_{\mathrm{st}}(\mathrm{SO}(2n+1)_+,T').$$

Proof. Let $w \in W_{\mathrm{st}}(\mathrm{Sp}(2n),T)$ and $t' \in T'$. Let $t \in T$ be such that $t' = \psi(t)$. Since $w \cdot t \in T$, we get that $\phi(w) \cdot t' = \phi(w) \cdot \psi(t) = \psi(w \cdot t) \in T'$. Therefore, $\phi(w) \in W_{\mathrm{st}}(\mathrm{SO}(2n+1)_+,T')$. The proof of the other inclusion is similar. □

LEMMA 7.4. *We have*

(1) $W_{\mathrm{st}}(\mathrm{Sp}(2n),T)/W_{\mathrm{st}}(M,T) \cong W(\mathrm{Sp}(2n),T)/W(M,T)$.

(2) $W_{\mathrm{st}}(\mathrm{SO}(2n+1)_+,T')/W_{\mathrm{st}}(M',T') \cong W(\mathrm{SO}(2n+1)_+,T')/W(M',T')$.

(3) $W_{\mathrm{st}}(M,T) \cong W_{\mathrm{st}}(M',T')$.

Proof. Recall the decomposition $T \cong T_F \times T_K \times T_N$. We have that $W_{\mathrm{st}}(\mathrm{Sp}(2a),T_F) = W(\mathrm{Sp}(2a), T_F)$ and (by the proof of Lemma 6.1) $W_{\mathrm{st}}(\mathrm{Sp}(2b),T_K) = W(\mathrm{Sp}(2b),T_K)$. Analogous statements are true for the components of $W(M,T)$. Now, (1) and (2) follow from the descriptions of the Weyl groups given in § 6. Statement (3) follows again from the explicit descriptions of $W_{\mathrm{st}}(M,T)$ and $W_{\mathrm{st}}(M',T')$ and from Lemma 7.3 applied to $W_{\mathrm{st}}(\mathrm{Sp}(2c),T_N)$ and $W_{\mathrm{st}}(\mathrm{SO}(2c+1)_+,T_N)$. □

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We summarize this section in the following lemma.

Lemma 7.5. *Let $T \subset \mathrm{Sp}(2n)$ and $T' \subset \mathrm{SO}(2n+1)_+$ be isomorphic Cartan subgroups. Let $g \in T$, $g' \in T'$ be a pair of strongly regular elements that stably correspond. Then there exist isomorphisms $\psi : T \to T'$ and $\phi : W_{\mathrm{st}}(\mathrm{Sp}(2n), T) \to W_{\mathrm{st}}(\mathrm{SO}(2n+1)_+, T')$ such that:*

(1) $\psi(g) = g'$;

(2) $\psi(w \cdot t) = \phi(w) \cdot \psi(t), w \in W_{\mathrm{st}}(\mathrm{Sp}(2n), T), t \in T$;

(3) *ϕ factors to* $\phi : W_{\mathrm{st}}(\mathrm{Sp}(2n), T)/W_{\mathrm{st}}(M, T) \xrightarrow{\sim} W_{\mathrm{st}}(\mathrm{SO}(2c+1)_+, T')/W_{\mathrm{st}}(M', T')$.

8. Stability in the metaplectic group

Our next goal is to define stability for the metaplectic group. We cannot simply generalize the definition we used in the linear case, since $\widetilde{\mathrm{Sp}}(2n)$ is not an algebraic group. The motivation is given by the following fact.

Lemma 8.1. *Let $\widetilde{g}, \widetilde{h} \in \widetilde{\mathrm{Sp}}(2n)$. If $p(\widetilde{g}) \sim_{\mathrm{st}} p(\widetilde{h})$, then $\Theta_{\omega_+ - \omega_-}(\widetilde{g}) = \pm\Theta_{\omega_+ - \omega_-}(\widetilde{h})$.*

Proof. Let $\widetilde{g}$ and $\widetilde{h}$ be as in the statement of the lemma. By [Tho08, Theorem 1C], the character of the oscillator representation satisfies

$$\Theta_{\omega_+ + \omega_-}(\widetilde{g}) = \pm \frac{\gamma(1)^{2n-1}\gamma(\det(g-1))}{|\det(g-1)|^{1/2}}.$$

Therefore, the lemma holds with $\Theta_{\omega_+ + \omega_-}$ in place of $\Theta_{\omega_+ - \omega_+}$, i.e. $\Theta_{\omega_+ + \omega_-}(\widetilde{g}) = \pm\Theta_{\omega_+ + \omega_-}(\widetilde{h})$. Let $x \in \widetilde{\mathrm{Sp}}(2n)$ be a lift of $-I \in \mathrm{Sp}(2n)$. By Corollary 3.3, we have that

$$\Theta_{\omega_+ - \omega_+}(\widetilde{g}) = \lambda^{-1}\, \Theta_{\omega_+ + \omega_-}(x\widetilde{g}) = \pm\lambda^{-1}\, \Theta_{\omega_+ + \omega_-}(x\widetilde{h}) = \pm\Theta_{\omega_+ - \omega_+}(\widetilde{h}). \qquad \square$$

Definition 8.2. Let $\widetilde{g}, \widetilde{h} \in \widetilde{\mathrm{Sp}}(2n)$. We say that $\widetilde{g}$ is stably conjugate to $\widetilde{h}$ if $p(\widetilde{g}) \sim_{\mathrm{st}} p(\widetilde{h})$ and $\Theta_{\omega_+ - \omega_-}(\widetilde{g}) = \Theta_{\omega_+ - \omega_-}(\widetilde{h})$. If $\widetilde{g}, \widetilde{h} \in \widetilde{M}$, then $\widetilde{g}$ is stably conjugate to $\widetilde{h}$ in $\widetilde{M}$ if $p(\widetilde{g}) \sim_{\mathrm{st}} p(\widetilde{h})$ in M and $\Theta_{\omega_+ - \omega_-}(\widetilde{g}) = \Theta_{\omega_+ - \omega_-}(\widetilde{h})$.

Equivalently, $\widetilde{g}$ is stably conjugate to $\widetilde{h}$ in $\widetilde{M}$ if there exist lifts of the elements $\widetilde{g}$ and $\widetilde{h}$ to elements $(\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_k)$ and $(\widetilde{h}_0, \widetilde{h}_1, \ldots, \widetilde{h}_k)$ in $\widetilde{\mathrm{Sp}}(2n_0) \times \widetilde{\mathrm{GL}}(n_1) \times \cdots \times \widetilde{\mathrm{GL}}(n_k)$, such that $\widetilde{g}_0$ is stably conjugate to $\widetilde{h}_0$ in $\widetilde{\mathrm{Sp}}(2n_0)$ and $\widetilde{g}_i$ is conjugate to $\widetilde{h}_i$ in $\widetilde{\mathrm{GL}}(n_i)$ for $i = 1, \ldots, k$.

Note that this definition is non-standard and one could for example use the character of the *sum* of the two halves of the oscillator representation instead of the difference. However, this definition matches our choice of the transfer factor and hence makes the results hold. It also agrees with the definition in the real case (see Adams [Ada98] for details).

Definition 8.3. If π is a representation of $\widetilde{\mathrm{Sp}}(2n)$ with a character Θ_π, then we say that Θ_π is stable if $\Theta_\pi(\widetilde{g}) = \Theta_\pi(\widetilde{h})$ for all $\widetilde{g}, \widetilde{h} \in \widetilde{\mathrm{Sp}}(2n)$ such that $\widetilde{g}$ is stably conjugate to $\widetilde{h}$. If ρ is a representation of $\widetilde{M}$ with a character Θ_ρ, then we say that Θ_ρ is stable if $\Theta_\rho(\widetilde{g}) = \Theta_\rho(\widetilde{h})$ for all $\widetilde{g}, \widetilde{h} \in \widetilde{M}$ such that $\widetilde{g}$ is stably conjugate to $\widetilde{h}$ in $\widetilde{M}$.

Let $\widetilde{T}$ be a Cartan subgroup in $\widetilde{\mathrm{Sp}}(2n)$. Let $T = p(\widetilde{T})$ and let $W_{\mathrm{st}}(\mathrm{Sp}(2n), T)$ be the stable Weyl group of T in $\mathrm{Sp}(2n)$. We define the action of $w \in W_{\mathrm{st}}(\mathrm{Sp}(2n), T)$ on $\widetilde{t} \in \widetilde{T}$ as follows:

$$w \cdot \widetilde{t} = \widetilde{h}, \quad \text{where } p(\widetilde{t}) = w \cdot p(\widetilde{h}) \text{ and } \widetilde{t} \sim_{\mathrm{st}} \widetilde{h}.$$

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To check that this is a group action, note that for any elements $w_1, w_2 \in W_{\mathrm{st}}(\mathrm{Sp}(2n), T)$ and any element $\widetilde{t} \in \widetilde{T}$ we have that $\Theta_{\omega_+ - \omega_-}((w_1 w_2) \cdot \widetilde{t}) = \Theta_{\omega_+ - \omega_-}(\widetilde{t}) = \Theta_{\omega_+ - \omega_-}(w_1 \cdot (w_2 \cdot \widetilde{t}))$.

We will be considering representations ρ' of M' and ρ of $\widetilde{M}$ that satisfy the condition

$$\Theta_\rho(\widetilde{g}) = \Phi_{\widetilde{M}}(\widetilde{g})\Theta_{\rho'}(g'),$$

for all elements $\widetilde{g} \in \widetilde{M}$, $g' \in M'$ such that $p(\widetilde{g}) \overset{M,M'}{\longleftrightarrow} g'$.

LEMMA 8.4. *The characters Θ_ρ and $\Theta_{\rho'}$ are stable on $\widetilde{M}$ and M', respectively.*

Proof. Let $g', h' \in M'$ be strongly regular semisimple elements that are stably conjugate in M'. Choose any element $\widetilde{g} \in \widetilde{\mathrm{Sp}}(2n)$ such that $p(\widetilde{g}) \overset{M,M'}{\longleftrightarrow} g'$. We also have that $p(\widetilde{g}) \overset{M,M'}{\longleftrightarrow} h'$; hence, $\Phi_{\widetilde{M}}(\widetilde{g})\Theta_{\rho'}(g') = \Theta_\rho(\widetilde{g}) = \Phi_{\widetilde{M}}(\widetilde{g})\Theta_{\rho'}(h')$. This implies that $\Theta_{\rho'}(g') = \Theta_{\rho'}(h')$. The proof of the assertion concerning Θ_ρ is analogous. □

Remark. Let $\pi = \mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(2n)} \rho$. Recall the formula (cf. Proposition 5.1) for the character of π:

$$\Theta_\pi(\widetilde{g}) = \sum_{w \in \bigcup W(\mathrm{Sp}(2n),H)/W(M,H)} \Theta_{w\rho}(\widetilde{g}) \frac{|D_{M^w}(g)|^{\frac{1}{2}}}{|D_{\mathrm{Sp}(2n)}(g)|^{\frac{1}{2}}}.$$

In Lemma 7.4, we showed that the Weyl group quotients that appear in this formula are isomorphic to the quotients of the stable Weyl groups. If $w \in W(\mathrm{Sp}(2n), H)/W(M, H)$ and $w_{\mathrm{st}} \in W_{\mathrm{st}}(\mathrm{Sp}(2n), H)/W_{\mathrm{st}}(M, H)$ are matched by this isomorphism, then, since the character Θ_ρ is stable, we have that $\Theta_\rho(w^{-1} \cdot g) = \Theta_\rho(w_{\mathrm{st}}^{-1} \cdot g)$. Therefore, we will replace those quotients without further referring to this isomorphism. The same is true in the orthogonal case.

The following fact is well known for the linear case. However, since the proofs for the non-linear case are similar, we present them both.

PROPOSITION 8.5. *If the character of a representation ρ is stable on M, then the character of the induced representation $\pi = \mathrm{Ind}_P^{\mathrm{Sp}(2n)} \rho$ is stable on $\mathrm{Sp}(2n)$. Analogous statements hold for $\widetilde{\mathrm{Sp}}(2n)$ and $\mathrm{SO}(2n+1)_+$.*

Proof. Let $g, h \in M$ be two strongly regular semisimple elements that are stably conjugate in $\mathrm{Sp}(2n)$. First we will show that there exists $x \in \mathrm{Sp}(2n)$ such that xgx^{-1} is stably conjugate in M to h. We decompose $g = (g_1, g_N)$, where g_1 belongs to $\mathrm{GL}(a+b)$ for some integer $a+b$ and g_N belongs to $T_N \subset M$, which is a product of norm-one tori. Similarly, we decompose $h = (h_1, h_N)$, with $h_1 \in \mathrm{GL}(a+b)$ and $h_N \in H_N$, a product of norm-one tori in M. We have that $x_1 g_1 x_1^{-1} = h_1$ for some $x_1 \in \mathrm{GL}(a+b)$. Let $x = x_1 \times I$. Then $xgx^{-1} = (h_1, g_N)$ is stably conjugate in M to $h = (h_1, h_N)$.

Now we use van Dijk's formula (see Proposition 5.1) to evaluate the character Θ_π at g and h. Let $T = \mathrm{Cent}_{\mathrm{Sp}(2n)}(h_1, g_N)$ and $H = \mathrm{Cent}_{\mathrm{Sp}(2n)}(h)$. We have that

$$\Theta_\pi(g) = \Theta_\pi(xgx^{-1}) = \sum_{s \in W(A,T)} \Theta_{s\rho}(h_1, g_N) \frac{|D_{M^s}(h_1, g_N)|^{\frac{1}{2}}}{|D_{\mathrm{Sp}(2n)}(h_1, g_N)|^{\frac{1}{2}}},$$

$$\Theta_\pi(h) = \sum_{s \in W(A,H)} \Theta_{s\rho}(h_1, h_N) \frac{|D_{M^s}(h_1, h_N)|^{\frac{1}{2}}}{|D_{\mathrm{Sp}(2n)}(h_1, h_N)|^{\frac{1}{2}}}.$$

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The Cartan subgroups T and H have the same split parts; therefore, $W(A,T)=W(A,H)$. Since (h_1,g_N) is stably conjugate in M to (h_1,h_N), we have that the values of the Weyl denominators at these two points are equal. Finally, Θ_ρ is stable on M and hence $\Theta_{s\rho}(h_1,g_N)=\Theta_{s\rho}(h_1,h_N)$. This completes the proof in the linear case.

The proof for the metaplectic case reduces to the proof for the linear case. Let $\widetilde{g},\widetilde{h}\in\widetilde{M}$ be two elements that are stably conjugate in $\widetilde{\mathrm{Sp}}(2n)$. We find an element $x\in\mathrm{Sp}(2n)$ such that $xp(\widetilde{g})x^{-1}$ is stably conjugate in M to $p(\widetilde{h})$. Choose any lift $\widetilde{x}\in\widetilde{\mathrm{Sp}}(2n)$ of x. Then $\Theta_{\omega_+-\omega_-}(\widetilde{x}\widetilde{g}\widetilde{x}^{-1})=\Theta_{\omega_+-\omega_-}(\widetilde{g})=\Theta_{\omega_+-\omega_-}(\widetilde{h})$ and hence the elements $\widetilde{x}\widetilde{g}\widetilde{x}^{-1}$ and $\widetilde{h}$ are stably conjugate in $\widetilde{M}$. The rest of the proof is similar to the linear case. □

9. Parabolic induction

We keep the notation from the previous sections. If ρ is a representation of $\widetilde{M}$, we will denote by $\widetilde{\rho}$ its lift to $\widetilde{\mathrm{Sp}}(2n_0)\times\widetilde{\mathrm{GL}}(n_1)\times\cdots\times\widetilde{\mathrm{GL}}(n_k)$.

Theorem 9.1. *Let ρ be a genuine representation of $\widetilde{M}$ and let ρ' be a representation of M'. We decompose $\widetilde{\rho}=\widetilde{\rho}_0\otimes\cdots\otimes\widetilde{\rho}_k$ and $\rho'=\rho'_0\otimes\cdots\otimes\rho'_k$ accordingly to the decomposition of the Levi factors. If $\Theta_\rho(\widetilde{g})=\Phi_{\widetilde{M}}(\widetilde{g})\Theta_{\rho'}(g')$ whenever $p(\widetilde{g})\overset{M,M'}{\longleftrightarrow}g'$, then:*

(1) $\Theta_{\widetilde{\rho}_0}(\widetilde{g}_0)=\Phi(\widetilde{g}_0)\Theta_{\rho'_0}(g'_0)$, $\widetilde{g}_0\in\widetilde{\mathrm{Sp}}(2n_0)$, $g'_0\in\mathrm{SO}(2n_0+1)_+$, *and* $p(\widetilde{g}_0)\overset{\text{stably}}{\longleftrightarrow}g'_0$;

(2) $\Theta_{\widetilde{\rho}_i}(\widetilde{g}_i)=\chi_i(\widetilde{g}_i)\Theta_{\rho'_i}(g'_i)$, *where* $\chi_i(g_i,\psi_\epsilon)=\epsilon\gamma(\det(g_i))\gamma(1)^{-1}$ *for* $i=1,\ldots,k$.

Proof. This is an immediate consequence of Lemma 4.5. □

Theorem 9.2. *Let ρ be a genuine admissible virtual representation of $\widetilde{M}$ and let ρ' be an admissible virtual representation of M'. We inflate ρ and ρ' to representations of $\widetilde{P}$ and P'. Let $\pi=\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(2n)}\rho$ and $\pi'=\mathrm{Ind}_{P'}^{\mathrm{SO}(2n+1)_+}\rho'$ and denote by Θ_π and $\Theta_{\pi'}$ the characters of these representations. If $\Theta_\rho(\widetilde{x})=\Phi_{\widetilde{M}}(\widetilde{x})\Theta_{\rho'}(x')$ whenever $p(\widetilde{x})\overset{M,M'}{\longleftrightarrow}x'$, then $\Theta_\pi(\widetilde{g})=\Phi(\widetilde{g})\Theta_{\pi'}(g')$ whenever $p(\widetilde{g})\overset{\text{stably}}{\longleftrightarrow}g'$.*

Remarks.

(1) Recall that we work with a fixed additive character η. The transfer factor and the correspondence of representations depend on η.

(2) The characters Θ_ρ and $\Theta_{\rho'}$ are stable (Lemma 8.4).

Proof. Assume that the characters of the representations ρ and ρ' satisfy the assumption of the theorem above. Let $\widetilde{g}\in\widetilde{\mathrm{Sp}}(2n)$ and $g'\in\mathrm{SO}(2n+1)_+$ be such that $g=p(\widetilde{g})\in\mathrm{Sp}(2n)$ and g' are strongly regular semisimple and $g\overset{\text{stably}}{\longleftrightarrow}g'$. Let $\widetilde{T}$ be the centralizer of $\widetilde{g}$ in $\widetilde{\mathrm{Sp}}(2n)$. Let T' be the centralizer of g' in $\mathrm{SO}(2n+1)_+$. If $\widetilde{T}$ (T') is not conjugate to a Cartan subgroup in $\widetilde{M}$ (M'), then the value of the character Θ_π $(\Theta_{\pi'})$ is zero. Therefore, without loss of generality we can assume that $\widetilde{T}$ is a Cartan subgroup in $\widetilde{M}$ and T' is a Cartan subgroup in M'. Then T' is isomorphic to

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the Cartan subgroup $T = p(\widetilde{T})$ in $\mathrm{Sp}(2n)$. By Proposition 5.1, we have

$$\Theta_\pi(\widetilde{g}) = \sum_{s \in W(A,T)} \Theta_{s\rho}(\widetilde{g}) \frac{|D_{M^s}(g)|^{\frac{1}{2}}}{|D_{\mathrm{Sp}(2n)}(g)|^{\frac{1}{2}}},$$

$$\Theta_{\pi'}(g') = \sum_{s' \in W(A',T')} \Theta_{s'\rho'}(g') \frac{|D_{M'^{s'}}(g')|^{\frac{1}{2}}}{|D_{\mathrm{SO}(2n+1)_+}(g')|^{\frac{1}{2}}}.$$

We use the methods of the proof of Lemma 6.6 to choose the representatives $H_1, \ldots, H_m$ of the set $\{H \subset M : H \sim_{\mathrm{Sp}(2n)} T\}/{\sim_M}$ and the representatives $H'_1, \ldots, H'_m$ of the set $\{H' \subset M : H' \sim_{\mathrm{SO}(2n+1)_+} T'\}/{\sim_{M'}}$ to be isomorphic Cartan subgroups. Therefore, we have

$$W(A,T) \cong \bigcup_{i=1,\ldots,m} W(\mathrm{Sp}(2n), H_i)/W(M, H_i),$$

$$W(A',T') \cong \bigcup_{i=1,\ldots,m} W(\mathrm{SO}(2n+1)_+, H'_i)/W(M', H'_i),$$

where H_i and H'_i are isomorphic Cartan subgroups for $i = 1, \ldots, m$. The characters Θ_ρ and $\Theta_{\rho'}$ are stable (cf. Lemma 8.4); therefore, by Lemma 7.4 we can replace the Weyl group quotients with their stabilized versions:

$$W(A,T) \cong \bigcup_{i=1,\ldots,m} W_{\mathrm{st}}(\mathrm{Sp}(2n), H_i)/W_{\mathrm{st}}(M, H_i),$$

$$W(A',T') \cong \bigcup_{i=1,\ldots,m} W_{\mathrm{st}}(\mathrm{SO}(2n+1)_+, H'_i)/W_{\mathrm{st}}(M', H'_i).$$

For each i we choose isomorphisms $\psi_i : H_i \to H'_i$ and

$$\phi_i : W_{\mathrm{st}}(\mathrm{Sp}(2n), H_i)/W_{\mathrm{st}}(M, H_i) \to W_{\mathrm{st}}(\mathrm{SO}(2n+1)_+, H'_i)/W_{\mathrm{st}}(M', H'_i)$$

such that $\psi_i(w \cdot t) = \phi_i(w) \cdot \psi_i(t)$, for $w \in W_{\mathrm{st}}(\mathrm{Sp}(2n), H_i)/W_{\mathrm{st}}(M, H_i)$ and $t \in H_i$ (see Lemma 7.5). If $w \in W_{\mathrm{st}}(\mathrm{Sp}(2n), H_i)/W_{\mathrm{st}}(M, H_i)$, then we will denote its image via the isomorphism ϕ_i by w'. We rewrite the character formula as follows:

$$\Theta_\pi(\widetilde{g}) = \sum_i \sum_w \Theta_{w\rho}(\widetilde{g}) \frac{|D_{M^w}(g)|^{\frac{1}{2}}}{|D_{\mathrm{Sp}(2n)}(g)|^{\frac{1}{2}}},$$

$$\Theta_{\pi'}(g') = \sum_i \sum_{w'} \Theta_{w'\rho'}(g') \frac{|D_{M'^{w'}}(g')|^{\frac{1}{2}}}{|D_{\mathrm{SO}(2n+1)_+}(g')|^{\frac{1}{2}}}.$$

We will compare these formulas term by term. We fix i, and for simplicity of the rest of the proof we assume that $H_i = T$, $H'_i = T'$, $\psi_i = \psi$, etc. Recall that $g = p(\widetilde{g}) \overset{\text{stably}}{\longleftrightarrow} g'$. We can find $w'' \in W_{\mathrm{st}}(\mathrm{SO}(2n+1)_+, T')$ such that $g \overset{M,M'}{\longleftrightarrow} w'' \cdot g'$. Since we average over the stable quotient $W_{\mathrm{st}}(\mathrm{SO}(2n+1)_+, T')/W_{\mathrm{st}}(M', T')$ and since the action of $W_{\mathrm{st}}(M', T')$ does not affect the stable conjugacy classes in M', we can assume without loss of generality that $g \overset{M,M'}{\longleftrightarrow} g'$. Therefore, also $p(w^{-1} \cdot \widetilde{g}) \overset{M,M'}{\longleftrightarrow} w'^{-1} \cdot g'$ and thus $\Theta_{w\rho}(\widetilde{g}) = \Phi_{\widetilde{M}}(w^{-1} \cdot \widetilde{g}) \Theta_{w'\rho'}(g')$. Recall that

$$\Phi_{\widetilde{M}}(w^{-1} \cdot \widetilde{g}) = \frac{|D_{\mathrm{Sp}(2n)}(g)|^{\frac{1}{2}}}{|D_{M^w}(g)|^{\frac{1}{2}}} \frac{|D_{M'^{w'}}(g')|^{\frac{1}{2}}}{|D_{\mathrm{SO}(2n+1)_+}(g')|^{\frac{1}{2}}} \Theta_{\omega_+ - \omega_-}(w^{-1} \cdot \widetilde{g}).$$

Therefore, since $\Theta_{\omega_+ - \omega_-}(w^{-1} \cdot \widetilde{g}) = \Theta_{\omega_+ - \omega_-}(\widetilde{g}) = \Phi(\widetilde{g})$, we have

$$\Theta_{w\rho}(\widetilde{g}) \frac{|D_{M^w}(g)|^{\frac{1}{2}}}{|D_{\mathrm{Sp}(2n)}(g)|^{\frac{1}{2}}} = \Phi(\widetilde{g}) \Theta_{w'\rho'}(g') \frac{|D_{M'^{w'}}(g')|^{\frac{1}{2}}}{|D_{\mathrm{SO}(2n+1)_+}(g')|^{\frac{1}{2}}}. \qquad \square$$

We showed that lifting of representations commutes with parabolic induction. The next step will be to compute lifting of depth-zero supercuspidal representations. Depth is an invariant of an admissible representation and it is a non-negative rational number. It does not change after parabolic induction and the depths of two irreducible admissible representations that are paired by the local theta correspondence are equal. Moreover, depth-zero characters are used to construct other supercuspidal representations of positive depths.

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