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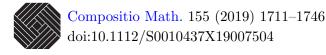
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On some incomplete theta integrals

Jens Funke and Stephen Kudla

Abstract

In this paper we construct indefinite theta series for lattices of arbitrary signature (p,q) as 'incomplete' theta integrals, that is, by integrating the theta forms constructed by the second author with J. Millson over certain singular q-chains in the associated symmetric space D. These chains typically do not descend to homology classes in arithmetic quotients of D, and consequently the theta integrals do not give rise to holomorphic modular forms, but rather to the non-holomorphic completions of certain mock modular forms. In this way we provide a general geometric framework for the indefinite theta series constructed by Zwegers and more recently by Alexandrov, Banerjee, Manschot, and Pioline, Nazaroglu, and Raum. In particular, the coefficients of the mock modular forms are identified with intersection numbers.

1. Introduction

The theory of theta series attached to an integral lattice L in rational quadratic space with bilinear form $(\,,\,)$ of signature (p,q), p, $q > 0$, has a long history including fundamental work by Hecke, Siegel, Maass, and others. In particular, Siegel constructed theta series for such indefinite lattices by using majorants and hence obtained functions depending on both an elliptic modular variable τ and a point $z \in D$, the space of oriented negative q-planes in $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. These Siegel theta series have weight $(p - q)/2$ in τ, but, unlike the classical theta series for positive definite lattices, they are non-holomorphic. In joint work of the second author and John Millson [KM86, KM87, KM90], a family of theta series valued in closed differential forms on D was constructed; we will refer to these as theta forms. The series obtained by passing to classes in the cohomology of the locally symmetric space $\Gamma \backslash D$, where Γ is a subgroup of finite index in the isometry group of L, were shown to be holomorphic modular forms of weight $(p + q)/2$ valued in $H^q(\Gamma \backslash D)$. The resulting theory provides one analogue of the classical holomorphic theta series in the indefinite case.

However, it is still an attractive challenge to define theta series for indefinite lattices more directly by restricting the summation to lattice vectors in suitable subsets $\mathcal{W}$ of V where the quadratic form is positive so that the series

$$\sum_{x \in h+L} \Phi(x; \mathcal{W})\, \mathbf{q}^{Q(x)}, \quad \mathbf{q} = e(\tau) = e^{2\pi i \tau}, \quad Q(x) = \tfrac{1}{2}(x,x), \tag{1.1}$$

is termwise absolutely convergent and hence defines a holomorphic function of τ. Here the coefficient function $\Phi(\cdot; \mathcal{W})$ is supported on $\mathcal{W}$, perhaps with values ± 1 on the interior and with rational values on the boundary. Unfortunately, such series are typically not modular.

In his thesis, Zwegers [Zwe02] introduced a series of this type for V of signature $(m-1, 1)$, where

$$\Phi(x; \mathcal{W}) = \tfrac{1}{2}(\text{sgn}(x, C') - \text{sgn}(x, C)),$$

for C and $C' \in V$ negative vectors in the same component of the cone of negative vectors in V. He showed that the resulting holomorphic series is not modular in general, but that it can be completed to a (non-holomorphic) modular form of weight $m/2$ by adding a suitable series constructed using the error function.

Alexandrov, Banerjee, Manschot and Pioline [ABMP18] proposed a generalization of Zwegers's construction to the case of arbitrary signature $(m-q, q)$ where $\Phi(x; \mathcal{W}) = \Phi_q^\square(x; \mathcal{C})$ is given by

$$\Phi_q^\square(x; \mathcal{C}) = 2^{-q} \prod_{j=1}^{q} (\text{sgn}(x, C_{j'}) - \text{sgn}(x, C_j)), \tag{1.2}$$

for a collection

$$\mathcal{C} = \mathcal{C}^\square = \{\{C_1, C_{1'}\}, \{C_2, C_{2'}\}, \ldots, \{C_q, C_{q'}\}\}$$

of pairs of negative vectors satisfying certain incidence relations. They introduced generalized error functions and, in the case $q = 2$, used them to construct a (non-holomorphic) modular completion of the series (1.1). Shortly thereafter, Nazaroglu [Naz18] handled the case of general signature along the lines suggested in [ABMP18]. In both [ABMP18] and [Naz18], the modularity of the non-holomorphic completion is established by using a result of Vignéras [Vig77] which asserts the modularity of theta-like series built from a certain class of functions. The essential step is to show that suitable combinations of generalized error functions define functions in this class and, at the same time, are suitably linked to the function $\Phi_q^\square(\cdot, \mathcal{C})$. Sums of lattice vectors in more general positive polyhedral cones were considered by Westerholt-Raum [Wes16]; he again uses the Vignéras criterion to deduce modularity. Both [ABMP18] and [Wes16] discuss some degenerate cases where edges of the cone are allowed to be rational isotropic vectors.

In this paper we show that the indefinite theta series of [Zwe02, ABMP18, Naz18] can be obtained by integrating the theta forms for V of signature (p, q) over certain singular q-cubes determined by a collection $\mathcal{C}$ which is in 'good position'. We also consider the analogous integrals over singular simplices, where the input data is now a collection $\mathcal{C} = \mathcal{C}^\triangle = \{C_0, C_1, \ldots, C_q\}$ of negative vectors in V in 'good position'. In particular, any q of them span a negative q-plane in V and these q-planes give the vertices of a singular simplex in D.

We refer to such integrals as incomplete theta integrals. The idea is that, since the theta forms are closed q-forms invariant under an arithmetic group Γ associated to the given lattice, it is most natural to consider their integrals over q-cycles in the quotient $\Gamma \backslash D$. As explained above, such integrals[1] produce holomorphic modular forms whose Fourier coefficients have a cohomological interpretation. The integrals over more general q-chains, for example, those arising from singular q-cubes or q-simplicies, can be viewed as 'incomplete' versions. The situation is analogous to the relation between the classical elliptic integrals, which are periods of holomorphic 1-forms, and their incomplete versions, which are integrals of such 1-forms over more general arcs.

In any case, our result shows that the holomorphic theta series of [KM86, KM87, KM90] and the indefinite theta series of [ABMP18, Naz18] have a common source.

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[1] Over compact cycles.

To state our results more precisely, we need some notation. Let L be an even integral lattice in V with dual lattice $L^\vee$. For $\tau = u + iv \in \mathfrak{H}$ and $\mu \in L^\vee/L$, the theta form is the closed Γ_L-invariant q-form on D given by

$$\theta_\mu(\tau; \varphi_{\mathrm{KM}}) = \sum_{x \in \mu + L} \varphi_{\mathrm{KM}}(\tau, x).$$

Here the Schwartz form

$$\varphi_{\mathrm{KM}}(\tau, x) = v^{-(p+q)/4} \left(\omega(g_\tau') \varphi_{\mathrm{KM}} \right)(x)$$

is obtained by the action $\omega(g_\tau')$ of the Weil representation on the basic Schwartz form $\varphi_{\mathrm{KM}}(x)$; cf. §2.2. A precise formula for $\varphi_{\mathrm{KM}}(x)$ is given in §5.

First consider the 'cubical' case. For a collection $\mathcal{C} = \mathcal{C}^\square$ of q pairs of negative vectors, we can define a q-tuple of vectors

$$B(s) = [(1 - s_1)C_1 + s_1 C_{1'}, \ldots, (1 - s_q)C_q + s_q C_{q'}] \in V^q,$$

for each $s = [s_1, \ldots, s_q] \in [0, 1]^q$. We say that $\mathcal{C}$ is in good position if the collection $B(s)$ spans a negative q-plane for all $s \in [0, 1]^q$. If $\mathcal{C}$ is in good position, we obtain an oriented singular q-cube

$$\rho_{\mathcal{C}} : [0, 1]^q \longrightarrow D, \quad s \mapsto \mathrm{span}\{B_1(s_1), \ldots, B_q(s_q)\}_{\mathrm{p.o.}},$$

where the subscript 'p.o.' indicates that the given ordered q-tuple defines the orientation. Let $S(\mathcal{C}^\square)$ be the resulting singular q-cube.

Next consider the simplicial case. In this case, we suppose that the set of vectors $\mathcal{C} = \mathcal{C}^\triangle$ is linearly independent over $\mathbb{R}$ and that any q of them span a negative q-plane. Their span U is an oriented $(q + 1)$-plane of signature $(1, q)$ and the dual basis $\mathcal{C}^\vee = \{C_0^\vee, \ldots, C_q^\vee\}$ consists of positive vectors. We say that $\mathcal{C}$ is in good position if, for all

$$s = [s_0, \ldots, s_q] \in \Delta^q = \left\{ s \in [0, 1]^{q+1} \,\bigg|\, \sum_{i=0}^q s_i = 1 \right\},$$

the vector

$$C^\vee(s) = \sum_i s_i \, C_i^\vee$$

is positive. For example, it suffices to require that all entries of the Gram matrix $((C_i^\vee, C_j^\vee))$ are non-negative.[2] For $\mathcal{C}$ in good position, we obtain a map

$$\rho_{\mathcal{C}} : \Delta^q \longrightarrow D, \quad s \mapsto C^\vee(s)^\perp,$$

where the $\perp$ is taken in U and the orientation of $\rho_{\mathcal{C}}(s)$ is determined by the normal vector $C^\vee(s)$. We write $S(\mathcal{C}^\triangle)$ for the resulting singular simplex. We also define

$$\Phi_q^\triangle(x; \mathcal{C}) = 2^{-q-1} \left(\prod_{j=0}^q (1 - \mathrm{sgn}(x, C_j)) + (-1)^q \prod_{j=0}^q (1 + \mathrm{sgn}(x, C_j)) \right). \tag{1.3}$$

For $S(\mathcal{C}) = S(\mathcal{C}^\square)$ or $S(\mathcal{C}^\triangle)$, we consider the theta integral

$$I_\mu(\tau; \mathcal{C}) = \int_{S(\mathcal{C})} \theta_\mu(\tau; \varphi_{\mathrm{KM}}). \tag{1.4}$$

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[2] This was pointed out to the second author by Sander Zwegers at the Dublin Conference in June 2017.

Note that, by construction, $I_\mu(\tau; \mathcal{C})$ is a (typically non-holomorphic) modular form of weight $(p+q)/2$ with transformation law inherited from that of the theta form.

Our explicit formulas involve the generalized error function introduced in [ABMP18] which is defined as follows. For $1 \leqslant r \leqslant q$ and for a collection of vectors $\boldsymbol{c} = \{c_1, \ldots, c_r\}$ spanning an oriented negative r-plane z in V, let

$$E_r(x; \boldsymbol{c}) = \int_z e^{\pi(y - \mathrm{pr}_z(x), y - \mathrm{pr}_z(x))} \, \mathrm{sgn}(y; \boldsymbol{c}) \, dy, \tag{1.5}$$

where $x \in V$, $\mathrm{pr}_z(x)$ is the projection of x onto z,

$$\mathrm{sgn}(y; \boldsymbol{c}) = \mathrm{sgn}(y, c_1) \, \mathrm{sgn}(y, c_2) \cdots \mathrm{sgn}(y, c_r), \tag{1.6}$$

and the measure dy is normalized so that

$$\int_z e^{\pi(y,y)} \, dy = 1.$$

Note that $E_r(x; \boldsymbol{c})$ is a C^∞-function of $x \in V$; cf. [ABMP18, § 6.1]. In various inductive arguments, it will be convenient to let $E_0(x; \boldsymbol{c}) = 1$.

Finally, for $x \in V$, $x \neq 0$, let

$$D_x = \{z \in D \mid x \perp z\},$$

and note that, if $Q(x) > 0$, then D_x is a totally geodesic sub-symmetric space in D of codimension q. Otherwise, D_x is empty.

Our main result is the following theorem.

Main Theorem. *Assume that $\mathcal{C}$ is in good position and let $\Phi_q(x; \mathcal{C})$ be $\Phi_q^\square(x; \mathcal{C})$ (respectively, $\Phi_q^\triangle(x; \mathcal{C})$) in the cubical (respectively, simplicial) case.*

(i) *The series*

$$\sum_{x \in \mu + L} \Phi_q(x; \mathcal{C}) \, \mathbf{q}^{Q(x)} \tag{1.7}$$

is termwise absolutely convergent and hence defines a holomorphic function of τ.

(ii) *If $\Phi_q(x; \mathcal{C}) \neq 0$, then*

$$D_x \cap S(\mathcal{C}) = \rho_{\mathcal{C}}(s(x))$$

for a unique point $s(x) \in [0,1]^q$ (respectively, Δ^q), the map $\rho_{\mathcal{C}}$ is immersive at $s(x)$, and

$$\Phi_q(x; \mathcal{C}) = I(D_x, S(\mathcal{C}))$$

is the intersection number[3] of D_x and $S(\mathcal{C})$ at $\rho_{\mathcal{C}}(s(x))$.

(iii) *In the cubical case, the theta integral (1.4) is given explicitly by*

$$I_\mu(\tau; \mathcal{C}^\square) = \sum_{x \in \mu + L} (-1)^q \, 2^{-q} \sum_I (-1)^{|I|} \, E_q(x\sqrt{2v}; C^I) \, \mathbf{q}^{Q(x)}, \tag{1.8}$$

where for a subset $I \subset \{1, \ldots, q\}$, C^I is the q-tuple with $C_j^I = C_j$ if $j \notin I$ and $C_j^I = C_j'$ if $j \in I$, ordered by the index j.

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[3] If $s(x)$ is on the boundary of $[0,1]^q$, this quantity is defined in (A.1) in the Appendix.

(iv) *In the simplicial case, the theta integral (1.4) is given by*

$$I_\mu(\tau; \mathcal{C}^\triangle) = \sum_{x \in \mu + L} (-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \\ |I|=2r+1}} E_{q-2r}(x\sqrt{2v}; \mathcal{C}^{(I)}) \, \mathbf{q}^{Q(x)}, \qquad (1.9)$$

where, for a subset $I \subset \{0, 1, \ldots, q\}$, $\mathcal{C}^{(I)}$ is the collection of $q + 1 - |I|$ elements obtained from $\mathcal{C}^\triangle$ by omitting the C_j for $j \in I$. Here $E_0(\cdots) = 1$.

We can view $I_\mu(\tau; \mathcal{C})$ as a *modular completion* of the series (1.7) in the sense that taking the limit $v \to \infty$ termwise in the Fourier coefficients of (1.8) (respectively, (1.9)) yields $\Phi_q(x; \mathcal{C})$; that is,

$$(-1)^q 2^{-q} \lim_{v \to \infty} \sum_I (-1)^{|I|} E_q(x\sqrt{2v}; C^I) = \Phi_q^\square(x; \mathcal{C}),$$

and the same for $\Phi_q^\triangle(x; \mathcal{C})$.

We also note that the mock modular forms $\sum_{x \in \mu + L} \Phi_q(x; \mathcal{C}) \, \mathbf{q}^{Q(x)}$ we are considering do *not* in general involve taking Γ-orbits of lattice vectors in a cone and hence are structurally different than the classical cases mentioned above which arise as 'complete' theta integrals.

Remark 1.1. (1) The series on the right-hand side of (1.8) coincides with that in [ABMP18] and [Naz18], at least when the collection $\mathcal{C}$ satisfies their incidence conditions. The incidence conditions they impose on $\mathcal{C}$ (i.e., conditions expressed as requirements on the entries of the Gram matrix of $\mathcal{C}$) imply that $\mathcal{C}$ is in good position. On the other hand, the 'good position' condition, which is a condition on the Gram matrix of the collection $B(s)$ for all $s \in [0,1]^q$, is sufficient for our results. We leave aside the, perhaps subtle, problem of expressing this condition on $B(s)$ in terms of necessary and sufficient conditions on the Gram matrix of $\mathcal{C}$.

(2) Part (ii) of the theorem provides a geometric interpretation of the coefficients of the holomorphic generating series as intersection numbers. It would be interesting to see if this interpretation has any significance in the physics context which was the original motivation for [ABMP18].

(3) The proof of (i) is already given in the general case in [Kud18]. That the right-hand side of (1.8) is a modular completion of the series (1.7) is, of course, a main result of [Zwe02, ABMP18, Naz18].

(4) It is interesting that generalized error functions for negative r-planes with $r < q$ occur in the explicit formula in the simplicial case. This phenomenon was pointed out by Westerholt-Raum for more general cones [Wes16]. The indefinite theta series associated to collections $\mathcal{C}^\triangle$ were also discussed by Zwegers in his talk at the Dublin conference on indefinite theta functions in June 2017.

Since the theta integral (1.4) can be computed termwise, the formulas of parts (iii) and (iv) follow immediately from the formulas for the integral of $\varphi_{\mathrm{KM}}(x)$ over $S(\mathcal{C})$ given in Theorems 4.1 and 8.3, respectively. These results are, in turn, proved by induction on q, where the case $q = 1$ is an elementary calculation. The key points are the following. First note that both sides of the identities in Theorems 4.1 and 8.3 are smooth functions of x and $\mathcal{C}$, so that it suffices to consider the case where x is regular with respect to $\mathcal{C}$, that is, where $(x, C) \neq 0$ for all $C \in \mathcal{C}$. As already noted in [FK17], the Schwartz form $\varphi_{\mathrm{KM}}(x)$ comes equipped with an explicit primitive $\Psi(x)$, defined on the set $D - D_x$. Taking care of the possible singularity, which under the regularity

assumption occurs at most at a unique interior point of $S(\mathcal{C})$, we can apply Stokes's theorem. The boundary of $S(\mathcal{C})$ consists of singular $(q-1)$-cubes (respectively, simplices) in totally geodesic subsymmetric spaces of the form

$$D'_y = \{z \in D \mid y \in z\}$$

for $y = C_j$ or C'_j in $\mathcal{C}$. Note that D'_y will then be isomorphic to the space of oriented negative $(q-1)$-planes in the space $V_y = y^\perp$, of signature $(p, q-1)$. Now the crucial (and remarkable!) fact is that the pullback of the primitive $\Psi(x)$ to such a subspace D'_y can be written as an integral transform of the Schwartz $(q-1)$-form $\varphi_{\mathrm{KM}}^{V_y}(\mathrm{pr}_{V_y} x)$ for V_y; cf. Proposition 6.3. By induction, we obtain an expression for the boundary integral as a sum of the corresponding signature $(p, q-1)$ theta integrals. Finally, we invoke an inductive identity for generalized error functions from [Naz18], Proposition 7.3, to conclude the proof.

Remark 1.2. (1) One can consider the theta integral $I(\tau; S)$ over any oriented q-chain S in D, and, if S is compact, this can again be computed termwise. If, moreover, the boundary of S consists of $q-1$ chains lying in D'_y, one can proceed by induction. In particular, our result gives an explicit formula for any q-chain written as a sum of simplices of the form $S(\mathcal{C}^\triangle)$. Moreover, since the theta forms are Γ_L-invariant, their integrals over Γ_L equivalent q-chains coincide.

(2) We can also consider the theta integral $I(\tau; \mathcal{C})$ in the degenerate case, when some of the elements in $\mathcal{C}$ are rational isotropic vectors. Geometrically, this amounts to the q-chain $S(\mathcal{C})$ going out to some of the rational cusps (of the arithmetic quotient) of D. However, while the theta integral over the non-compact region $S(\mathcal{C})$ still is convergent by the results of [FM13] (for signature $(m-1, 1)$, see [FM02]), it is in general no longer termwise absolutely convergent (unless one imposes a 'non-singularity' condition as in [Kud81]; see also [Wes16]). One interesting example is signature $(1, 2)$, where one can realize the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ as a surface $S(\mathcal{C})$ for a certain $\mathcal{C}$, and the associated theta integral $I(\tau; \mathcal{C})$ gives Zagier's non-holomorphic Eisenstein series of weight $3/2$; see [Fun02, BF06].

(3) In the companion paper [FK17], we consider the theta integral $\int_D \eta \wedge \theta_\mu(\tau; \varphi_{\mathrm{KM}})$ against a compactly supported $(p-1)q$ differential form η on D. In particular, we establish the properties of the primitive $\Psi(x)$ as a current on D.

Our construction yields a formula for the image of the (typically non-holomorphic) modular form $I_\mu(\tau; \mathcal{C})$ under the lowering operator $-2iv^2(\partial/\partial\bar\tau)$ and hence for its shadow, its image under the operator $\xi_k = 2iv^k\overline{(\partial/\partial\bar\tau)}$, with $k = \frac{1}{2}(p+q)$. This formula implies the following corollary; cf. § 9.

COROLLARY 1.3. *Suppose that $\mathcal{C}$ is rational collection (i.e., that $C \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $C \in \mathcal{C}$). Then the shadow of $I_\mu(\tau; \mathcal{C})$ is a linear combination of products of unary theta series of weight $\frac{3}{2}$ and complex conjugates of indefinite theta series for the spaces $V_C = C^\perp$ for $C \in \mathcal{C}$.*

Here is an outline of the contents of the various sections. Section 2 contains an overview of the construction of theta forms, their modular transformation properties, and their relation to geodesic cycles. There is considerable overlap with the material in [FK17], although our notation and perspective here differ somewhat. Section 3 explains the singular q-cubes associated to collections $\mathcal{C}$ in good position and their intersection with the cycles D_x in the regular case. It should be noted that the role of the symmetric space D and the singular q-cubes is not so evident in [ABMP18, Naz18]. The use of the 'good position' condition streamlines the treatment,

although the important problem of finding equivalent incidence relations is left open. The explicit formula for the 'cubical' integrals of $\varphi_{\mathrm{KM}}(x)$ is given in Theorem 4.1 of §4. In §5 we give a more detailed discussion of the Schwartz forms φ_{KM} and their primitives. In §6 we prove the key formulas for the pullbacks of these forms to the spaces D'_y. Section 7 contains the proof of Theorem 4.1. Section 8 contains the computation of the shadows. Section 9 contains the analogous computations in the simplicial case, where the are several crucial and interesting differences. Some technical details are provided in the Appendix.

1.1 Notation

For vectors x and y in a non-degenerate inner product space V, $(\,,\,)$ with $Q(y) = \frac{1}{2}(y, y) \neq 0$, we write

$$x_{\perp y} = x - \frac{(x, y)}{(y, y)}y. \qquad (1.10)$$

Note that

$$(x_{\perp y}, x'_{\perp y}) = (x, x') - \frac{(x, y)(x', y)}{(y, y)}. \qquad (1.11)$$

For a non-degenerate subspace z in V, we write pr_z for the orthogonal projection to z.

We write $e(x) = e^{2\pi i x}$.

For our collection of vectors

$$\mathcal{C} = \mathcal{C}^{\square} = \{\{C_1, C_{1'}\}, \{C_2, C_{2'}\}, \ldots, \{C_q, C_{q'}\}\}$$

we are following the convention of [ABMP18] and writing $C_{j'}$ for the second vector of the jth pair. This is convenient as it allows us, for example, to write $C_{i \perp j'}$ for the projection of C_i to the orthogonal complement of $C_{j'}$.

2. Theta forms and their integrals

2.1 Preliminaries

We begin by reviewing some standard notation and constructions. A good reference is [Shi75]. Suppose that L, $(\,,\,)$ is a lattice of rank $m = p + q$ with an even integral symmetric bilinear form of signature (p, q) with p, $q > 0$. Let $L^{\vee} \supset L$ be the dual lattice and set $Q(x) = \frac{1}{2}(x, x)$. Let $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ and let $G = O(V)$ be the orthogonal group of V. Let

$$\Gamma_L = \{\gamma \in G \mid \gamma L = L, \ \gamma|_{L^{\vee}/L} = \mathrm{id}\}.$$

We denote by $\mathrm{Gr}_q^o(V)$ the Grassmannian of *oriented* q-planes in V and let

$$D = D(V) = \{z \in \mathrm{Gr}_q^o(V) \mid (\,,\,)|_z < 0\}$$

be the space of oriented negative q-planes in V. For $z \in D$, the associated Gaussian is

$$\varphi_0(x, z) = e^{-\pi(x, x)_z},$$

where, for $R(x, z) = -(\mathrm{pr}_z(x), \mathrm{pr}_z(x))$,

$$(x, x)_z = (x, x) + 2R(x, z)$$

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is the majorant determined by z. For fixed z, $\varphi_0(\cdot, z) = \varphi_0(z) \in \mathcal{S}(V)$ is a Schwartz function on V, while, for fixed $x \in V$, $\varphi_0(x, \cdot) = \varphi_0(x) \in A^0(D)$ is a smooth function on D satisfying the equivariance

$$\varphi_0(gx, gz) = \varphi_0(x, z)$$

for $g \in G$, or equivalently

$$g^* \varphi_0(x) = \varphi_0(g^{-1}x) =: \omega(g)\varphi_0(x),$$

where g^* denotes the pullback of functions on D and $\omega(g)$ denotes the action of g on $\mathcal{S}(V)$. Thus

$$\varphi_0 \in [\mathcal{S}(V) \otimes A^0(D)]^G. \tag{2.1}$$

The action ω of G on $\mathcal{S}(V)$ commutes with the Weil representation[4] action of the two-fold cover $G' = \mathrm{Mp}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{S}(V)$, and hence there is a representation of $G \times G'$ on this space, which we also denote by ω. Recall that for $b \in \mathbb{R}$ and $a \in \mathbb{R}^\times$, there are elements $n'(b)$, $m'(a)$, and w' in G' projecting to

$$n(b) = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}, \quad m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad \text{and} \quad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

in $\mathrm{SL}_2(\mathbb{R})$ whose Weil representation action is given by

$$\omega(n'(b))\varphi(x) = e(bQ(x))\,\varphi(x),$$
$$\omega(m'(a))\varphi(x) = |a|^{m/2}\varphi(ax),$$
$$\omega(w')\varphi(x) = e\left(\frac{p-q}{8}\right)\hat{\varphi}(x) = e\left(\frac{p-q}{8}\right)\int_V \varphi(y)\,e(-(x,y))\,dy.$$

Then, for $\tau = u + iv \in \mathfrak{H}$ and $g'_\tau = n'(u)m'(v^{1/2})$, we have

$$\omega(g'_\tau)\varphi_0(x, z) = v^{(p+q)/4}e^{-2\pi v R(x,z)}\,\mathbf{q}^{Q(x)}, \quad \mathbf{q} = e(\tau) = e^{2\pi i\tau}.$$

The following invariance property gives rise to the modularity of the theta series. Define a vector-valued tempered distribution

$$\Theta_L : \mathcal{S}(V) \longrightarrow \mathbb{C}[L^\vee/L], \quad \varphi \mapsto \Theta(\varphi; L) = \sum_{\mu \in L^\vee/L} \theta_\mu(\varphi)\,e_\mu,$$

where $e_\mu \in \mathbb{C}[L^\vee/L]$ is the characteristic function of the coset $\mu + L$ and

$$\theta_\mu(\varphi) = \sum_{x \in \mu+L} \varphi(x).$$

Let Γ' be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in G'. Then there is a finite Weil representation ρ_L of Γ' acting on $\mathbb{C}[L^\vee/L]$, and the theta distribution Θ_L satisfies

$$\Theta_L(\omega(\gamma')\varphi) = \rho_L(\gamma')\Theta_L(\varphi).$$

Let K' be the inverse image of $\mathrm{SO}(2)$ in G', and suppose that φ is an eigenfunction of weight $\ell \in \frac{1}{2}\mathbb{Z}$ for the Weil representation action of K', that is,

$$\omega(k'_\theta)\varphi = e(\ell\theta)\,\varphi, \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

for a suitable preimage k'_θ of k_θ in G'.

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[4] Of course, if $m = \dim V$ is even, this representation factors through $\mathrm{SL}_2(\mathbb{R})$ and we can dispense with the metaplectic group.

Then the invariance of the theta distribution together with a standard calculation [Shi75, pp. 90–98] implies that the $\mathbb{C}[L^\vee/L]$-valued theta series

$$\sum_{\mu \in L^\vee/L} \theta_\mu(\tau; \varphi) \, e_\mu = v^{-\ell/2} \, \Theta_L(\omega(g'_\tau)\varphi)$$

is a (non-holomorphic) vector-valued modular form of weight ℓ and type $(\rho_L, \mathbb{C}[L^\vee/L])$.

The Gaussian φ_0 is an eigenfunction of K' of weight $(p-q)/2$ so that the Siegel theta series $\theta_\mu(\tau, z; \varphi_0)$ are components of vector-valued modular forms and, moreover, via equivariance (2.1), are Γ_L-invariant as functions of z, that is,

$$\theta_\mu(\tau, \cdot; \varphi_0) \in A^0(D)^{\Gamma_L}.$$

For this semi-classical reformulation of Weil's construction of theta functions we are following Shintani [Shi75]; cf. also [BF04].

2.2 Theta forms

The basic idea is to replace equivariant families of Schwartz functions by equivariant families of Schwartz forms, that is, Schwartz functions valued in differential forms on D. Let $A^r(D)$ be the space of smooth r-forms on D. A main result of [KM86, KM87] is the explicit construction of a family of Schwartz forms

$$\varphi_{\mathrm{KM}} \in [\mathcal{S}(V) \otimes A^q(D)]^G. \tag{2.2}$$

Thus, for $x \in V$ and $g \in G$,

$$g^* \varphi_{\mathrm{KM}}(x) = \varphi_{\mathrm{KM}}(g^{-1}x) \in A^q(D).$$

In particular, for fixed $x \in V$, $\varphi_{\mathrm{KM}}(x)$ is a G_x-invariant q-form on D. For example, $\varphi_{\mathrm{KM}}(0)$ is a G-invariant form. Under K',

$$\omega(k'_\theta)\varphi_{\mathrm{KM}} = e\left(\frac{p+q}{2}\,\theta\right)\varphi_{\mathrm{KM}}.$$

Note the shift in weight! Moreover, the q-form $\varphi_{\mathrm{KM}}(x)$ is closed,

$$d\varphi_{\mathrm{KM}} = 0,$$

where $d : A^q(D) \to A^{q+1}(D)$ is the exterior derivative.

Define the *theta form*

$$\theta_\mu(\tau; \varphi_{\mathrm{KM}}) := v^{-(p+q)/4}\theta_\mu(\omega(g'_\tau)\varphi_{\mathrm{KM}}).$$

Then, by construction, $\theta_\mu(\tau; \varphi_{\mathrm{KM}})$ is a closed Γ_L-invariant q-form on D and hence defines a closed q-form on the (orbifold) quotient $M_L = [\Gamma_L \backslash D]$. Moreover, as a function of τ, $\theta_\mu(\tau; \varphi_{\mathrm{KM}})$ is a component of a (non-holomorphic) modular form of weight $(p+q)/2$ and type $(\rho_L, \mathbb{C}[L^\vee/L])$.

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2.3 Relation to geodesic cycles

To avoid orientation issues, we take $\Gamma \subset \Gamma_L$ to be a neat subgroup of finite index. The theta forms define cohomology classes for the locally symmetric space $M_\Gamma = \Gamma \backslash D$ which are related to totally geodesic cycles. This was the original motivation for their construction. We briefly recall the basic facts. For $x \in V$ with $x \neq 0$, let $V_x = x^\perp$, and let

$$D_x = \{ z \in D \mid R(x, z) = 0, \text{i.e.}, z \subset V_x \}.$$

In particular, D_x is empty if $Q(x) \leqslant 0$, and is a totally geodesic sub-symmetric space of codimension q if $Q(x) > 0$.

Let $\mathrm{pr}_\Gamma : D \to \Gamma \backslash D = M_\Gamma$ and, for x with $Q(x) > 0$, let

$$Z(x) = \mathrm{pr}_\Gamma(D_x),$$

a totally geodesic codimension q-cycle in M_Γ with an immersion

$$i_x : \Gamma_x \backslash D_x \longrightarrow Z(x) \subset \Gamma \backslash D.$$

Notice that $Z(x)$ depends only on the Γ-orbit of x.

The following results are special cases of those obtained in [KM86, KM87, KM90].

(i) Suppose that η is a closed and compactly supported $(p-1)q$-form on M_Γ. Then

$$\int_{M_\Gamma} \eta \wedge \theta_\mu(\tau; \varphi_{\mathrm{KM}}) = \int_{M_\Gamma} \eta \wedge \varphi_{\mathrm{KM}}(0) + \sum_{\substack{x \in \mu + L \\ Q(x) > 0 \\ \bmod \Gamma}} \left(\int_{Z(x)} \eta \right) \mathbf{q}^{Q(x)}.$$

(ii) Suppose that S is a compact closed (i.e., $\partial S = 0$) oriented q-cycle on M_Γ. Then

$$\int_S \theta_\mu(\tau; \varphi_{\mathrm{KM}}) = \int_S \varphi_{\mathrm{KM}}(0) + \sum_{\substack{x \in \mu + L \\ Q(x) > 0 \\ \bmod \Gamma}} I(Z(x), S) \, \mathbf{q}^{Q(x)},$$

where $I(Z(x), S)$ is the intersection number of the cycles $Z(x)$ and S.

In particular, both series are termwise absolutely convergent and define *holomorphic* modular forms of weight $(p+q)/2$.

Note that these results exactly fit into the framework of (1.1). Additional discussion is given in [FK17]. Many interesting variations are possible! For example, the case of certain non-compact cycles S in M_Γ is considered in joint work of the first author with John Millson [FM02, FM11, FM14].

2.4 Non-closed compact cycles

Suppose that S is a piecewise smooth compact oriented q-chain in the symmetric space D. Then, from the general machinery sketched in the previous sections, we obtain (non-holomorphic) modular forms, which we will refer to as indefinite theta series, or incomplete theta integrals, as explained in the introduction,

$$I_\mu(\tau; S) := \int_S \theta_\mu(\tau; \varphi_{\mathrm{KM}}) \tag{2.3}$$

of weight $m/2$. Since S is compact, we can compute such integrals termwise. Define an operator

$$I_S : \mathcal{S}(V) \otimes A^q(D) \longrightarrow \mathcal{S}(V), \quad \varphi \mapsto \int_S \varphi \tag{2.4}$$

from Schwartz forms to Schwartz functions by integrating out the form part. This operator commutes with the Weil representation action of G'. Thus, we have

$$
\begin{aligned}
I_\mu(\tau; S) &= \int_S v^{-(p+q)/4} \theta_\mu(\omega(g'_\tau)\varphi_{\mathrm{KM}}) \\
&= v^{-(p+q)/4} \theta_\mu(\omega(g'_\tau) I_S(\varphi_{\mathrm{KM}})),
\end{aligned}
$$

so that the indefinite theta series (2.3) is just the theta series defined by the Schwartz function $I_S(\varphi_{\mathrm{KM}})$. We obtain explicit formulas for the modular forms $I_\mu(\tau; S)$ whenever we can compute the Schwartz function

$$
I_S(\varphi_{\mathrm{KM}}) \in \mathcal{S}(V) \tag{2.5}
$$

for a given q-chain S.

The remainder of this paper is devoted to the explicit computation of the Schwartz functions $I_S(\varphi_{\mathrm{KM}})$ in two cases: (1) the case of the singular q-cubes defined in the next section; and (2) the case of singular q-simplices defined in § 8.

3. Singular q-cubes

The data[5] $\mathcal{C} = \mathcal{C}^\square$ introduced in [ABMP18, § 6] and recalled in (3.1) below determines a singular q-cube $S(\mathcal{C})$ in D, whose geometry we discuss in this section. We give an explicit formula in terms of generalized error functions for the integral (2.5) in the case when $S = S(\mathcal{C})$ for $\mathcal{C}$ in 'good position'.

3.1 The singular q-cube $S(\mathcal{C})$ and its faces
Let

$$
\mathcal{C} = \mathcal{C}^\square = \{\{C_1, C_{1'}\}, \{C_2, C_{2'}\}, \ldots, \{C_q, C_{q'}\}\} \tag{3.1}
$$

be a collection of q pairs of negative vectors in V. For a subset $I \subset \{1, \ldots, q\}$, let C^I be the ordered set $\{C_1^I, \ldots, C_q^I\}$ of q vectors where we take $C_j^I = C_j$ if $j \notin I$ and $C_j^I = C_{j'}$ if $j \in I$. The vectors are ordered according to the index j. Thus, $C^\emptyset = \{C_1, \ldots, C_q\}$, etc. We would like to have the following 'incidence relations'.

(Inc-1) Each collection C^I spans an oriented negative q-plane

$$
z^I = \mathrm{span}\{C^I\}_{\mathrm{p.o.}}.
$$

(Inc-2) The oriented negative q-planes z^I all lie on the same component of D.

These relations, which can be achieved by imposing conditions on the determinants of minors of Gram matrices, should allow us to construct a singular q-cube with the points z^I as the vertices. However, as already seen in [Kud18], it will be more convenient to work with the following formalism.

For $s = [s_1, \ldots, s_q] \in [0,1]^q$, let

$$
B(s) = [B_1(s_1), \ldots, B_q(s_q)],
$$

where

$$
B_j(s_j) = (1 - s_j)C_j + s_j C_{j'}.
$$

---

[5] We abbreviate $\mathcal{C}^\square$ to $\mathcal{C}$ unless it is useful to emphasize the case at hand.

Definition 3.1. A collection $\mathcal{C}$ is said to be in *good position* if, for all $s \in [0,1]^q$,

$$\text{span}\{B(s)\}_{\text{p.o.}} = \text{span}\{B_1(s_1), \ldots, B_q(s_q)\}_{\text{p.o.}} \in D.$$

If $\mathcal{C}$ is in good position, then relations (Inc-1) and (Inc-2) hold, and we obtain an oriented singular q-cube

$$\rho_{\mathcal{C}} : [0,1]^q \longrightarrow D, \quad s = [s_1, \ldots, s_q] \mapsto \text{span}\{B_1(s_1), \ldots, B_q(s_q)\}_{\text{p.o.}} \in D \tag{3.2}$$

with the z^I as its vertices. Let $S(\mathcal{C}) = \rho_{\mathcal{C}}([0,1]^q)$ be its image in D. Note that the most degenerate case, in which $C_j = C_{j'}$ for all j and $S(\mathcal{C})$ is a point in D, is allowed.

From now on, unless stated otherwise, we assume that $\mathcal{C}$ is in good position, so that $\rho_{\mathcal{C}}$ and $S(\mathcal{C})$ are defined.

As in [Mas91], we define the front j-face

$$\alpha_j \rho_{\mathcal{C}} : [0,1]^{q-1} \longrightarrow D, \quad \alpha_j \rho_{\mathcal{C}}(s_1, \ldots, s_{q-1}) = \rho_{\mathcal{C}}(s_1, \ldots, s_{j-1}, 0, s_j, \ldots, s_{q-1}),$$

and back j-face

$$\beta_j \rho_{\mathcal{C}} : [0,1]^{q-1} \longrightarrow D, \quad \beta_j \rho_{\mathcal{C}}(s_1, \ldots, s_{q-1}) = \rho_{\mathcal{C}}(s_1, \ldots, s_{j-1}, 1, s_j, \ldots, s_{q-1}).$$

We write $\partial_j^+ S(\mathcal{C})$ (respectively, $\partial_j^- S(\mathcal{C})$) for the image of $\alpha_j \rho_{\mathcal{C}}$ (respectively, $\beta_j \rho_{\mathcal{C}}$), viewed as an oriented $(q-1)$-cube. With this convention, the boundary of the oriented q-cube $S(\mathcal{C})$ is given by

$$\partial S(\mathcal{C}) = \sum_{j=1}^{q} (-1)^j (\partial_j^+ S(\mathcal{C}) - \partial_j^- S(\mathcal{C})). \tag{3.3}$$

Note that, if $y \in V$ with $(y, y) < 0$, then $V_y := y^\perp$ has signature $(p, q-1)$. We write $D(V_y)$ for the space of oriented negative $(q-1)$-planes in V_y, and we have an embedding

$$\kappa_y : D(V_y) \longrightarrow D, \quad z' \mapsto \text{span}\{y, \zeta'\}_{\text{p.o.}}, \quad z' = \text{span}\{\zeta'\}_{\text{p.o.}}, \tag{3.4}$$

whose image D_y' is the space of $z \in D$ such that $y \in z$.

We define collections

$$\mathcal{C}[j] = \{\{C_{1\perp j}, C_{1'\perp j}\}, \ldots, \{\widehat{C_j, C_{j'}}\}, \ldots, \{C_{q\perp j}, C_{q'\perp j}\}\} \tag{3.5}$$

and

$$\mathcal{C}[j'] = \{\{C_{1\perp j'}, C_{1'\perp j'}\}, \ldots, \{\widehat{C_j, C_{j'}}\}, \ldots, \{C_{q\perp j'}, C_{q'\perp j'}\}\} \tag{3.6}$$

of $(q-1)$ pairs of negative vectors in $V_j = C_j^\perp$ and $V_{j'} = C_{j'}^\perp$, respectively. Here recall that $C_{i\perp j}$ is the orthogonal projection of C_i to V_j; cf. (1.10).

The following easy fact illustrates the advantage of the 'good position' formalism.

Lemma 3.2. *If the collection $\mathcal{C}$ is in good position for V and D, then the collections $\mathcal{C}[j]$ and $\mathcal{C}[j']$ are in good position for V_j, $D(V_j)$ and for $V_{j'}$, $D(V_{j'})$, respectively.*

Proof. Note that, if we set $s' = [s_1, \ldots, s_{q-1}] \in [0,1]^{q-1}$ and write $\alpha_j s' = [s_1, \ldots, s_{j-1}, 0, s_j, \ldots, s_{q-1}]$, then, since $\mathcal{C}$ is in good position,

$$\begin{aligned}
\alpha_j \rho_{\mathcal{C}}(s') &= \rho_{\mathcal{C}}(\alpha_j s') \\
&= \text{span}\{B_1(s_1'), \ldots, B_{j-1}(s_{j-1}'), C_j, B_{j+1}(s_j'), \ldots, B_q(s_{q-1}')\}_{\text{p.o.}} \\
&= \text{span}\{B_1(s_1')_{\perp j}, \ldots, B_{j-1}(s_{j-1}')_{\perp j}, C_j, B_{j+1}(s_j')_{\perp j}, \ldots, B_q(s_{q-1}')_{\perp j}\}_{\text{p.o.}} \in D,
\end{aligned}$$

which implies that $\mathcal{C}[j]$ is in good position for V_j and $D(V_j)$. Similarly for $\mathcal{C}[j']$. $\qquad\square$

We write $S(\mathcal{C}[j])$ and $S(\mathcal{C}[j'])$ for the corresponding oriented singular $(q-1)$-cubes in $D(V_j)$ and $D(V_{j'})$ with parametrizations analogous to (3.2),

$$\rho_{\mathcal{C}[j]} : [0,1]^{q-1} \longrightarrow D(V_j)$$

and

$$\rho_{\mathcal{C}[j']} : [0,1]^{q-1} \longrightarrow D(V_{j'}).$$

We let $\kappa_j = \kappa_{(-1)^{j-1}C_j}$ and $\kappa_{j'} = \kappa_{(-1)^{j-1}C_{j'}}$ so that

$$\kappa_j \circ \rho_{\mathcal{C}[j]} = \alpha_j \rho_{\mathcal{C}} \tag{3.7}$$

and

$$\kappa_{j'} \circ \rho_{\mathcal{C}[j']} = \beta_j \rho_{\mathcal{C}}. \tag{3.8}$$

Here the key point to note is that

$$
\begin{aligned}
\mathrm{span}&\{B_1(s_1), \ldots, B_q(s_q)\}_{\mathrm{p.o.}}|_{s_j=0} \\
&= \mathrm{span}\{B_{1\perp j}(s_1), \ldots, B_{(j-1)\perp j}(s_{j-1}), \underline{C}_j, B_{(j+1)\perp j}(s_{j+1}), \ldots, B_{q\perp j}(s_q)\}_{\mathrm{p.o.}} \\
&= \mathrm{span}\{(-1)^{j-1}\underline{C}_j, B_{1\perp j}(s_1), \ldots, B_{(j-1)\perp j}(s_{j-1}), B_{(j+1)\perp j}(s_{j+1}), \ldots, B_{q\perp j}(s_q)\}_{\mathrm{p.o.}} \\
&= \kappa_j \circ \rho_{\mathcal{C}[j]}(s_1, \ldots, \widehat{s_j}, \ldots, s_q),
\end{aligned}
$$

where, for example,

$$B_{1\perp j}(s_1) = (1-s_1)C_{1\perp j} + s_1 C_{1'\perp j}.$$

3.2 The regular case

Recall from [Kud18] that a vector $x \in V$ is said to be *regular* with respect to $\mathcal{C}$ if $(x, C) \neq 0$ for all $C \in \mathcal{C}$. Parts (i) and (ii) of the following are an analogue of [Kud18, Lemma 4.2] and the proofs given there extend immediately to the general case. Part (iii) will be proved in the Appendix, where the definition of the local intersection number will also be reviewed.

LEMMA 3.3. *Let $\mathcal{C}$ be a collection in good position. For a vector $x \in V$, let $\Phi_q(x; \mathcal{C}) = \Phi_q^{\square}(x; \mathcal{C})$ be as in (1.2).*

(i) *If $x \in V$ is regular with respect to $\mathcal{C}$, then $D_x \cap S(\mathcal{C})$ is non-empty if and only if $\Phi_q(x; \mathcal{C}) \neq 0$, and in this case $D_x \cap S(\mathcal{C}) = \rho_{\mathcal{C}}(s(x))$ for a unique point $s(x) \in (0,1)^q$ given by*

$$s(x)_j = \frac{(x, C_j)}{(x, C_j) - (x, C_{j'})}. \tag{3.9}$$

(ii) *If $x \in V$ is any vector with $\Phi_q(x; \mathcal{C}) \neq 0$, then $D_x \cap S(\mathcal{C})$ consists of a single point $\rho_{\mathcal{C}}(s(x))$ with $s(x) \in [0,1]^q$ given by (3.9).*

(iii) *If $x \in V$ is any vector with $\Phi_q(x; \mathcal{C}) \neq 0$ and $s(x)$ is as in (ii), then the map $\rho_{\mathcal{C}}$ is immersive at $s(x)$, and the quantity $\Phi_q(x; \mathcal{C})$ is the local intersection number of D_x and $S(\mathcal{C})$ at $s(x)$. A precise definition of this quantity is given in (A.1) in the Appendix.*

4. Cubical integrals and generalized error functions

In this section we state our main result, an explicit expression for the Schwartz function (2.5) defined by the integral

$$I(x;\mathcal{C}) := \int_{S(\mathcal{C})} \varphi_{\mathrm{KM}}(x)$$

of the q-form $\varphi_{\mathrm{KM}}(x)$ over the singular q-cube $S(\mathcal{C})$ in D in terms of generalized error functions, as suggested in [Kud18, § 5].

THEOREM 4.1. *Suppose that* $\mathcal{C} = \mathcal{C}^{\square}$ *is in good position. Then*

$$I(x;\mathcal{C}) = (-1)^q \, 2^{-q} \sum_I (-1)^{|I|} \, E_q(x\sqrt{2}; C^I) \, e^{-\pi(x,x)}, \tag{4.1}$$

where, as in § 3.1*, for a subset* $I \subset \{1, \ldots, q\}$*,* C^I *is the* q*-tuple with* $C_j^I = C_j$ *if* $j \notin I$ *and* $C_j^I = C_{j'}$ *if* $j \in I$*, ordered by the index* j*.*

The 2^q terms in the sum on the right-hand side of (4.1) are generalized error functions associated to the vertices $z^I = \mathrm{span}\{C^I\}_{\mathrm{p.o.}}$ of $S(\mathcal{C})$ of the singular q-cube evaluated on the projections of x to those q-planes.

Remark 4.2. In the case $q = 2$, the expression given in Theorem 4.1 is the negative of the expression found in [Kud18]. But there is a simple explanation, namely that the orientation of $S(\mathcal{C})$ is defined by the 'loop' (3.11) in [Kud18], but this is the opposite of the orientation we use here, defined by the singular square $\rho_{\mathcal{C}}$.

The proof of Theorem 4.1 by induction on q is given in § 7.

5. Review of the Schwartz form φ_{KM} and its relatives

In this section we review the basic facts about the Schwartz forms $\varphi_{\mathrm{KM}}(x)$ that we need.

5.1 Local formulas

We fix a base point $z_0 \in D$ and an orthonormal basis $\{e_1, \ldots, e_m\}$, $m = p + q$, $(e_r, e_s) = \epsilon_r \delta_{rs}$, $\epsilon_r = +1$ for $1 \leqslant r \leqslant p$ and $\epsilon_r = -1$ for $r > p$, with

$$z_0 = \mathrm{span}\{e_{p+1}, \ldots, e_m\}_{\mathrm{p.o.}}.$$

In particular $V \simeq \mathbb{R}^m$, and the associated to z_0 Gaussian is given by

$$\varphi_0(x) = \varphi_0(x, z_0) = e^{-\pi \sum_j x_j^2} \in \mathcal{S}(V), \quad x = \sum_i x_i e_i. \tag{5.1}$$

Let K be the stabilizer of z_0 in G and write $\mathfrak{g}_o = \mathrm{Lie}(G) = \mathfrak{k}_o + \mathfrak{p}_o$ where $\mathfrak{k}_o = \mathrm{Lie}(K)$ and $\mathfrak{p}_o$ are the $+1$ and -1 eigenspace for the Cartan involution at z_0. There is a canonical isomorphism $T_{z_0}(D) \simeq \mathfrak{p}_o$. Under the identification

$$V \otimes V \xrightarrow{\sim} \mathrm{End}(V), \quad (v_1 \otimes v_2)(v) = (v_2, v)v_1,$$

a basis for $\mathfrak{p}_o$ is given by

$$X_{\alpha\mu} = e_\alpha \otimes e_\mu + e_\mu \otimes e_\alpha, \quad 1 \leqslant \alpha \leqslant p < \mu \leqslant p + q.$$

Let $\omega_{\alpha\mu}$ be the dual basis for $\mathfrak{p}_o^*$.

By the equivariance property (2.2), $\varphi_{\mathrm{KM}}(x)$ is determined by the element of the complex[6]

$$[\mathcal{S}(V) \otimes \wedge^\bullet(\mathfrak{p}_o^*)]^K$$

obtained by restriction to the point z_0.

For $1 \leqslant s, t \leqslant q$, and for $x = \sum_j x_j e_j \in V$, let

$$\omega(s) = \omega(x; s) = \sum_{j=1}^p x_j\, \omega_{j, p+s} \in \mathfrak{p}_o^*$$

and

$$\Omega(s, t) = \sum_{j=1}^p \omega_{j, p+s} \wedge \omega_{j, p+t} \in \wedge^2(\mathfrak{p}_o^*).$$

For λ with $0 \leqslant \lambda \leqslant [q/2]$, we define q-forms

$$\mathbf{AO}_\lambda(q) = A[\omega(1) \wedge \cdots \wedge \omega(q - 2\lambda) \wedge \Omega(q - 2\lambda + 1, q - 2\lambda + 2) \wedge \cdots \wedge \Omega(q - 1, q)], \quad (5.2)$$

where A is the alternation

$$A[\omega(1) \wedge \cdots \wedge \Omega(t - 1, t)] = \frac{1}{t!} \sum_{\sigma \in S_t} \operatorname{sgn}(\sigma)\, \omega(\sigma(1)) \wedge \cdots \wedge \Omega(\sigma(t - 1), \sigma(t)). \quad (5.3)$$

Note that these are homogeneous of degree $q - 2\lambda$ in x, and it will sometimes be useful to write $\mathbf{AO}_\lambda(q)(x)$ to indicate this dependence. With this notation, we have the following formula for the restriction of $\varphi_{\mathrm{KM}}(x)$ at the point z_0 (cf. [KM86, p. 371]):

$$\varphi_{\mathrm{KM}}(x) = 2^{q/2} \sum_{\lambda=0}^{[q/2]} C(q, \lambda)\, \mathbf{AO}_\lambda(q)(x)\, \varphi_0(x), \quad (5.4)$$

where

$$C(t, \lambda) = \left(-\frac{1}{4\pi}\right)^\lambda \frac{t!}{2^\lambda \lambda!(t - 2\lambda)!}. \quad (5.5)$$

There are two auxiliary $(q-1)$-forms associated to $\varphi_{\mathrm{KM}}(x)$ that will play a fundamental role in our calculations. We will recall their relation to $\varphi_{\mathrm{KM}}(x)$ in a moment. The first of these is given by

$$\psi_{\mathrm{KM}}(x) = 2^{q/2-1} \sum_{\lambda=0}^{[(q-1)/2]} \sum_{s=1}^q (-1)^s x_{p+s}\, C(q - 1, \lambda)\, \mathbf{AO}_\lambda(q; s)(x)\, \varphi_0(x), \quad (5.6)$$

where the $(q-1)$-form $\mathbf{AO}_\lambda(q; s)$ is defined by the alternation analogous to $\mathbf{AO}_\lambda(q-1)$ but for the index set $\{1, \ldots, \hat{s}, \ldots, q\}$ replacing $\{1, \ldots, q-1\}$. For example, $\mathbf{AO}_\lambda(q; q) = \mathbf{AO}_\lambda(q-1)$.

Now we include the parameter $\tau = u + iv$. Writing

$$\varphi_{\mathrm{KM}}(x) = \varphi_{\mathrm{KM}}^0(x)\, e^{-\pi(x,x)},$$
$$\psi_{\mathrm{KM}}(x) = \psi_{\mathrm{KM}}^0(x)\, e^{-\pi(x,x)},$$

---

[6] Note that $K = \mathrm{O}(z_0^\perp) \times \mathrm{SO}(z_0)$, so that the character α appearing in [KM86, Theorem 3.1] is trivial on K and hence does not appear here.

we have, for $\mathbf{q} = e(\tau)$ and $Q(x) = \frac{1}{2}(x, x)$,

$$\varphi_{\mathrm{KM}}(\tau, x) = \varphi_{\mathrm{KM}}^0(v^{1/2}x)\, \mathbf{q}^{Q(x)} = v^{-(p+q)/4}\omega(g_\tau')\varphi_{\mathrm{KM}}(x) \tag{5.7}$$

and

$$\psi_{\mathrm{KM}}(\tau, x) = v\, \psi_{\mathrm{KM}}^0(v^{1/2}x)\, \mathbf{q}^{Q(x)}. \tag{5.8}$$

Note that

$$-2i\frac{\partial}{\partial\bar{\tau}}\varphi_{\mathrm{KM}}(\tau, x) = \frac{\partial}{\partial v}\{\varphi_{\mathrm{KM}}^0(v^{1/2}x)\}\, \mathbf{q}^{Q(x)}.$$

On the set of x such that $R(x, z_0) > 0$, let

$$\Psi_{\mathrm{KM}}^0(x) = -\int_1^\infty \psi^0(t^{1/2}x)\, t^{-1}\, dt. \tag{5.9}$$

The point here is that

$$\psi^0(t^{1/2}x) = (\text{form valued polynomial in } t^{1/2}x) \cdot e^{-2\pi t R(x, z_0)},$$

so that the integral only makes sense when $R(x, z_0) > 0$. For x with $R(x, z_0) > 0$, let

$$\Psi_{\mathrm{KM}}(\tau, x) := \Psi_{\mathrm{KM}}^0(v^{1/2}x)\, \mathbf{q}^{Q(x)} = -\int_v^\infty \psi^0(t^{1/2}x)\, t^{-1}\, dt\, \mathbf{q}^{Q(x)}. \tag{5.10}$$

The following basic relations between the primitives $\psi_{\mathrm{KM}}(\tau, x)$, $\Psi_{\mathrm{KM}}(\tau, x)$ and the form $\varphi_{\mathrm{KM}}(\tau, x)$ are given in [FK17, §3, Proposition 3.2]; cf. also [KM90, §8].

Lemma 5.1. (i) $-2iv^2\dfrac{\partial}{\partial\bar{\tau}}\varphi_{\mathrm{KM}}(\tau, x) = d\psi_{\mathrm{KM}}(\tau, x) = v\, d\psi_{\mathrm{KM}}^0(v^{1/2}x)\, \mathbf{q}^{Q(x)}$.

(ii) $d\Psi_{\mathrm{KM}}(\tau, x) = \varphi_{\mathrm{KM}}(\tau, x)$, $\quad R(x, z_0) > 0$,
    and
    $d\Psi_{\mathrm{KM}}^0(x) = \varphi_{\mathrm{KM}}^0(x)$, $\quad R(x, z_0) > 0$.

Taking homogeneity in x of various terms into account and writing $R = R(x, z_0)$, we have the explicit formulas

$$\Psi_{\mathrm{KM}}(\tau, x) = 2^{q/2-1}\sum_{\lambda=0}^{[(q-1)/2]}\sum_{s=1}^{q}(-1)^{s-1}x_{p+s}\, C(q-1, \lambda)\, \mathbf{AO}_\lambda(q; s)$$
$$\times (2\pi R)^{-(q-2\lambda)/2}\, \Gamma\left(\frac{1}{2}(q-2\lambda), 2\pi Rv\right)\mathbf{q}^{Q(x)} \tag{5.11}$$

and

$$\varphi_{\mathrm{KM}}(\tau, x) = 2^{q/2}\sum_{\lambda=0}^{[q/2]}C(q, \lambda)\, \mathbf{AO}_\lambda(q)\, v^{(q-2\lambda)/2}\, e^{-2\pi vR}\, \mathbf{q}^{Q(x)}. \tag{5.12}$$

Here $\Gamma(s, a) = \int_a^\infty e^{-t}\, t^{s-1}\, dt$ is the incomplete Γ-function.

5.2 Global formulas

We now explain how the formulas of the previous section define global differential forms on D. We will use the notation and conventions explained in [Kud18], especially the Appendix, which we now briefly recall.

Let

$$\mathrm{FD} = \{\zeta = [\zeta_1, \ldots, \zeta_q] \in V^q \mid (\zeta, \zeta) := ((\zeta_i, \zeta_j)) < 0\}$$

be the bundle of oriented negative frames, and let

$$\mathrm{OFD} = \{\zeta = [\zeta_1, \ldots, \zeta_q] \in V^q \mid (\zeta, \zeta) = -1_q\}$$

be the bundle of oriented orthonormal negative frames. Let $\pi : \mathrm{FD} \to D$ be the natural projection, taking ζ to its oriented span. Then, for $\zeta \in \mathrm{OFD}$, we have an identification of tangent spaces

$$V^q \simeq T_\zeta(\mathrm{FD}) \supset T_\zeta(\mathrm{OFD}) = \{\eta = [\eta_1, \ldots, \eta_q] \in V^q \mid (\eta, \zeta) + (\zeta, \eta) = 0\}.$$

For $z \in D$, we let $U(z) = z^\perp$. Then the 'horizontal' subspace $U(z)^q \subset T_\zeta(\mathrm{OFD})$ is identified with $T_z(D)$ under $d\pi_\zeta$. Note that, while the space $U(z)^q$ depends only on z, the identification with $T_z(D)$ depends on ζ. The identifications for different choices of ζ differ by the action of $\mathrm{SO}(q)$.

A priori, the expressions given in (5.12) and (5.11) are elements of $S(V) \otimes \wedge^r(\mathfrak{p}_o^*)$ with $r = q$ and $q - 1$ respectively, where $\mathfrak{p}_o$ is identified with the tangent space to D at the base point

$$z_0 = \mathrm{span}\{e_{p+1}, \ldots, e_{p+q}\}_{\text{p.o.}} \in D$$

determined by our chosen orthonormal basis. They yield global formulas as follows. For any $\zeta \in \mathrm{OFD}$, the function $R(x, z)$ is defined by $R(x, z) = (x, \zeta)(\zeta, x)$. For vectors $\eta = [\eta_1, \ldots, \eta_q]$ and $\mu = [\mu_1, \ldots, \mu_q]$ in $U(z)^q$, define

$$\omega(s)(\eta) = (x, \eta_s), \quad \Omega(s, t)(\eta, \mu) = (\eta_s, \mu_t) - (\eta_t, \mu_s). \tag{5.13}$$

Also note that, in the global version of (5.11),

$$x_{p+s} = -(x, \zeta_s). \tag{5.14}$$

LEMMA 5.2. *With these definitions, the q-forms $\mathbf{AO}_\lambda(q)$ and $(q-1)$-forms $\mathbf{AO}_\lambda(q; s)$ on $U(z)^q$ are invariant under $\mathrm{SO}(q)$ and hence define forms on $T_z(D)$.*

Proof. We observe that for some non-zero constant c,

$$\mathbf{AO}_\lambda(q)(\eta^1, \ldots, \eta^q)$$
$$= c \frac{1}{q!} \sum_{\sigma \in S_q} \mathrm{sgn}(\sigma) \det \begin{pmatrix} (x, \eta^1_{\sigma(1)}) & \cdots & (x, \eta^1_{\sigma(q-2\lambda)}) & \eta^1_{\sigma(q-2\lambda+1)} & \cdots & \eta^1_{\sigma(q)} \\ \vdots & & \vdots & \vdots & & \vdots \\ (x, \eta^q_{\sigma(1)}) & \cdots & (x, \eta^q_{\sigma(q-2\lambda)}) & \eta^q_{\sigma(q-2\lambda+1)} & \cdots & \eta^q_{\sigma(q)} \end{pmatrix}, \tag{5.15}$$

where, in expanding the determinant, the product of vectors is taken using $(,)$. $\qquad\square$

Thus (5.12) (respectively, (5.11)) defines a global q-form $\varphi_{\mathrm{KM}}(\tau, x)$ on D (respectively, a global $(q-1)$-form $\Psi_{\mathrm{KM}}(\tau, x)$ on $D - D_x$) and these forms satisfy

$$d\Psi_{\mathrm{KM}}(\tau, x) = \varphi_{\mathrm{KM}}(\tau, x)$$

on $D - D_x$.

Remark 5.3. The formula for the pullback for these forms to OFD involves additional terms determined by the requirement that the forms vanish if one of the input tangent vectors is vertical (i.e., in the kernel of $d\pi_\zeta$). We will not need these expressions.

6. The pullback to certain sub-symmetric spaces

Recall that for a negative vector $y \in V$ we have $V_y = y^\perp$,

$$D'_y = \{z \in D \mid y \in z\}, \tag{6.1}$$

and

$$D(V_y) = \{z = \text{oriented neg. } (q-1)\text{-plane in } V_y\}.$$

For the properly oriented orthogonal frame bundle $\mathrm{OFD}(V_y) \to D(V_y)$, there is an embedding

$$\kappa_y : \mathrm{OFD}(V_y) \hookrightarrow \mathrm{OFD}, \quad \zeta \mapsto [\underline{y}, \zeta], \tag{6.2}$$

where $\underline{y} = y|(y,y)|^{-1/2}$, and the resulting embedding

$$\kappa_y : D(V_y) \xrightarrow{\sim} D'_y \subset D. \tag{6.3}$$

A fundamental result is the following pullback formula, which we find rather striking.

PROPOSITION 6.1. *For $x \in V$, write $x = -(x,\underline{y})\,\underline{y} + x_{\perp y}$, so that $x_{\perp y}$ is the V_y-component of x. Then we have the following statements.*

(i) $\kappa_y^* \psi_{\mathrm{KM}}^0(x) = 2^{-1/2}\,(x,\underline{y})\,e^{-2\pi(x,\underline{y})^2}\,\varphi_{\mathrm{KM}}^{V_y,0}(x_{\perp y}).$

(ii) $\kappa_y^*(\psi_{\mathrm{KM}}(\tau, x)) = 2^{-1/2}\,v^{3/2}\,(x,\underline{y})\,e^{-2\pi v(x,\underline{y})^2}\,\mathbf{q}^{-(x,\underline{y})^2/2}\,\varphi_{\mathrm{KM}}^{V_y}(\tau, x_{\perp y}).$
   Here $\varphi_{\mathrm{KM}}^{V_y,0}(\tau, \cdot)$ is the φ_{KM}^0 Schwartz $(q-1)$-form on $D(V_y)$.

(iii) $\kappa_y^*(\varphi_{\mathrm{KM}}(\tau, x)) = 0.$

Remark 6.2. The vanishing in (iii) is a fundamental property of $\varphi_{\mathrm{KM}}(\tau, x)$ which does not seem to have been observed before.

Proof. The map on tangent spaces is given by

$$d\kappa_y : T_\zeta(\mathrm{OFD}(V_y)) \longrightarrow T_{\kappa_y(\zeta)}(\mathrm{OFD}), \quad \eta = [\eta_1, \ldots, \eta_{q-1}] \mapsto [0, \eta_1, \ldots, \eta_{q-1}], \tag{6.4}$$

and this map is compatible with the 'horizontal' subspaces. It follows that any term in $\psi_{\mathrm{KM}}^0(x)$ involving an index $s = 1$ in the differential form will vanish under pullback. Thus, by (5.6), we have

$$\kappa_y^* \psi_{\mathrm{KM}}^0(x) = 2^{q/2-1}\,(x,\underline{y})\,e^{-2\pi(x,\underline{y})^2} \sum_{\lambda=0}^{[(q-1)/2]} C(q-1,\lambda)\,\mathbf{AO}_\lambda(q-1)(x_{\perp y})\,e^{-2\pi R(x_{\perp y},\zeta)}$$
$$= 2^{-1/2}\,(x,\underline{y})\,e^{-2\pi(x,\underline{y})^2}\,\varphi_{\mathrm{KM}}^{V_y,0}(x_{\perp y}).$$

Passing to $\psi_{\mathrm{KM}}(\tau, x)$ via (5.8) and noting that

$$Q(x) = -(x,\underline{y})^2 + Q(x_{\perp y}),$$

we obtain the formula claimed.

Finally, (iii) is immediate due to (5.4) and (5.15), since when we evaluate on a q-tuple of tangent vectors in the image of the map (6.4), there will be a null column in every term in (5.15)! $\qquad \square$

Next consider the $(q-1)$-form $\Psi_{\mathrm{KM}}^0(x)$. Using the expressions just found and Lemma 5.1, we have the following result.

COROLLARY 6.3. *On the subset of $D(V_y)$ for which $\kappa_y(z) \notin D_x$,*

$$\kappa_y^* \Psi_{\mathrm{KM}}^0(x) = -2^{1/2}\,(x,\underline{y}) \int_1^\infty e^{-2\pi t^2(x,\underline{y})^2}\, \varphi_{\mathrm{KM}}^{V_y,0}(t x_{\perp y})\, dt.$$

In the next section, it will be useful to have the following variant, which involves a shift in the orientations. For an index j, $1 \leqslant j \leqslant q$, define

$$\kappa_y[j] : \mathrm{OFD}(V_y) \hookrightarrow \mathrm{OFD}, \quad \zeta \mapsto [\zeta_1,\ldots,\zeta_{j-1},\underline{y},\zeta_j,\ldots,\zeta_{q-1}], \tag{6.5}$$

and write $\kappa_y[j] : D(V_y) \longrightarrow D$ for the corresponding embedding of symmetric spaces. Here note that

$$\mathrm{span}\{\zeta_1,\ldots,\zeta_{j-1},\underline{y},\zeta_j,\ldots,\zeta_{q-1}\}_{\mathrm{p.o.}} = \mathrm{span}\{(-1)^{j-1}\underline{y},\zeta_1,\ldots,\zeta_{j-1},\zeta_j,\ldots,\zeta_{q-1}\}_{\mathrm{p.o.}},$$

so that $\kappa_y[j]$ is well defined on $D(V_y)$. Of course, $\kappa_y = \kappa_y[1]$ and the embeddings of symmetric spaces only depend on the parity of j.

COROLLARY 6.4. (i) *On the subset of D_y' for which $\kappa_j(z) \notin D_x$,*

$$\kappa_y[j]^* \Psi_{\mathrm{KM}}^0(x) = (-1)^j\, 2^{1/2}\,(x,\underline{y}) \int_1^\infty e^{-2\pi t^2(x,\underline{y})^2}\, \varphi_{\mathrm{KM}}^{V_y,0}(t x_{\perp y})\, dt.$$

(ii) *On D_y',*

$$\kappa_y[j]^* \psi_{\mathrm{KM}}(\tau,x) = (-1)^{j-1} 2^{-1/2}\, v^{3/2}\,(x,\underline{y})\, e^{-2\pi v(x,\underline{y})^2} \mathbf{q}^{-(x,\underline{y})^2/2}\, \varphi_{\mathrm{KM}}^{V_y}(\tau, x_{\perp y}).$$

7. Proof of Theorem 4.1

As before, for convenience, we remove a factor independent of z and write

$$\varphi_{\mathrm{KM}}(x) = \varphi_{\mathrm{KM}}^0(x)\, e^{-\pi(x,x)}.$$

In this section we compute the cubical integrals

$$I^0(x;\mathcal{C}) = \int_{S(\mathcal{C})} \varphi_{\mathrm{KM}}^0(x).$$

Note that both sides of the identity (4.1) to be proved are C^∞-functions of $x \in V$. Since the set of $x \in V$ that are regular with respect to $\mathcal{C}$ is open and dense, it suffices to prove the identity for x regular.

7.1 The regular case

Suppose that x is regular with respect to $\mathcal{C}$, so that, by Lemma 3.3, the intersection $D_x \cap S(\mathcal{C})$ is either empty or consists of a single interior point $\rho_{\mathcal{C}}(s(x))$ depending on whether $\Phi_q(x;\mathcal{C})$ vanishes or not. If $\Phi_q(x;\mathcal{C}) \neq 0$ and for $\epsilon > 0$ sufficiently small, define a collection

$$\mathcal{C}^\epsilon(x) = \{\{B_1(s(x)_1 - \epsilon), B_1(s(x)_1 + \epsilon)\},\ldots,\{B_q(s(x)_q - \epsilon), B_q(s(x)_q + \epsilon)\}\}.$$

For simplicity, we will abbreviate this as

$$\mathcal{C}^\epsilon = \mathcal{C}^\epsilon(x) = \{\{C_1^\epsilon, C_{1'}^\epsilon\},\ldots,\{C_1^\epsilon, C_{q'}^\epsilon\}\}.$$

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LEMMA 7.1. *The collection $\mathcal{C}^\epsilon(x)$ is in good position.*

Proof. We note that, for $t \in [0, 1]$,

$$(1-t)C_j^\epsilon + tC_{j'}^e = (1 - s(x)_j + \epsilon - 2t\epsilon)C_j + (s(x)_j - \epsilon + 2t\epsilon)C_{j'}$$

so that, for $t \in [0, 1]^q$,

$$\rho_{\mathcal{C}^\epsilon(x)}(t) = \rho_{\mathcal{C}}(s(x) - \epsilon + 2\epsilon t) \in D,$$

that is, $\mathcal{C}^\epsilon(x)$ is in good position. $\qquad \square$

By construction, the singular q-cube $S(\mathcal{C}^\epsilon(x))$ contains the point $D_x \cap S(\mathcal{C})$. For x regular with respect to $\mathcal{C}$ and $\Phi_q(x; \mathcal{C}) = 0$, we let $S(\mathcal{C}^\epsilon(x))$ be the empty set. In general, we let

$$S^\epsilon(x; \mathcal{C}) = S(\mathcal{C}) - \text{int } S(\mathcal{C}^\epsilon).$$

Then Stokes's theorem and the inductive relation of Corollary 6.3 imply the following inductive formula.

PROPOSITION 7.2. *Suppose that x is regular with respect to $\mathcal{C}$. Then the set D_x does not meet $\partial S(\mathcal{C})$, the integral*

$$I^{00}(x; \mathcal{C}) := \int_{\partial S(\mathcal{C})} \Psi_{\mathrm{KM}}^0(x)$$

is well defined, and

$$I^0(x; \mathcal{C}) = I^{00}(x; \mathcal{C}) - \lim_{\epsilon \downarrow 0} I^{00}(x; \mathcal{C}^\epsilon(x)).$$

Moreover,

$$I^{00}(x; \mathcal{C}) = 2^{1/2} \sum_{j=1}^q (x, \underline{C}_j) \left(\int_1^\infty e^{-2\pi t^2 (x, \underline{C}_j)^2} I^0(tx_{\perp j}; \mathcal{C}[j]) \, dt \right)$$
$$- (x, \underline{C}_{j'}) \left(\int_1^\infty e^{-2\pi t^2 (x, \underline{C}_{j'})^2} I^0(tx_{\perp j'}; \mathcal{C}[j']) \, dt \right), \tag{7.1}$$

where $\mathcal{C}[j]$ and $\mathcal{C}[j']$ are given by (3.5) and (3.6).

Proof. Combining (3.3), (3.7), (3.8), and Corollary 6.4, we obtain

$$I^{00}(x; \mathcal{C}) = \sum_{j=1}^q (-1)^j \left(\int_{\partial_j^+ S(\mathcal{C})} \Psi_{\mathrm{KM}}^0(x) - \int_{\partial_j^- S(\mathcal{C})} \Psi_{\mathrm{KM}}^0(x) \right)$$
$$= \sum_{j=1}^q (-1)^j \left(\int_{S(\mathcal{C}[j])} \kappa_j^* \Psi_{\mathrm{KM}}^0(x) - \int_{S(\mathcal{C}[j'])} \kappa_{j'}^* \Psi_{\mathrm{KM}}^0(x) \right)$$
$$= 2^{1/2} \sum_{j=1}^q (x, \underline{C}_j) \left(\int_1^\infty e^{-2\pi t^2 (x, \underline{C}_j)^2} I^0(tx_{\perp j}; \mathcal{C}[j]) \, dt \right)$$
$$- (x, \underline{C}_{j'}) \left(\int_1^\infty e^{-2\pi t^2 (x, \underline{C}_{j'})^2} I^0(tx_{\perp j'}; \mathcal{C}[j']) \, dt \right),$$

as claimed. $\qquad \square$

7.2 The case $q = 1$

As a basis for the inductive proof of Theorem 4.1, we first suppose that $q = 1$, so that $\mathrm{sig}(V) = (m-1, 1)$. This case is discussed in several places, among them [Kud13, FK17, Liv16], but we give the calculation for convenient reference. We have

$$D \simeq \{\zeta \in V \mid Q(\zeta) = -1\}, \quad z = \mathrm{span}\{\zeta\}_{\mathrm{p.o.}},$$

and the tangent space at $z \in D$ is

$$T_z(D) \simeq U(z) := z^{\perp}.$$

For any $x \in V$, the 1-form $\omega(1)$ on D is defined by

$$\omega(1)_z(\eta) = (x, \eta), \quad \eta \in U(z) \simeq T_z(D),$$

and the Schwartz form is given by

$$\varphi_{\mathrm{KM}}^0(x) = 2^{1/2} \, \omega(1) \, e^{-2\pi R(x,z)},$$

with $R(x, z) = (x, \zeta)^2$. Take $C, C' \in V$ such that

$$Q(C) < 0, \quad Q(C') < 0, \quad (C, C') < 0,$$

where the third condition ensures that

$$\{C\}_{\mathrm{p.o.}} \simeq \underline{C} = C \, |(C, C)|^{-1/2}, \quad \{C'\}_{\mathrm{p.o.}} \simeq \underline{C}'$$

lie on the same component of D. For $s \in [0, 1]$, we define

$$B(s) = (1 - s)C + sC',$$

and note that

$$(B(s), B(s)) = (1 - s)^2 (C, C) + 2s(1 - s)(C, C') + s^2 (C', C') < 0,$$

so that the collection $\mathcal{C} = \{\{C, C'\}\}$ is in good position. Writing

$$\zeta = \zeta(s) = B(s) |(B(s), B(s))|^{-1/2},$$

we obtain a geodesic curve

$$\rho_{\mathcal{C}} : [0, 1] \longrightarrow D, \quad s \mapsto \{B(s)\}_{\mathrm{p.o.}} \simeq \zeta(s)$$

joining $\underline{C}$ and $\underline{C}'$. The tangent vector to this curve will be $\dot{\zeta} = (d/ds)\zeta$, and

$$
\begin{aligned}
I^0(x; \mathcal{C}) &= 2^{1/2} \int_0^1 (x, \dot{\zeta}(s)) \, e^{-2\pi(x, \zeta(s))^2} \, ds \\
&= 2^{1/2} \int_0^1 \frac{\partial}{\partial s} \left(-\int_{(x, \zeta(s))}^{\infty} e^{-2\pi t^2} \, dt \right) ds \\
&= 2^{1/2} \left(\int_{(x, \underline{C})}^{\infty} e^{-2\pi t^2} \, dt - \int_{(x, \underline{C}')}^{\infty} e^{-2\pi t^2} \, dt \right).
\end{aligned}
$$

Since

$$\int_u^{\infty} e^{-2\pi t^2} \, dt = 2^{-3/2}(1 - E(u\sqrt{2})),$$

for

$$E(u) = 2 \int_0^u e^{-\pi t^2} \, dt = 2 \operatorname{sgn}(u) \int_0^{|u|} e^{-\pi t^2} \, dt,$$

as in [Zag10], we obtain the expression

$$
\begin{aligned}
I^0(x; \mathcal{C}) &= \tfrac{1}{2}(E((x, \underline{C'})\sqrt{2}) - E((x, \underline{C})\sqrt{2})) \\
&= \tfrac{1}{2}(E_1(x\sqrt{2}; C') - E_1(x\sqrt{2}; C)),
\end{aligned}
$$

which is the $q = 1$ case of Theorem 4.1. Here we use the fact that, for $C \in V$ with $Q(C) < 0$, a simple calculation shows that $E_1(x; C) = E((x, \underline{C}))$. Note that in this calculation we have not used the Stokes's theorem argument. However, it is instructive to note that

$$\psi_{\mathrm{KM}}^0(x) = 2^{-1/2} \, (x, \zeta) \, e^{-2\pi(x,\zeta)^2},$$

so that, for $z = \operatorname{span}\{C\}_{\mathrm{p.o.}} \in D - D_x$, the primitive is given by

$$
\begin{aligned}
\Psi_{\mathrm{KM}}^0(x) &= -2^{-1/2} \, (x, \underline{C}) \int_1^\infty e^{-2\pi t(x,\underline{C})^2} \, t^{-1/2} \, dt \\
&= -2^{1/2} \, (x, \underline{C}) \int_1^\infty e^{-2\pi t^2 (x, \underline{C})^2} \, dt \\
&= -\operatorname{sgn}(x, \underline{C}) \int_{\sqrt{2}|(x,\underline{C})|}^\infty e^{-\pi t^2} \, dt \\
&= \frac{1}{2} \operatorname{sgn}(x, \underline{C}) \left(2 \int_0^{\sqrt{2}|(x,\underline{C})|} e^{-\pi t^2} \, dt - 1 \right) \\
&= \frac{1}{2}(E_1(x\sqrt{2}; C) - \operatorname{sgn}(x, C)).
\end{aligned}
$$

Thus the Stokes' theorem calculation gives

$$I^{00}(x; \mathcal{C}) = \int_{\partial S(\mathcal{C})} \Psi_{\mathrm{KM}}^0(x) = \frac{1}{2}(E_1(x\sqrt{2}; C_{1'}) - \operatorname{sgn}(x, C_{1'}) - E_1(x\sqrt{2}; C_1; x) + \operatorname{sgn}(x, C_1)),$$

so that the basis for Zwegers's 'completion' construction emerges.

7.3 Induction

Next we consider the inductive step. Note that we are assuming that x is regular with respect to $\mathcal{C}$ so that (7.1) holds, and we suppose that the identity (4.1) holds for all $q' < q$ and all $\mathcal{C}'$ in good position. Let $I[j]$ and $I[j']$ be subsets of $\{1, \ldots, \hat{j}, \ldots, q\}$ and let $C[j]^{I[j]}$ (respectively, $C[j']^{I[j']}$) be obtained by the recipe defining C^I in Theorem 4.1, starting with the set $\mathcal{C}[j]$ defined in (3.5) (respectively, the set $\mathcal{C}[j']$ defined in (3.6)). Then (7.1) becomes

$$
\begin{aligned}
I^{00}(x/\sqrt{2}; \mathcal{C}) &= \sum_{j=1}^q (x, \underline{C}_j) \left(\int_1^\infty e^{-\pi t^2 (x, \underline{C}_j)^2} \, I^0(t x_{\perp j}/\sqrt{2}; \mathcal{C}[j]) \, dt \right) \\
&\quad - (x, \underline{C}_{j'}) \left(\int_1^\infty e^{-\pi t^2 (x, \underline{C}_{j'})^2} \, I^0(t x_{\perp j'}/\sqrt{2}; \mathcal{C}[j']) \, dt \right) \\
&= (-1)^{q-1} 2^{1-q} \sum_{j=1}^q \left(\sum_{I[j]} (-1)^{|I[j]|} \left((x, \underline{C}_j) \int_1^\infty e^{-\pi t^2 (x, \underline{C}_j)^2} \, E_{q-1}(t x_{\perp j}; C[j]^{I[j]}) \, dt \right) \right. \\
&\quad \left. - \sum_{I[j']} (-1)^{|I[j']|} \left((x, \underline{C}_{j'}) \int_1^\infty e^{-\pi t^2 (x, \underline{C}_{j'})^2} \, E_{q-1}(t x_{\perp j'}; C[j']^{I[j']}) \, dt \right) \right).
\end{aligned}
\tag{7.2}
$$

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We want to compare this to the expression

$$-2^{-q} \sum_I (-1)^{|I|} E_q(x; C^I).$$

The key is to relate the individual quantities $E_q(x; C^I)$ in this sum and the terms on the right-hand side of (7.2) where $I = I[j]$ or $I = \{j\} \cup I[j']$. Note that if $I = I[j]$ then the collection $C[j]^{I[j]}$ spans a negative $(q-1)$-plane in V_j which maps to z^I under κ_j. Similarly, if $I = \{j\} \cup I[j']$ then the collection $C[j']^{I[j']}$ spans a negative $(q-1)$-plane in $V_{j'}$ which maps to z^I under $\kappa_{j'}$. Thus, we are collecting all of the terms which 'correspond to' a given vertex of the q-cube $S(\mathcal{C})$. The required identities are all consequences of that for $I = \emptyset$, and thus the main identity needed is as given in the following proposition.

PROPOSITION 7.3. *Suppose that the set of vectors $\boldsymbol{c} = \{c_1, \ldots, c_r\}$ spans an oriented negative r-plane in V and that x is regular with respect to $\boldsymbol{c}$. Then*

$$E_r(x; \boldsymbol{c}) - \mathrm{sgn}(x; \boldsymbol{c}) = -2 \sum_{j=1}^r (x, \underline{c}_j) \int_1^\infty e^{-\pi t^2 (x, \underline{c}_j)^2} E_{r-1}(C[j]; tx_{\perp j}) \, dt, \qquad (7.3)$$

where $\boldsymbol{c}[j] = \{c_{1 \perp j}, \ldots, \widehat{c}_j, \ldots, c_{r \perp j}\}$, and $\mathrm{sgn}(x; \boldsymbol{c})$ is defined in (1.6).

Remark 7.4. This result is just an integrated version of equation (25) in [Naz18, Proposition 3.6]. For convenience, we give the proof, taken from [Naz18], in our notation.

Proof. Let z be the negative r-plane spanned by $\boldsymbol{c}$, and, for $y, y' \in z$, let $((y, y')) = -(y, y')$. We also suppose that $x = \mathrm{pr}_z(x)$. If f is a smooth function on z, then

$$-\int_1^\infty ((\nabla f(t\, x), x)) \, dt = -\int_1^\infty \frac{d}{dt} \{f(tx)\} \, dt$$
$$= f(x) - \lim_{t \to \infty} f(tx). \qquad (7.4)$$

Here ∇ is the gradient operator and we assume that the radial limit of f exists. On the other hand, by equation (25) in [Naz18, Proposition 3.6],

$$-((\nabla E_r(x; \boldsymbol{c}), x)) = 2 \sum_{j=1}^r ((x, \underline{c}_j)) \, e^{-\pi ((x, \underline{C}_j))^2} E_{r-1}(x_{\perp j}; \boldsymbol{c}[j]). \qquad (7.5)$$

Moreover, for x regular with respect to $\boldsymbol{c}$, we have, [ABMP18, (6.2)] and [Naz18], that

$$\lim_{t \to \infty} E_r(tx; \boldsymbol{c}) = \mathrm{sgn}(x; \boldsymbol{c}). \qquad (7.6)$$

For convenience, we will give the proof of (7.5) in the Appendix. Combining them and noting that the identity (7.4) is valid for the function $f(x) = E_q(x; \boldsymbol{c})$ when x is regular with respect to $\boldsymbol{c}$, we have

$$E_r(x; \boldsymbol{c}) - \mathrm{sgn}(x; \boldsymbol{c}) = -2 \sum_j (x, \underline{c}_j) \int_1^\infty e^{-\pi t^2 (x, \underline{c}_j)^2} E_{r-1}(tx_{\perp j}; \boldsymbol{c}[j]) \, dt,$$

as required. $\qquad \square$

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Corollary 7.5.

$$I^{00}(x;\mathcal{C}) = (-1)^q 2^{-q} \sum_I (-1)^{|I|}(E_q(x\sqrt{2};C^I) - \operatorname{sgn}(x;C^I))$$

and

$$I^0(x;\mathcal{C}) = I^{00}(x;\mathcal{C}) + (-1)^q \Phi_q(x;\mathcal{C}) = (-1)^q 2^{-q} \sum_I (-1)^{|I|} E_q(x\sqrt{2};C^I).$$

Note that the second identity in Corollary 7.5 follows from Proposition 7.2, since the first identity implies that

$$\lim_{\epsilon \downarrow 0} I^{00}(x;\mathcal{C}^\epsilon(x)) = -(-1)^q \Phi_q(x;\mathcal{C}).$$

The identity of Theorem 4.1 follows immediately from this and the continuity of $E(x;C^I)$ with respect to C^I.

Summarizing, if we include the parameter τ by using (5.7), we have now established the basic formula

$$I(\tau,x;\mathcal{C}) := \int_{S(\mathcal{C}^\square)} \varphi_{\mathrm{KM}}(\tau,x)$$
$$= (-1)^q 2^{-q} \sum_I (-1)^{|I|} E_q(x\sqrt{2v};C^I)\,\mathbf{q}^{Q(x)}. \tag{7.7}$$

8. The case of a simplex

In this section we work out the theta integral over a simplex. The general inductive procedure is the same as in the cubical case, but some interesting differences arise.

8.1 Some geometry
For V of signature (p,q), we consider a collection of vectors

$$\mathcal{C} = \mathcal{C}^\triangle = [C_0,\ldots,C_q],$$

$C_i \in V$ with $(C_i,C_i) < 0$. We suppose that, for all j,

$$P_j = \operatorname{span}\{C_0,\ldots,\widehat{C_j},\ldots,C_q\}$$

is a negative q-plane. We assume that the collection $\mathcal{C}$ is linearly independent and let $U = \operatorname{span}(\mathcal{C})$. Note that $\operatorname{sig}(U) = (1,q)$, and let $D(U)$ be the space of oriented negative q-planes in U.

Let

$$\mathcal{C}^\vee = [C_0^\vee,\ldots,C_q^\vee] = \mathcal{C}\,(\mathcal{C},\mathcal{C})^{-1}$$

be the dual basis of U with respect to (,). Since $C_j^\vee$ then spans $P_j^\perp$, we have $(C_j^\vee,C_j^\vee) > 0$. Let

$$\Delta_q = \left\{ s = [s_0,\ldots,s_q] \in \mathbb{R} \ \middle|\ s_j \geqslant 0, \text{for all } j, \sum_j s_j = 1 \right\},$$

and, for $s \in \Delta_q$, let

$$C^\vee(s) = \sum_{j=0}^q s_j C_j^\vee = \mathcal{C}^{\vee\,t} s.$$

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Note that $s_j = (C^\vee(s), C_j)$. We say that $\mathcal{C}$ is in good position if

$$0 < (C^\vee(s), C^\vee(s)) = s(\mathcal{C}^\vee, \mathcal{C}^\vee)^t s = s(\mathcal{C}, \mathcal{C})^{-1t} s$$

for all $s \in \Delta_q$. For example, if all entries of $(\mathcal{C}^\vee, \mathcal{C}^\vee) = (\mathcal{C}, \mathcal{C})^{-1}$ are non-negative, then $\mathcal{C}$ is in good position.

Given $\mathcal{C}$ in good position, we define

$$z(s) = C^\vee(s)^\perp \in D,$$

with orientation $\nu_{z(s)} \in \wedge^q z(s)$ defined by

$$C^\vee(s) \wedge \nu_{z(s)} = \nu_U, \tag{8.1}$$

where we have fixed an orientation

$$\nu_U = C_0 \wedge C_1 \wedge \cdots \wedge C_q$$

in $\wedge^{q+1} U$. For example,

$$z_j = z(0, \ldots, 1, \ldots, 0) = (C_j^\vee)^\perp = \text{span}\{C_0, \ldots, \widehat{C_j}, \ldots, C_q\}$$

with orientation given as follows. Let R_j be the jth column of the matrix $(C, C)^{-1}$, so that

$$C_j^\vee = \mathcal{C} \, R_j = \sum_{i=0}^q R_{ij} C_i. \tag{8.2}$$

Then

$$C_j^\vee \wedge C_0 \wedge \cdots \wedge \widehat{C_j} \wedge \cdots \wedge C_q = (-1)^j R_{jj} \, C_0 \wedge C_1 \wedge \cdots \wedge C_q.$$

Since $R_{jj} = (C_j^\vee, C_j^\vee) > 0$,

$$z_j = \text{span}\{C_0, \ldots, \widehat{C_j}, \ldots, C_q\}[j] \tag{8.3}$$

where the 'twist' $[j]$ indicates that the given basis gives $(-1)^j \nu_{z(s)}$.

For example, for $q = 1$ we have

$$z_0 = \text{span}\{C_1\}_{\text{p.o.}}, \quad z_1 = \text{span}\{-C_0\}_{\text{p.o.}}. \tag{8.4}$$

In particular, good position requires $(C_0, C_1) > 0$ in this case! For $q = 2$, we have

$$z_0 = \text{span}\{C_1, C_2\}_{\text{p.o.}}, \quad z_1 = \text{span}\{-C_0, C_2\}_{\text{p.o.}}, \quad z_2 = \text{span}\{C_0, C_1\}_{\text{p.o.}}. \tag{8.5}$$

By construction, all the z_j lie in the same component of D and, by linear independence, the map

$$\rho_{\mathcal{C}} : \Delta_q \longrightarrow D, \quad s \mapsto z(s)$$

is an embedding. Let $S(\mathcal{C}) = \rho_{\mathcal{C}}(\Delta_q)$ be its image. The jth face of this tetrahedron is given by restricting to the subset of s with $s_j = 0$, so that it is given as

$$\{z \in S(\mathcal{C}) \mid (C^\vee(s), C_j) = 0\} = \{z \in S \mid C_j \in z\}.$$

Moreover, in the image U_j of U under the projection to $V_j = C_j^\perp$, we have that

$$[C_0^\vee, \ldots, \widehat{C_j^\vee}, \ldots, C_q^\vee]$$

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is the dual basis to

$$\mathcal{C}_{\perp j} := [C_{0 \perp j}, \ldots, C_{q \perp j}].$$

Thus, up to orientation, to be discussed in a moment, the restriction of $\rho_{\mathcal{C}}$ to a face of Δ_q is again a simplex $\rho_{\mathcal{C}_{\perp j}}$ in $D(V_j)$! Note that, in particular, $\mathcal{C}$ in good position implies that $\mathcal{C}_{\perp j}$ is in good position for all j.

Next consider $D_x \cap S(\mathcal{C})$. This set depends only on $\mathrm{pr}_U(x)$ and is given by

$$D_x \cap S(\mathcal{C}) = \begin{cases} D(U)_{\mathrm{pr}_U(x)} \cap S(\mathcal{C}) & \text{if } Q(\mathrm{pr}_U(x)) > 0, \\ \emptyset & \text{if } \mathrm{pr}_U(x) \neq 0 \text{ and } Q(\mathrm{pr}_U(x)) \leqslant 0, \\ S(\mathcal{C}) & \text{if } \mathrm{pr}_U(x) = 0. \end{cases}$$

Here, when $Q(\mathrm{pr}_U(x)) > 0$ so that $\mathrm{pr}_U(x)$ is a positive vector in U, $D(U)_{\mathrm{pr}_U(x)}$ is a pair of oriented negative q-planes in U given by the orthogonal complement to $\mathrm{pr}_U(x)$ with its two orientations. One of these has orientation determined by $\mathrm{pr}_U(x)$ by the analogue of the recipe (8.1). Then $D_x \cap S(\mathcal{C}) = \rho_{\mathcal{C}}(s(x))$ is the same q-plane with orientation shifted by

$$\mathrm{sgn}(\mathrm{pr}_U(x), C^\vee(s(x))^q = \mathrm{sgn}(x, C^\vee(s(x)))^q.$$

To determine $s(x)$, we solve

$$\mathrm{pr}_U(x) = \lambda \, C^\vee(s), \quad s \in \Delta_q, \ \lambda \in \mathbb{R}^\times,$$

that is,

$$(x, C_j) = \lambda \, s_j, \quad 0 \leqslant j \leqslant q.$$

The existence of a solution implies that $\mathrm{sgn}(x, C_j)$, if non-zero, is independent of j and that

$$\sum_{j=0}^q (x, C_j) = \lambda. \tag{8.6}$$

Thus we have the following simple description.

LEMMA 8.1. *Suppose that $Q(\mathrm{pr}_U(x)) > 0$. If $\mathrm{sgn}(x, C_j)$ is independent of j when it is non-zero, then*

$$D_x \cap S(\mathcal{C}) = \rho_{\mathcal{C}}(s(x)),$$

where

$$s(x)_j = (x, C_j)\lambda(x; \mathcal{C})^{-1}$$

with

$$\lambda(x, \mathcal{C}) = \sum_j (x, C_j).$$

Otherwise $D_x \cap S(\mathcal{C})$ is empty.

When $D_x \cap S(\mathcal{C})$ is non-empty, we determine the intersection number of the oriented q-simplex $S(\mathcal{C})$ with the oriented codimension q cycle D_x. The claim is that this is determined by the sign of the inner product of $\mathrm{pr}_U(x)$ with $C^\vee(s(x))$.

PROPOSITION 8.2. *Let $\Phi_q^\triangle(x; \mathcal{C})$ be as in (1.3). Then, if x is regular with respect to $\mathcal{C}$,*

$$I(D_x, S(\mathcal{C})) = \Phi_q^\triangle(x; \mathcal{C}). \tag{8.7}$$

Suppose that $\mathrm{pr}_U(x) \neq 0$. Then $\Phi_q^\triangle(x; \mathcal{C})$ is non-zero precisely when all of the non-zero $\mathrm{sgn}(x, C_i)$ coincide. Suppose further that $s(x)$ lies on r 'walls', that is, that exactly r of the inner products (x, C_j) vanish. Then

$$\Phi_q^\triangle(x; \mathcal{C}) = 2^{-r}(-1)^q \mathrm{sgn}(\lambda(x, \mathcal{C}))^q.$$

When $\mathrm{pr}_U(x) = 0$, we have that $\Phi_q^\triangle(x; \mathcal{C}) = 2^{-q}$ for q even and vanishes for q odd. Note that if x is not regular with respect to $\mathcal{C}$, then the intersection number is not defined.

Proof. Recall that if $\zeta \in \mathrm{OFD}$ is a properly oriented q-frame projecting to $z \in D$, then $T_z(D) \simeq U(z)^q$, where $U(z) = z^\perp$ in V. Also note that, under this isomorphism, the natural metric on $T_z(D)$ is given by $((\eta, \eta')) = -\mathrm{tr}((\eta_i, \eta_j'))$ where $\eta = [\eta_1, \ldots, \eta_q]$ and $\eta' = [\eta_1', \ldots, \eta_q']$. For our fixed collection $\mathcal{C}$ with $U = \mathrm{span}\{\mathcal{C}\}$, we have an embedding $D(U) \longrightarrow D$, where $D(U)$ is the space of oriented negative q-planes in U. Recall that $\mathrm{sig}(U) = (1, q)$. For $z \in D(U)$, write $W(z)$ for its orthogonal complement in U. Again supposing that $\zeta \in \mathrm{OFD}$ with projection z is given, we have

$$T_z(D(U)) \simeq W(z)^q.$$

Note that $\dim W(z) = 1$, and suppose that $w = w(z)$ is a properly oriented basis vector. Then $T_z(D(U))$ is spanned by the vectors $\tau_1(w) = [w, 0, \ldots, 0]$, $\tau_2(w) = [0, w, 0, \ldots, 0]$, etc. Similarly, if $z \in D_x$, then the normal subspace to $T_z(D_x)$ is spanned by the vectors $\tau_i(x)$, $1 \leqslant i \leqslant q$. For $z = \rho_{\mathcal{C}}(s(x))$, we have $w = C^\vee(s(x))$, and the intersection number of these two cycles is then given by

$$\mathrm{sgn}((\tau_1(x) \wedge \cdots \wedge \tau_q(x), \tau_1(w) \wedge \cdots \wedge \tau_q(w))) = (-1)^q \det((\tau_i(x), \tau_j(w)))$$
$$= (-1)^q \mathrm{sgn}(x, C^\vee(s(x)))^q.$$

But now

$$C^\vee(s(x)) = \lambda(x; \mathcal{C})^{-1} \sum_j (x, C_j) \, C_j^\vee,$$

and, recalling (8.2),

$$(x, C^\vee(s(x))) = \lambda(x; \mathcal{C})^{-1} \sum_j (x, C_j)(x, C_j^\vee) = \lambda(x; \mathcal{C})^{-1} \sum_{i,j} (x, C_j) R_{i,j}(x, C_i).$$

If we assume that all of the non-zero (x, C_i) have the same sign, and recalling that $R_{i,j} \geqslant 0$, we see that

$$\mathrm{sgn}(x, C^\vee(s(x))) = \mathrm{sgn}(\lambda(x; \mathcal{C})). \qquad \square$$

For $q = 1$, and x regular with respect to $\mathcal{C}$,

$$I(D_x, S(\mathcal{C})) = -\tfrac{1}{2}(\mathrm{sgn}(x, C_0) + \mathrm{sgn}(x, C_1)).$$

Note that, due to the 'twist' occurring in (8.3), our negative lines are $z_0 = \mathrm{span}\{C_1\}_{\mathrm{p.o.}}$ and $z_1 = \mathrm{span}\{-C_0\}_{\mathrm{p.o.}}$ Thus the 'cubical' data is $\mathcal{C}^\square = \{C_1, -C_0\}$, and $I(D_x, S(\mathcal{C}))$ coincides with

$$\Phi_1^\square(x; \mathcal{C}^\square) = \tfrac{1}{2}(\mathrm{sgn}(x, -C_0) - \mathrm{sgn}(x, C_1)).$$

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8.2 The integral of the theta form

We now compute

$$I^0(x; \mathcal{C}^{\triangle}) = \int_{S(\mathcal{C})} \varphi_{\mathrm{KM}}^0(x).$$

The case $q = 1$ coincides with the Zwegers case for $\mathcal{C}^{\square} = \{C_1, -C_0\}$, and we have

$$I^0(x/\sqrt{2}; \mathcal{C}) = -\tfrac{1}{2}(E_1(x; C_0) + E_1(x; C_1)). \tag{8.8}$$

As a check on signs, note that, since

$$\lim_{t \to \infty} E_1(tx; C) = \mathrm{sgn}(x, C),$$

this is consistent with the value of $I(D_x, S(\mathcal{C}))$ for $q = 1$ above.

For the general case, we suppose that x is regular with respect to $\mathcal{C}$ and proceed by induction. Due to regularity, $D_x \cap S(\mathcal{C})$ either is empty or is a single point $\rho_{\mathcal{C}}(s(x))$ on the interior of $S(\mathcal{C})$. Recall that

$$\partial S(\mathcal{C}) = \sum_{j=0}^{q} (-1)^j S(\mathcal{C}_{\perp j}). \tag{8.9}$$

Then, by [FK17, Remark 3.4], we have

$$I^0(x; \mathcal{C}) = \int_{S(\mathcal{C})} \varphi_{\mathrm{KM}}^0(x) = I(D_x, S(\mathcal{C})) + \int_{\partial S(\mathcal{C})} \Psi_{\mathrm{KM}}^0(x). \tag{8.10}$$

Since $\lim_{t \to \infty} \Psi_{\mathrm{KM}}^0(tx) = 0$, this identity gives the limiting value

$$\lim_{t \to \infty} I^0(tx; \mathcal{C}) = \lim_{t \to \infty} \int_{S(\mathcal{C})} \varphi_{\mathrm{KM}}^0(tx) = I(D_x, S(\mathcal{C})).$$

Now using Corollary 6.3, we have the inductive formula

$$\int_{\partial S(\mathcal{C})} \Psi_{\mathrm{KM}}^0(x) = \sum_{j=0}^{q} (-1)^j \int_{S(\mathcal{C}_{\perp j})} \kappa_j^* \Psi_{\mathrm{KM}}^0(x)$$

$$= \sum_{j=0}^{q} 2^{1/2} (x, \underline{C}_j) \int_1^{\infty} e^{-2\pi t^2 (x, \underline{C}_j)^2} I^0(tx_{\perp j}; \mathcal{C}_{\perp j}) \, dt. \tag{8.11}$$

Using this, we obtain the following explicit formula.

THEOREM 8.3. *For a subset* $I \subset \{0, 1, \ldots, q\}$, *let* $\mathcal{C}^{(I)}$ *be the collection of* $q + 1 - |I|$ *elements where the* C_i *with* $i \in I$ *have been omitted. Then*

$$I^0(x/\sqrt{2}; \mathcal{C}) = (-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \\ |I|=2r+1}} E_{q-2r}(x; \mathcal{C}^{(I)}).$$

Here $E_0(\cdots) = 1$.

Remark 8.4. (i) Note that if this formula is proved for x regular, then it holds for all x by continuity.

(ii) Substituting tx for x and letting t go to infinity, we obtain the 'holomorphic' part,

$$(-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \\ |I|=2r+1}} \mathrm{sgn}(x; \mathcal{C}^I), \tag{8.12}$$

where $\mathrm{sgn}(x; \emptyset) = 1$. In the case of x regular, (8.10) implies that this must coincide with $I(D_x, S(\mathcal{C}))$. In fact, it is easily checked that (8.12) is equal to $\Phi_q^{\triangle}(x; \mathcal{C})$ for all x. Thus our theta integral is the non-holomorphic completion of the series

$$\sum_{x \in \mu + L} \Phi_q^{\triangle}(x; \mathcal{C}) \, \mathbf{q}^{Q(x)}.$$

Proof. The case $q = 1$ is (8.8). In the induction, we use the notation

$$\mathcal{C}[j] = [C_{0 \perp j}, \ldots, C_{j-1 \perp j}, C_{j+1 \perp j}, \ldots, C_{q \perp j}].$$

Let $A = \{0, 1, \ldots, q\}$ and for a subset $I \subset A$, let $\mathcal{C}^I$ be the collection of $q+1-|I|$ vectors obtained by omitting the C_i with $i \in I$. Also denote by $I[j]$ a subset of $A[j] := \{0, 1, \ldots, \hat{j}, \ldots, q\}$.

We have

$$I^0(x/\sqrt{2}; \mathcal{C}) - I(D_x, S(\mathcal{C}))$$

$$= \int_{\partial S(\mathcal{C})} \Psi_{\mathrm{KM}}^0(x/\sqrt{2})$$

$$= \sum_{j=0}^{q} (x, \underline{C}_j) \int_1^{\infty} e^{-\pi t^2 (x, \underline{C}_j)^2} I^0(tx_{\perp j}/\sqrt{2}; \mathcal{C}[j]) \, dt$$

$$= (-1)^q 2^{-q} \sum_{j=0}^{q} -2(x, \underline{C}_j) \int_1^{\infty} e^{-\pi t^2 (x, \underline{C}_j)^2} \sum_{r=0}^{[(q-1)/2]} \sum_{\substack{I[j] \subset A[j] \\ |I[j]|=2r+1}} E_{q-1-2r}(tx_{\perp j}; \mathcal{C}[j]^{I[j]}) \, dt$$

$$= (-1)^q 2^{-q} \sum_{r=0}^{[(q-1)/2]} \sum_{j \in A} \sum_{\substack{I \subset A \\ |I|=2r+1 \\ j \notin I}} -2(x, \underline{C}_j) \int_1^{\infty} e^{-\pi t^2 (x, \underline{C}_j)^2} E_{q-1-2r}(tx_{\perp j}; \mathcal{C}[j]^I) \, dt$$

$$= (-1)^q 2^{-q} \sum_{r=0}^{[(q-1)/2]} \sum_{\substack{I \subset A \\ |I|=2r+1}} \sum_{\substack{j \in A \\ j \notin I}} -2(x, \underline{C}_j) \int_1^{\infty} e^{-\pi t^2 (x, \underline{C}_j)^2} E_{q-1-2r}(tx_{\perp j}; (\mathcal{C}^I)[j]) \, dt$$

$$= (-1)^q 2^{-q} \sum_{r=0}^{[(q-1)/2]} \sum_{\substack{I \subset A \\ |I|=2r+1}} (E_{q-2r}(x; \mathcal{C}^I) - \mathrm{sgn}(x; \mathcal{C}^I))$$

$$= (-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \subset A \\ |I|=2r+1}} E_{q-2r}(x; \mathcal{C}^I)$$

$$\quad - (-1)^q 2^{-q} \sum_{r=0}^{[(q-1)/2]} \sum_{\substack{I \subset A \\ |I|=2r+1}} \mathrm{sgn}(x; \mathcal{C}^I) - (-1)^q 2^{-q} \delta_{q, \mathrm{even}}.$$

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Thus, to finish the proof, we note that

$$I(D_x, S(\mathcal{C})) = (-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \subset A \\ |I| = 2r+1}} \operatorname{sgn}(x; \mathcal{C}^I), \tag{8.13}$$

where we use the convention that $\operatorname{sgn}(x; \emptyset) = 1$. Here recall that we are assuming that x is regular with respect to $\mathcal{C}$. To check this, observe that

$$(-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \subset A \\ |I| = 2r+1}} \operatorname{sgn}(x; \mathcal{C}^I) = (-1)^q 2^{-q} \sum_{\substack{J \subset A \\ |J| \equiv q(2)}} \prod_{j \in J} \sigma_j$$

$$= (-1)^q 2^{-q-1} \bigg(\prod_{j \in A} (1 + \sigma_j) + (-1)^q \prod_{j \in A} (1 - \sigma_j) \bigg). \qquad \square$$

9. Shadows of indefinite theta series

In this section we discuss the shadow of $I_\mu(\tau; S)$, that is, the complex conjugate of its image under the lowering operator $\mathbb{L} = -2iv^2(\partial/\partial\bar\tau)$. Of course, the lowering operator can be applied directly to the explicit expressions given in the main theorem. Alternatively, a more conceptual formula can be obtained by applying the operator $\mathbb{L}$ inside the integral. The key point is relation (i) of Lemma 5.1,

$$\mathbb{L}\varphi_{\mathrm{KM}}(\tau, x) = d\psi_{\mathrm{KM}}(\tau, x).$$

Then, for any compact oriented q-cell S in D with nice boundary, we have

$$\mathbb{L}\, I_\mu(\tau; S) = \sum_{x \in \mu + L} \int_{\partial S} \psi_{\mathrm{KM}}(\tau, x) = \int_{\partial S} \theta_\mu(\tau; \psi_{\mathrm{KM}}). \tag{9.1}$$

Here note that the $(q-1)$-form $\psi_{\mathrm{KM}}(\tau, x)$ defined by (5.6) and (5.8) is again a Schwartz form so that the integral is again a type of indefinite theta series of weight $(p+q)/2 - 2$. In the cubical (respectively, simplicial) case, one can say more since the faces are themselves singular cubes (respectively, simplices) lying in D'_y. Then the pullback identity (ii) of Lemma 6.4 and the argument from the proof of Proposition 7.2 yield, in the cubical case,

$$\int_{\partial S(\mathcal{C}^\square)} \psi(\tau, x) = 2^{-1/2} v^{3/2} \sum_{j=1}^q ((x, \underline{C}_{j'}) \, e^{-2\pi v(x, \underline{C}_{j'})^2} \, \mathbf{q}^{-(x, \underline{C}_{j'})^2/2} \, I(\tau, x_{\perp j'}; \mathcal{C}[j'])$$

$$- (x, \underline{C}_j) \, e^{-2\pi v(x, \underline{C}_j)^2} \, \mathbf{q}^{-(x, \underline{C}_j)^2/2} \, I(\tau, x_{\perp j}; \mathcal{C}[j])), \tag{9.2}$$

where we use the notation introduced in (7.7). The combination of (9.1), (9.2) and (7.7) yields an explicit formula for the shadow of $I_\mu(\tau; \mathcal{C})$, a (typically non-holomorphic) modular form of weight $2 - (p+q)/2$.

Now suppose that the collection $\mathcal{C}$ is rational. For each j, write $M_j = L \cap \mathbb{Q}C_j$ and $N_j = L \cap V_j$ so that

$$M_j + N_j \subset L \subset L^\vee \subset M_j^\vee + N_j^\vee. \tag{9.3}$$

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Thus

$$\mu + L = \bigsqcup_{\substack{(\nu_j,\mu_j)\in M_j^\vee/M_j \times N_j^\vee/N_j \\ \nu_j+\mu_j\equiv\mu \ \mathrm{mod} \ L}} ((\nu_j + M_j) \oplus (\mu_j + N_j)), \tag{9.4}$$

and similarly for $C_{j'}$. Using this decomposition and writing $I_\mu(\tau;\mathcal{C},L)$ in place of $I_\mu(\tau;\mathcal{C})$ to make explicit the dependence on the lattice L, we obtain the following proposition.

PROPOSITION 9.1. *If $\mathcal{C}^\square$ is rational, and with the notation of (9.3) and (9.4),*

$$\mathbb{L}\,I_\mu(\tau;\mathcal{C},L) = 2^{-1/2} \sum_{j=1}^q \bigg(\sum_{\nu_{j'},\mu_{j'}} v^{3/2}\,\overline{\theta_{\nu_{j'}}(\tau;M_{j'})}\,I_{\mu_{j'}}(\tau;\mathcal{C}[j'],N_{j'})$$
$$- \sum_{\nu_j,\mu_j} v^{3/2}\,\overline{\theta_{\nu_j}(\tau;M_j)}\,I_{\mu_j}(\tau;\mathcal{C}[j],N_j) \bigg),$$

where

$$\theta_{\nu_j}(\tau;M_j) = \sum_{x_0\in\nu_j+M_j} (x_0,\underline{C}_j)\,\mathbf{q}^{(x_0,\underline{C}_j)^2/2}$$

is a unary theta series of weight $\frac{3}{2}$.

Thus, the image of $I_\mu(\tau;\mathcal{C})$ under the ξ-operator is a linear combination of products of unary theta series of weight $\frac{3}{2}$ and the conjugates of cubical indefinite theta series for spaces of signature $(p, q-1)$, as asserted in Corollary 1.3.

Analogously, in the simplicial case, we have, using (8.9),

$$\int_{\partial S(\mathcal{C}^\triangle)} \psi_{\mathrm{KM}}(\tau,x) = \sum_{j=0}^q (-1)^j \int_{S(\mathcal{C}_{\perp j})} \kappa_j^* \psi_{\mathrm{KM}}(\tau,x)$$
$$= 2^{-1/2}\,v^{3/2} \sum_{j=0}^q (x,\underline{C}_j)\,e^{-2\pi v(x,\underline{C}_j)^2}\,\mathbf{q}^{-(x,\underline{C}_j)^2/2}\,I(\tau,x_{\perp j};\mathcal{C}_{\perp j}). \tag{9.5}$$

Note that there is a sign shift due to the fact that j now runs from 0 to q.

In the rational case we obtain the following result.

PROPOSITION 9.2. *If $\mathcal{C}=\mathcal{C}^\triangle$ is rational, and with the notation of (9.3) and (9.4),*

$$\mathbb{L}\,I_\mu(\tau;\mathcal{C}^\triangle,L) = 2^{-1/2} \sum_{j=0}^q \sum_{\nu_j,\mu_j} v^{3/2}\,\overline{\theta_{\nu_j}(\tau,M_j)}\,I_{\mu_j}(\tau;\mathcal{C}_{\perp j},N_j).$$

Thus, the shadow of $I_\mu(\tau;\mathcal{C}^\triangle,L)$ is again a linear combination of products of unary theta series of weight $\frac{3}{2}$ with conjugates of indefinite theta series associated to $(q-1)$-simplices in the spaces D_j'.

10. An example

In this section we write out a very simple example, which illustrates the relation between the (degenerate) cubical formula and the simplicial formula in the case $q = 2$.

Let $\mathcal{A} = \{A_0, A_1, A_2\}$ be the data for a 2-simplex. The vertices are

$$z_0 = \mathrm{span}\{A_1, A_2\}_{\mathrm{p.o.}}, \quad z_1 = \mathrm{span}\{-A_0, A_2\}_{\mathrm{p.o.}}, \quad z_2 = \mathrm{span}\{A_0, A_1\}_{\mathrm{p.o.}},$$

and the theta integral is

$$\tfrac{1}{4}(E_2(x; A_1, A_2) + E_2(x; A_0, A_2) + E_2(x; A_0, A_1) + 1).$$

We can consider the related cubical data $\mathcal{C} = \{\{C_1, C_{1'}\}, \{C_2, C_{2'}\}\}$, where

$$C_1 = A_0, \quad C_2 = A_1, \quad C_{2'} = -A_2, \quad C_{1'} = C_{2'} - C_2 = -A_1 - A_2,$$

so that the associated (degenerate) 2-cube has vertices

$$z_2 = \{C_1, C_2\}, \quad z_1 = \{C_1, C_{2'}\}, \quad z_0 = \{C_{1'}, C_{2'}\} = \{C_{1'}, C_2\},$$

and theta integral

$$\begin{aligned}
&\tfrac{1}{4}(E_2(x; C_1, C_2) - E_2(x; C_1, C_{2'}) - E_2(x; C_{1'}, C_2) + E_2(x; C_{1'}, C_{2'})) \\
&= \tfrac{1}{4}(E_2(x; A_0, A_1) + E_2(x; A_0, A_2) + E_2(x; A_1 + A_2, A_1) + E_2(x; A_1 + A_2, A_2)).
\end{aligned}$$

Coincidence of the two theta integrals is equivalent to the identity

$$E_2(x; A_1 + A_2, A_1) + E_2(x; A_1 + A_2, A_2) = E_2(x; A_1, A_2) + 1,$$

where all terms are given by integrals over the negative 2-plane z_0. Writing $y \in z_0$ as $y = aA_1^\vee + bA_2^\vee$, with respect to the dual basis, and noting that

$$\mathrm{sgn}(a + b)(\mathrm{sgn}(a) + \mathrm{sgn}(b)) = \mathrm{sgn}(a)\mathrm{sgn}(b) + 1,$$

for a and b not both 0, the identity follows. Note that there are a vast number of such identities for combinations of generalized error functions.

Appendix. Some proofs and details

A.1 Proof of part (iii) of Lemma 3.3

Suppose that $\mathcal{C}$ is in good position and that $x \in V$ with $\Phi_q(x; \mathcal{C}) \neq 0$. Let $s_0 = s(x)$ be the unique point of $[0, 1]^q$ such that $\rho_{\mathcal{C}}(s_0) = D_x \cap S(\mathcal{C})$. Note that the map $\rho_{\mathcal{C}}$ extends to an open neighborhood of $[0, 1]^q$ so that, even if s_0 lies on the boundary, we can define $\rho_{\mathcal{C}}$ on an open set $\mathcal{U}$ around s_0. We lift $\rho_{\mathcal{C}}$ to a map $\tilde{\rho}_{\mathcal{C}} : \mathcal{U} \to \mathrm{OFD}$, defined by

$$\tilde{\rho}_{\mathcal{C}} : s \mapsto \zeta(s) = B(s)P^{-1}, \quad P \in \mathrm{Sym}_q(\mathbb{R})_{>0}, \quad P^2 = -(B(s), B(s)).$$

For convenience, we write $B = [B_1, \ldots, B_q] = B(s)$. Then

$$(\tilde{\rho}_{\mathcal{C}})_* \left(\frac{\partial}{\partial s_j} \right) = \dot{B}_j P^{-1} - \zeta \dot{P}_j P^{-1}, \quad \dot{B}_j := \frac{\partial}{\partial s_j} B = [0, \ldots, -C_j + C_{j'}, \ldots, 0], \quad \dot{P}_j := \frac{\partial}{\partial s_j} P.$$

The components in the connection subspace $U(z)^q$ of $T_\zeta(\mathrm{OFD})$ are then

$$(\rho_{\mathcal{C}})_* \left(\frac{\partial}{\partial s_j} \right) = \tau_j P^{-1}, \quad \tau_j = [0, \ldots, \mathrm{pr}_{U(z)}(-C_j + C_{j'}), \ldots, 0],$$

and these are linearly independent provided $\mathrm{pr}_{U(z)}(-C_j + C_{j'}) \neq 0$ for all j. But at the point $z_0 = \rho_{\mathcal{C}}(s_0)$, we have $x \in U(z_0)$, and the q vectors

$$\eta(x, j) = [0, \ldots, 0, x, 0, \ldots, 0],$$

with x in the jth component, span the normal to $T_{z_0}(D_x)$. Note that the metric g on $T_z(D) \simeq U(z)^q$ is given by

$$g(\eta, \eta') = \mathrm{tr}((\eta_i, \eta'_j)).$$

Then we have

$$g(\eta(x, i), \tau_j) = [(x, C_{j'}) - (x, C_j)] \, \delta_{ij}.$$

This shows that $\tau_j \neq 0$ for all j and hence $\rho_{\mathcal{C}}$ is immersive at $s(x)$. We can choose the open neighborhood $\mathcal{U}$ of $s(x)$ in $\mathbb{R}^q$ so that the restriction of $\rho_{\mathcal{C}}$ to $\mathcal{U}$ is an embedding. The orientation of the cycle D_x of codimension q is defined by an element of $\nu_{z,x} \in \bigwedge^{(p-1)q} T_z(D_x)$ such that

$$\nu_x \wedge \nu_{z,x} \in \bigwedge^{pq}(T_z(D))$$

is properly oriented, where

$$\nu_x = \eta(x, 1) \wedge \cdots \wedge \eta(x, q).$$

Here we have fixed an orientation of D. Thus the intersection number at z_0 of D_x with $\rho_{\mathcal{C}}(\mathcal{U})$ is

$$I(D_x, \rho_{\mathcal{C}}(\mathcal{U})) = \mathrm{sgn} \det(g(\eta(x, i), \tau_j)) = \prod_j \mathrm{sgn}((x, C_{j'}) - (x, C_j)).$$

If x is regular with respect to $\mathcal{C}$, then this quantity is

$$2^{-q} \prod_{j=1}^{q} (\mathrm{sgn}(x, C_{j'}) - \mathrm{sgn}(x, C_j)) = (-1)^q \Phi_q(x; \mathcal{C}).$$

In general, we have

$$(-1)^q \Phi_q(x; \mathcal{C}) = 2^{-r} I(D_x, \rho_{\mathcal{C}}(\mathcal{U})), \tag{A.1}$$

where r, $0 \leqslant r \leqslant q$, is the number of walls passing through $s(x)$. Thus, $\Phi_q(x; \mathcal{C})$ is a 'weighted' intersection number.

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A.2 Proof of (7.5)

For $y, y' \in Z = \operatorname{span}\{C\}$, we write $((y,y')) = -(y,y')$, and we assume that $x \in Z$. We let

$$C^\vee = [C_1^\vee, \ldots, C_q^\vee] = C((C,C))^{-1}$$

be the dual basis. We write

$$x = \sum_i x_i \, C_i^\vee, \quad x_i = ((x, C_i)).$$

For a fixed index j, we write

$$x = x_{\perp j} + x' C_j, \quad x_{\perp j} = \sum_{i \neq j} x_i C_i^\vee, \quad x_j = ((x, C_j)) = x'((C_j, C_j)),$$

and similarly for our variable of integration $y \in Z$. Note that, in particular,

$$\operatorname{sgn}((y, C_j)) = \operatorname{sgn}(y').$$

We can write

$$dy = dy_{\perp j} \, dy'$$

where

$$1 = \int_Z e^{-\pi((y,y))} \, dy = \int_{Z_{\perp j}} \int_{\mathbb{R}} e^{-\pi((y_{\perp j}, y_{\perp j}))} \, e^{-\pi(y')^2((C_j, C_j))} \, dy_{\perp j} \, dy',$$

where dy' is $((C_j, C_j))^{1/2}$ times Lebesque measure, so that

$$\int_{\mathbb{R}} e^{-\pi(y')^2((C_j, C_j))} \, dy' = 1.$$

We write[7]

$$(-1)^q E_q(x; C) = \int_Z e^{-\pi((y-x, y-x))} \prod_i \operatorname{sgn}((y, C_i)) \, dy$$

$$= \int_{Z_{\perp j}} \int_{\mathbb{R}} e^{-\pi((y_{\perp j} - x_{\perp j}, y_{\perp j} - x_{\perp j}))} \, e^{-\pi(y'-x')^2((C_j, C_j))} \prod_{i \neq j} \operatorname{sgn}((y, C_i)) \operatorname{sgn}(y') \, dy_{\perp j} \, dy'$$

$$= (-1)^{q-1} E_{q-1}(x_{\perp j}; C[j]) \int_{\mathbb{R}} e^{-\pi(y'-x')^2((C_j, C_j))} \operatorname{sgn}(y') \, dy'$$

$$= (-1)^{q-1} E_{q-1}(x_{\perp j}; C[j]) \int_{\mathbb{R}} e^{-\pi(y')^2((C_j, C_j))} \operatorname{sgn}(y_j + x_j) \, dy'.$$

But then, taking into account that $dy' = ((C_j, C_j))^{-1/2} \, d_{\mathrm{Leb}} y_j$, we have

$$x_j \frac{\partial}{\partial x_j} \{(-1)^q E_q(x; C)\} = ((x, C_j)) \, (-1)^{q-1} E_{q-1}(x_{\perp j}; C[j])$$

$$\times \int_{\mathbb{R}} e^{-\pi y_j^2((C_j, C_j))^{-1}} 2\delta(y_j + x_j) \, ((C_j, C_j))^{-1/2} \, d_{\mathrm{Leb}} y_j$$

$$= 2((x, \underline{C}_j))(-1)^{q-1} E_{q-1}(x_{\perp j}; C[j];) \, e^{-\pi((x, \underline{C}_j))^2}.$$

Here recall that $\underline{C}_j = C_j((C_j, C_j))^{-1/2}$. Summing over j, we obtain (7.5).

[7] Note that the extra factor of $(-1)^q$ etc. is due to our temporary change in the sign of the inner product on Z, so that our E_q differs from that in [Naz18] by this sign.

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