Nagoya Math. J., **228** (2017), 186–221
DOI 10.1017/nmj.2016.57

WEIGHT CHANGING OPERATORS FOR AUTOMORPHIC FORMS ON GRASSMANNIANS AND DIFFERENTIAL PROPERTIES OF CERTAIN THETA LIFTS

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Abstract. We define weight changing operators for automorphic forms on Grassmannians, that is, on orthogonal groups, and investigate their basic properties. We then evaluate their action on theta kernels, and prove that theta lifts of modular forms, in which the theta kernel involves polynomials of a special type, have some interesting differential properties.

Introduction

The classical Shimura–Maaß operators $\frac{\partial}{\partial_\tau}+\frac{k}{2iy}$ and $y^2\frac{\partial}{\partial_{\overline{\tau}}}$ are well known for taking (elliptic, real-analytic) modular forms of weight k to modular forms of weight $k+2$ and $k-2$, respectively. In addition, [Ma1, Ma2] consider differential operators which have a similar effect on Siegel modular forms, a work which was generalized in [Sh2]. The following paper [Sh3] concerns differential operators on functions on unitary groups which have related properties. All these operators have number-theoretic as well as representation-theoretic (or Lie-algebraic) interpretations, and are therefore the subject of many research papers (see, e.g., the reference [Sh1], which is strongly related to the case considered in this paper, as well as [Sh4] for some generalizations of the results of the previously mentioned references or the investigation of invariant differential operators appearing in [Sh5]).

Our first goal is to define similar operators for modular (or automorphic) forms on another type of Shimura varieties, namely quotients of Grassmannians of vector spaces of signature $(2, b_-)$. These are obtained by interpolating the square of the Shimura–Maaß operators from the case

Received December 20, 2014. Accepted March 28, 2016.
2010 Mathematics subject classification. 11F55, 11F60.
This work was carried out while I was working at the Technical University of Darmstadt, Germany. The initial part of this work was supported by the Minerva Fellowship (Max-Planck-Gesellschaft).

$b_- = 1$, the multiple Shimura–Maaß operators obtained in the case $b_- = 2$, and the operators for Siegel modular forms appearing in the case $b_- = 3$. One may use Lie-theoretic considerations in order to establish the existence of such operators, but obtaining their explicit formula in this way is very tedious, because of the change of coordinates between the tube domain model and the transitive free action of an appropriate parabolic subgroup of $\mathrm{SO}^+(V)$. We also remark that [Sh1] also considers differential operators on automorphic forms on orthogonal groups. However, the operators defined in that reference take automorphic forms of some weight (i.e., a representation of the maximal compact subgroup) ρ to automorphic forms having weight $\rho \otimes \eta$ for some b_--dimensional representation η, hence in particular take scalar-valued automorphic forms on Grassmannians to vector-valued functions. Moreover, since that reference works with the coordinates arising from the bounded model while we consider the tube domain model (since the explicit formulas for the theta functions are more neatly presented in this model), an appropriate change of coordinates must be employed. It is true that after this change of coordinates, using the natural bilinear form on the tangent space of the Grassmannian in the tube domain model we may indeed obtain differential operators which remain in the scalar-valued realm. Indeed, after some additional normalization we obtain the operators defined in this paper using this method. However, the calculations involved are very delicate, laborious, long, and unenlightening, for which reason we have chosen to state and prove the formulas for the operators directly.

The second goal of this paper is to present two applications of these weight changing operators, in the theory of theta lifts. We recall the generalization, defined in [B], for the Doi–Naganuma lifting first introduced in [DN, Ng]. This map is given in [B] in terms of a singular theta lift, and takes weakly holomorphic elliptic modular forms to meromorphic modular forms on Grassmannians. On the other hand, [Ze2] defines a similar theta lift, using the same theta functions with polynomials. The first result of this paper states that in the case of an even dimension, a power of our weight raising operator sends the theta lift from [Ze2] to the generalized Doi–Naganuma lifting of [B].

In addition, recall that the theta lift from [B, Section 13] (which is also studied extensively in [Bru] and others) is a *real* function. No automorphic forms of nonzero weight can be real. As a second application for our operators we define a notion of m-real automorphic forms of positive weight m, and show that in case one applies the theta lift from [B, Section 14] (or

https://doi.org/10.1017/nmj.2016.57 Published online by Cambridge University Press

from [Ze2]) to a modular form with real Fourier coefficients, the resulting theta lift is m-real.

The first half of the paper contains numerous statements whose proofs are delayed to later sections. We choose this way of presentation since most of the proofs consist of direct calculations, which may divert the reader's attention from the main ideas. Specifically, the paper is divided into 4 sections. In Section 1 we define the weight raising and weight lowering operators and state their properties. Section 2 presents the images of theta functions with special polynomials under the weight changing operators, and proves the main theorems. Section 3 presents the proofs for the assertions of Section 1, while Section 4 contains the missing proofs of Section 2.

§1. Weight changing operators for automorphic forms on orthogonal groups

In this section, we present automorphic forms on complex manifolds arising as orthogonal Shimura varieties of signature $(2, b_-)$, introduce the weight raising and weight lowering operators on such forms, and give some of their properties. The proofs of most assertions are postponed to Section 3.

Let V be a real vector space with a nondegenerate bilinear form of signature (b_+, b_-). The pairing of x and y in V is written (x, y), and x^2 stands for the norm (x, x) of x. For $S \subseteq V$, $S^\perp$ denotes the subspace of V which is perpendicular to S. The *Grassmannian* $G(V)$ of V is defined to be the set of all decompositions of V into the orthogonal direct sum of a positive definite space v_+ and a negative definite space v_-. In the case $b_+ = 2$ (which is the only case we consider in this paper), it is shown in [B, Section 13], [Bru, Sections 3.2 and 3.3], or [Ze2, Section 2.2] (among others), that $G(V)$ carries a complex structure and has several equivalent models, which we now briefly present. Let

$$P = \left\{Z_V = X_V + iY_V \in V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C} \middle| Z_V^2 = 0, (Z_V, \overline{Z_V}) > 0\right\}.$$

$Z_V \in V_\mathbb{C}$ lies in P if and only if X_V and Y_V are orthogonal and have the same positive norm. P has two connected components (which are interchanged by complex conjugation), and let P^+ be one component. The map

$$P^+ \to G(V), \qquad Z_V \mapsto \mathbb{R}X_V \oplus \mathbb{R}Y_V$$

is surjective, and C^* acts freely and transitively on each fiber of this map by multiplication. This realizes $G(V)$ as the image of P^+ in the projective

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space $\mathbb{P}(L_{\mathbb{C}})$, which is an analytically open subset of the (algebraic) quadric $Z_V^2=0$, yielding a complex structure on $G(V)$. This is the *projective model* of $G(V)$.

Let z be a nonzero vector in V which is *isotropic*, that is, $z^2=0$. The vector space $K_{\mathbb{R}}=z^{\perp}/\mathbb{R}z$ is nondegenerate and Lorentzian of signature $(1,b_{-}-1)$. Choosing some $\zeta\in V$ with $(z,\zeta)=1$ and restricting the projection $z^{\perp}\to K_{\mathbb{R}}$ to $\{z,\zeta\}^{\perp}$ gives an isomorphism. We thus write V as $K_{\mathbb{R}}\times\mathbb{R}\times\mathbb{R}$, in which

$$(\alpha,a,b)=a\zeta+bz+\big(\alpha\in\{z,\zeta\}^{\perp}\cong K_{\mathbb{R}}\big),\quad (\alpha,a,b)^2=\alpha^2+2ab+a^2\zeta^2.$$

A (holomorphic) section $s:G(V)\to P^{+}$ is defined by the pairing with z being 1. Subtracting ζ from any s-image and taking the $K_{\mathbb{C}}$-image of the result yields a biholomorphism between $G(V)\cong s\big(G(V)\big)$ and the *tube domain* $K_{\mathbb{R}}+iC$, where C is a cone of positive norm vectors in the Lorentzian space $K_{\mathbb{R}}$. C is called the *positive cone*, and it is determined by the choice of z and the connected component P^{+}. The inverse biholomorphism takes $Z=X+iY\in K_{\mathbb{C}}$ to

$$Z_{V,Z}=\left(Z,1,\frac{-Z^2-\zeta^2}{2}\right)=\left(X,1,\frac{Y^2-X^2-\zeta^2}{2}\right)+i\big(Y,0,-(X,Y)\big),$$

with the real and imaginary parts denoted by $X_{V,Z}$ and $Y_{V,Z}$, respectively. They are orthogonal and have norm $Y^2>0$. This identifies $G(V)$ with the *tube domain model* $K_{\mathbb{R}}+iC$. Taking the other connected component of P corresponds to taking the other cone $-C$ to be the positive cone, and to the conjugate complex structure.

The subgroup $O^{+}(V)$ consisting of elements of $O(V)$ preserving the orientation on the positive definite part acts on P^{+} and $G(V)$, respecting the projection. Elements of $O(V)\setminus O^{+}(V)$ interchange the connected components of P. The action of $O^{+}(V)$ (and also of the connected component $\mathrm{SO}^{+}(V)$) on $G(V)$ is transitive, with the stabilizer K (or $SK\leqslant\mathrm{SO}^{+}(V)$) of a point being isomorphic to $\mathrm{SO}(2)\times O(n)$ (resp. $\mathrm{SO}(2)\times\mathrm{SO}(n)$). Therefore $G(V)$ is isomorphic to $O^{+}(V)/K$ and to $\mathrm{SO}^{+}(V)/SK$. Given an isotropic z as above, the action of $O^{+}(V)$ transfers to $K_{\mathbb{R}}+iC$, and for $M\in O^{+}(V)$ and $Z\in K_{\mathbb{R}}+iC$ we have

$$MZ_{V,Z}=J(M,Z)Z_{V,MZ},\quad \text{with } J(M,Z)=(MZ_{V,Z},z)\in\mathbb{C}^{*}.$$

J is a *factor of automorphy*, namely the equality

$$J(MN, Z) = J(M, NZ)J(N, Z)$$

holds for all $Z \in K_{\mathbb{R}} + iC$ and M and N in $O^+(V)$. For such M we define the *slash operator* of weight m, and more generally of weight (m, n), by

$$\Phi[M]_{m,n}(Z) = J(M, Z)^{-m}\overline{J(M, Z)}^{-n}\Phi(MZ), \qquad [M]_m = [M]_{m,0}.$$

The fact that $(Z_V, \overline{Z_V}) = 2Y^2$ and the definition of $J(M, Z)$ yield the equalities

(1)
$$\big(\Im(MZ)\big)^2 = \frac{Y^2}{|J(M, Z)|^2} \qquad \text{and} \qquad \big(F(Y^2)^t\big)[M]_{m,n} = F[M]_{m+t,n+t}(Y^2)^t$$

the latter holding for every m, n, t, and function F on $K_{\mathbb{R}} + iC$ (see [Bru, Lemma 3.20] for the first equality in Equation (1), and the second one follows immediately).

The invariant measure on $K_{\mathbb{R}} + iC$ is $\frac{dX\,dY}{(Y^2)^{b_-}}$ (see [Bru, Section 4.1], but one can also prove this directly, using the generators of $O^+(V)$ considered in Section 3 below). Note that this measure depends on the choice of a basis for $K_{\mathbb{R}} + iC$, but changing the basis only multiplies this measure by a positive global scalar. Let Γ be a discrete subgroup of $O^+(V)$ of cofinite volume. In most of the interesting cases Γ will be either the O^+ or the SO^+ part of the orthogonal group of an even lattice L in V, or the discriminant kernel of such a group. Given $m \in \mathbb{Z}$, an *automorphic form of weight m with respect to* Γ is defined to be a (complex valued) function Φ on $K_{\mathbb{R}} + iC$ for which the equation

$$\Phi(MZ) = J(M, Z)^m\Phi(Z), \qquad \text{or equivalently} \qquad \Phi[M]_m(Z) = \Phi(Z),$$

holds for all $M \in \Gamma$ and $Z \in K_{\mathbb{R}} + iC$. Using the standard argument, such a function is equivalent to a function on P^+ which is $-m$-homogeneous (with respect to the action of $\mathbb{C}^*$) and Γ-invariant, as considered, for example, in [B].

We now consider some differential operators on functions on $K_{\mathbb{R}} + iC$. Given a basis for $K_{\mathbb{R}}$, we write ∂_{x_k} for $\frac{\partial}{\partial x_k}$ (for $1 \leqslant k \leqslant b_-$). Similarly, ∂_{y_k} stands for the coordinates of the imaginary part from C. The notation for the derivatives $\partial_{z_k} = \frac{1}{2}(\partial_{x_k} - i\partial_{y_k})$ and $\partial_{\overline{z_k}} = \frac{1}{2}(\partial_{x_k} + i\partial_{y_k})$ will be shortened further to ∂_k and $\partial_{\overline{k}}$, respectively.

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The operator $I=\sum_k x_k\partial_{x_k}$ multiplies a homogeneous function on $K_{\mathbb{R}}$ by its homogeneity degree, and is thus independent of the choice of basis (indeed, it has an intrinsic Lie-theoretic description). The operators

$$D^*=\sum_k y_k\partial_k \qquad \text{and} \qquad \overline{D^*}=\sum_k y_k\partial_{\overline{k}}$$

from [Na] are intrinsic as well, and they are also invariant under translations in the real part of $K_{\mathbb{R}}+iC$. If the basis for $K_{\mathbb{R}}$ is *orthonormal*, that is, orthogonal with the first vector having norm 1 and the rest having norm -1, then the *Laplacian of* $K_{\mathbb{R}}$, denoted $\Delta_{K_{\mathbb{R}}}$, is defined to be $\partial_{x_1}^2-\sum_{k=2}^{b_-}\partial_{x_k}^2$. It is independent of the choice of the orthonormal basis (though using a basis which is not orthonormal it takes a different form), and it is invariant under the action of $O(K_{\mathbb{R}})$ as well as under translations in $K_{\mathbb{R}}$. With complex coordinates it has three counterparts,

$$\Delta_{K_{\mathbb{C}}}^{h}=\partial_1^2-\sum_{k=2}^{b_-}\partial_k^2, \qquad \Delta_{K_{\mathbb{C}}}^{\overline{h}}=\partial_{\overline{1}}^2-\sum_{k=2}^{b_-}\partial_{\overline{k}}^2, \qquad \text{and}$$

$$\Delta_{K_{\mathbb{C}}}^{\mathbb{R}}=\partial_1\partial_{\overline{1}}-\sum_{k=2}^{b_-}\partial_k\partial_{\overline{k}},$$

which we call the *holomorphic Laplacian of* $K_{\mathbb{C}}$ (of Hodge weight $(2,0)$), the *anti-holomorphic Laplacian of* $K_{\mathbb{C}}$ (of Hodge weight $(0,2)$), and the *real Laplacian of* $K_{\mathbb{C}}$ (of Hodge weight $(1,1)$), respectively. These operators have the same invariance and independence properties as $\Delta_{K_{\mathbb{R}}}$. Note that the appropriate combinations appearing in [Bru, Na] can be identified as our operators $\frac{1}{2}\Delta_{K_{\mathbb{C}}}^{h}$, $\frac{1}{2}\Delta_{K_{\mathbb{C}}}^{\overline{h}}$, and $\Delta_{K_{\mathbb{C}}}^{\mathbb{R}}$, respectively, expressed in a basis which is not orthonormal. We shall indeed discuss and generalize the operators Δ_1 and Δ_2 of [Na] in Proposition 1.5 below.

The weight changing operators and their defining property are given in

THEOREM 1.1. *For any integer m define $R_m^{(b_-)}$ to be the operator*

$$(Y^2)^{\frac{b_-}{2}-m-1}\Delta_{K_{\mathbb{C}}}^{h}(Y^2)^{m+1-\frac{b_-}{2}}=\Delta_{K_{\mathbb{C}}}^{h}-\frac{i(2m+2-b_-)}{Y^2}D^* - \frac{m(2m+2-b_-)}{2Y^2}.$$

In addition, define

$$L^{(b_-)}=(Y^2)^2\overline{R_0}=(Y^2)^{\frac{b_-}{2}+1}\Delta_{K_{\mathbb{C}}}^{\overline{h}}(Y^2)^{1-\frac{b_-}{2}}=(Y^2)^2\Delta_{K_{\mathbb{C}}}^{\overline{h}}+iY^2(2-b_-)\overline{D^*}.$$

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Then the equalities

$$(R_m^{(b_-)}F)[M]_{m+2} = R_m^{(b_-)}\big(F[M]_m\big), \qquad (L^{(b_-)}F)[M]_{m-2} = L^{(b_-)}\big(F[M]_m\big)$$

hold for every $\mathcal{C}^2$ function F on $K_\mathbb{R} + iC$ and any $M \in O^+(V)$.

The different descriptions of $R_m^{(b_-)}$ and $L^{(b_-)}$ coincide by Lemma 3.1 below. Theorem 1.1 has the following standard

Corollary 1.2. *If Φ is an automorphic form of weight m on $G(V) \cong K_\mathbb{R} + iC$ then $R_m^{(b_-)}\Phi$ and $L^{(b_-)}\Phi$ are automorphic forms on $K_\mathbb{R} + iC$ which have weights $m+2$ and $m-2$, respectively.*

In correspondence with Theorem 1.1 and Corollary 1.2 we call $R_m^{(b_-)}$ and $L^{(b_-)}$ the *weight raising operator of weight m* and the *weight lowering operator* for automorphic forms on Grassmannians of signature $(2, b_-)$, respectively. As already mentioned in the Introduction, these operators may also be given a Lie-theoretic description (see Section 3 for more details). However, the explicit operators appearing in Theorem 1.1 are more useful for our applications.

We shall make use of the operator

$$D^*\overline{D^*} - \frac{\overline{D^*}}{2i} = \overline{D^*}D^* + \frac{D^*}{2i} = \sum_{k,l} y_k y_l \partial_k \partial_{\bar{l}},$$

which we denote by $|D^*|^2$. Lemma 2.2 of [Ze2] shows that

$$\Delta_{m,n}^{(b_-)} = 8|D^*|^2 - 4Y^2\Delta_{K_\mathbb{C}}^{\mathbb{R}} - 4im\overline{D^*} + 4inD^* + 2n(2m - b_-)$$

is the *weight (m, n) Laplacian* on $K_\mathbb{R} + iC$, and the *weight m Laplacian* $\Delta_m^{(b_-)}$ is just $\Delta_{m,0}^{(b_-)}$ (this extends the corresponding assertion of [Na], since his operator Δ_1 is our $\Delta_0^{(b_-)}$ divided by 8). The constants are normalized such that

$$\Delta_{m,n}^{(b_-)}(Y^2)^t = (Y^2)^t\Delta_{m+t,n+t}^{(b_-)} \tag{2}$$

holds for every m, n, and t (see the remark after Lemma 3.1 below). The relations between $R_m^{(b_-)}$, $L^{(b_-)}$, and the corresponding Laplacians are given by

PROPOSITION 1.3. *The equalities*

$$\Delta_{m+2}^{(b_-)}R_m^{(b_-)} - R_m^{(b_-)}\Delta_m^{(b_-)} = (2b_- - 4m - 4)R_m^{(b_-)}$$

and

$$\Delta_{m-2}^{(b_-)}L^{(b_-)} - L^{(b_-)}\Delta_m^{(b_-)} = (4m - 2b_- - 4)L^{(b_-)}$$

hold for every $m \in \mathbb{Z}$.

We recall that an automorphic form of weight m on $K_{\mathbb{R}} + iC$ is said to have eigenvalue λ if it is annihilated by $\Delta_m^{(b_-)} + \lambda$ (i.e., eigenvalues are of $-\Delta_m^{(b_-)}$). Hence Proposition 1.3 has the following

COROLLARY 1.4. *If F is an automorphic form of weight m on $K_{\mathbb{R}} + iC$ which has eigenvalue λ then the automorphic forms $R_m^{(b_-)}F$ and $L^{(b_-)}F$ have eigenvalues $\lambda + 4m - 2b_- + 4$ and $\lambda - 4m + 2b_- + 4$, respectively.*

By evaluating compositions of the weight changing operators one shows

PROPOSITION 1.5. *The combination*

$$\Xi_m^{(b_-)} = (Y^2)^2\Delta_{K_{\mathbb{C}}}^h\Delta_{K_{\mathbb{C}}}^{\overline{h}} - iY^2(2m+2-b_-)D^*\Delta_{K_{\mathbb{C}}}^{\overline{h}} + iY^2(2-b_-)\overline{D^*}\Delta_{K_{\mathbb{C}}}^h + \frac{(2-b_-)(2m+2-b_-)}{2}Y^2\Delta_{K_{\mathbb{C}}}^{\mathbb{R}} - \frac{m(2m+2-b_-)}{2}Y^2\Delta_{K_{\mathbb{C}}}^{\overline{h}}$$

commutes with all the weight m slash operators as well as with the Laplacian $\Delta_m^{(b_-)}$. The commutator of the global weight raising operator and the weight lowering operator is

$$\left[R^{(b_-)}, L^{(b_-)}\right]_m = R_{m-2}^{(b_-)}L^{(b_-)} - L^{(b_-)}R_m^{(b_-)} = \frac{m\Delta_m^{(b_-)}}{2} - \frac{mb_-(2m-2-b_-)}{4}.$$

Proposition 1.5 provides another proof to [Ze2, Lemma 2.2] about $\Delta_m^{(b_-)}$. It also implies that $\Xi_m^{(b_-)}$ preserves the spaces of automorphic forms of weight m for all $m \in \mathbb{Z}$ and for every discrete subgroup Γ of cofinite volume in $O^+(V)$. It also commutes with $\Delta_m^{(b_-)}$, hence preserves eigenvalues of such automorphic forms. By rank considerations, one can probably show that the ring of differential operators which commute with all the slash operators of weight m is generated by $\Delta_m^{(b_-)}$ and $\Xi_m^{(b_-)}$, hence is a polynomial ring in two variables (if $b_- > 1$). This assertion should also follow from [Sh5, part (3) of Theorem 3.3] (since the rank of the symmetric space $G(V)$ is 2 if $b_- > 1$), though I have not verified this in detail. As $\Delta_0^{(b_-)}$ is $8\Delta_1$ and $\Xi_0^{(b_-)}$ is $16\Delta_2$

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in the notation of [Na], Proposition 1.5 generalizes the main result of that reference to other weights. A similar argument yields results of the same sort for (m,n), where a possible normalization for $\Xi^{(b_-)}_{m,n}$ is $(Y^2)^{-n}\Xi^{(b_-)}_{m-n}(Y^2)^n$, for which an equality similar to Equation (2) holds. We shall not need these results in what follows.

We now consider compositions of the weight raising operators. The natural lth power of $R^{(b_-)}_m$ is the composition

$$(R^{(b_-)}_m)^l = R^{(b_-)}_{m+2l-2} \circ \cdots \circ R^{(b_-)}_m.$$

The general formula for the resulting operator seems too complicated to write as a combination of $\Delta^h_{K_{\mathbb{C}}}$, D^*, and $\frac{1}{Y^2}$ with explicit coefficients. However, we can establish the properties given in the following

PROPOSITION 1.6. (i) *The operator* $(R^{(b_-)}_m)^l$ *takes automorphic forms of weight* m *on* $G(L_{\mathbb{R}})$ *to automorphic forms of weight* $m+2l$. (ii) *In case the former automorphic form is an eigenfunction with eigenvalue* λ, *the latter is also an eigenfunction, and the corresponding eigenvalue is* $\lambda + l(4m+4l-2b_-)$. (iii) *The operator* $(R^{(b_-)}_m)^l$ *can be written as*

$$(R^{(b_-)}_m)^l = \sum_{c=0}^{l}\sum_{a=0}^{c} A^{(l)}_{a,c}\frac{(iD^*)^{c-a}(\Delta^h_{K_{\mathbb{C}}})^{l-c}}{(-Y^2)^c},$$

where $A^{(0)}_{0,0}=1$ *and given the coefficients* $A^{(l)}_{a,c}$ *for given* l, *the coefficient* $A^{(l+1)}_{a,c}$ *of the next power* $l+1$ *is defined recursively as*

$$\sum_{s=0}^{a}\binom{c-s}{a-s}A^{(l)}_{s,c} + (2m+4l-2c+4-b_-) \times\left(A^{(l)}_{a,c-1} + \frac{m+2l-c+1}{2}A^{(l)}_{a-1,c-1}\right).$$

(iv) *For* $a=0$ *the coefficients* $A^{(l)}_{0,c}$ *are given by the explicit formula*

$$A^{(l)}_{0,c} = \frac{l!\cdot 2^c}{(l-c)!}\binom{m+l-\frac{b_-}{2}}{c}.$$

The binomial symbol appearing in part (iv) of Proposition 1.6 is the *extended binomial coefficient*: Indeed, for two nonnegative integers x and n

we have

$$\binom{x}{n} = \frac{1}{n!}\prod_{j=0}^{n-1}(x-j),$$

a formula which makes sense for $x \in \mathbb{R}$ (as well as x in any $\mathbb{Q}$-algebra).

Part (i) of Proposition 1.6 follows immediately from Corollary 1.2. For part (ii) Corollary 1.4 shows that the application of R_{m+2r} (for $0 \leqslant r \leqslant l-1$) to an eigenfunction adds $4m+8r+4-2b_-$ to the eigenvalue, so the assertion follows from evaluating

$$\sum_{r=0}^{l-1}(4m+8r+4-2b_-) = l(4m+4l-2b_-).$$

The proofs of parts (iii) and (iv) are given in Section 3.

We recall that $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ defines the holomorphic map

$$M : \tau \in \Big[\mathcal{H} = \big\{\tau = x+iy \in \mathbb{C} \big| y > 0\big\}\Big] \mapsto \frac{a\tau+b}{c\tau+d}, \quad \text{with } j(M,\tau) = c\tau+d,$$

the latter being the factor of automorphy of this action. Modular forms of weight (k,l) (or just weight k if $l=0$) with respect to a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ with cofinite volume (with respect to the invariant measure $\frac{dxdy}{y^2}$) are functions $f : \mathcal{H} \to \mathbb{C}$ which are invariant under the corresponding weight (k,l) slash operators for elements of Γ. The weight (k,l) Laplacian is

$$\Delta_{k,l} = 4y^2\partial_\tau\partial_{\overline{\tau}} - 2iky\partial_{\overline{\tau}} + 2ily\partial_\tau + l(k-1),$$

normalized such that $\Delta_k = \Delta_{k,0}$ annihilates holomorphic functions and the Laplacians commute with powers of y as in Equation (2). The *Shimura–Maaß operators*

$$\delta_k = y^{-k}\partial_\tau y^k = \partial_\tau + \frac{k}{2iy} \qquad \text{and} \qquad y^2\partial_{\overline{\tau}}$$

(note the different normalization from [Bru, Ze2]!) take modular forms of weight k to modular forms of weight $k+2$ and $k-2$, respectively, or more precisely, satisfy an appropriate commutation relation with the slash operators for all the elements of $\mathrm{SL}_2(\mathbb{R})$. They also change Laplacian eigenvalues (again, with respect to $-\Delta_k$ rather than Δ_k): δ_k adds k to the eigenvalue, while $y^2\partial_{\overline{\tau}}$ subtracts $k-2$ from it. Moreover, the powers of the Shimura–Maaß operators are given by, for example, [Za, equation (56)], stating that

$$\delta_k^l = \delta_{k+2l-2} \circ \cdots \circ \delta_k = \sum_{r=0}^{l} \frac{l!}{(l-r)!} \binom{k+l-1}{r} \frac{\partial_\tau^{l-r}}{(2iy)^r}$$

(for arbitrary k, not necessarily integral and nonnegative). Theorem 1.1 and Proposition 1.3 show that our weight changing operators $R_m^{(b_-)}$ and $L^{(b_-)}$ have similar properties. However, our operators are differential operators of order 2 while the Shimura–Maaß operators are of order 1. This is why the results of Propositions 1.5 and 1.6 are more complicated than the fact that $\delta_{k-2} y^2 \partial_{\overline{\tau}}$ is just $\frac{\Delta_k}{4}$, the commutator $[\delta, y^2 \partial_{\overline{\tau}}]_k$ being simply $\frac{k}{4}$, and [Za, equation (56)].

Nonetheless, the operators $R_m^{(b_-)}$ and $L^{(b_-)}$ for small values of b_- are closely related to the Shimura–Maaß operators. Indeed, for $b_- = 1$ the group $\mathrm{SO}_{2,1}^+$ is $\mathrm{PSL}_2(\mathbb{R})$ and the tube domain $K_\mathbb{R} + iC$ is just $\mathcal{H}$. We have

$$J(M, \tau) = j^2(M, \tau), \quad \text{hence } [M]_m = [M]_{2m}^{\mathcal{H}} \qquad \text{and} \qquad \Delta_m^{(1)} = \Delta_{2m}$$

(the same assertions hold for the operators involving anti-holomorphic weights). Our operators $R_m^{(1)}$ and $L^{(1)}$ are *squares* of the Shimura–Maaß operators, namely

$$R_m^{(1)} = \delta_{2m}^2 = \delta_{2m+2}\delta_{2m} \qquad \text{and} \qquad L^{(1)} = (y^2 \partial_{\overline{\tau}})^2.$$

Note that in this case

$$\Xi_m^{(1)} = \frac{(\Delta_{2m})^2}{16} - \frac{m\Delta_{2m}}{8} \in \mathbb{C}[\Delta_m^{(1)} = \Delta_{2m}],$$

in accordance with the rank of the group being 1 rather than 2 (in particular, in the notation of [Na] we have $\Delta_2 = \frac{\Delta_1}{4}$ in this case).

For $b_- > 1$ many authors (including [Bru, Na]) take the basis for $K_\mathbb{R}$ as two elements spanning a hyperbolic plane together with an orthogonal basis of elements of norm -2. In elements of the positive cone C, the first two coordinates are positive. In particular, for $b_- = 2$ we have $K_\mathbb{R} + iC \cong \mathcal{H} \times \mathcal{H}$, with $\tau = x + iy$ and $\sigma = s + it$ being the two coordinates. The group $\mathrm{SO}_{2,2}^+$ is an order 2 quotient of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, acting on $G(V) \cong \mathcal{H} \times \mathcal{H}$ through

$$(M, N) : (\tau, \sigma) \mapsto (M\tau, N\sigma) \quad \text{with } J\big((M, N), (\tau, \sigma)\big) = j(M, \tau) j(N, \sigma).$$

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It follows that

$$[M,N]_m=[M]^{\mathcal{H}}_{m,\tau}[N]^{\mathcal{H}}_{m,\sigma}\qquad\text{and}\qquad\Delta^{(2)}_m=2\Delta_{m,\tau}+2\Delta_{m,\sigma}$$

(which extend to the operators with anti-holomorphic weights as well). Our operators are

$$R^{(2)}_m=2\delta_{m,\tau}\delta_{m,\sigma},\qquad L^{(2)}=8y^2t^2\partial_{\overline{\tau}}\partial_{\overline{\sigma}}\qquad\text{and}\qquad\Xi^{(2)}_m=\Delta_{m,\tau}\Delta_{m,\sigma}.$$

In both cases $b_-=1$ and $b_-=2$ the assertions of this section follow from properties of the Shimura–Maaß operators (note that Y^2 is $2y^2$ for $b_-=1$). When $b_-=2$ the special orthogonal group of a negative definite subspace is also $\mathrm{SO}(2)$, which makes the theory of automorphic forms more symmetric.

Working with $b_-=3$ in this model yields another coordinate $z=u+iv$. The positivity of y, t, and $yt-v^2$ is equivalent to

$$\Pi=\begin{pmatrix}\tau & z\\ z & \sigma\end{pmatrix}\quad\text{being in }\mathcal{H}_2=\left\{\Pi=X+iY\in M_2(\mathbb{C})\middle|\Pi=\Pi^t,\,Y\gg 0\right\}.$$

Hence $K_{\mathbb{R}}+iC$ is identified with the Siegel upper half-plane of degree 2. The group $\mathrm{SO}^+_{2,3}$ is $\mathrm{PSp}_4(\mathbb{R})$, with the symplectic action and the factor of automorphy (hence the slash operators) from the theory of Siegel modular forms. In this case

$$R^{(3)}_m=-\frac{M_m}{Y^2},\qquad L^{(3)}=-Y^2N_0,\qquad\text{and}\qquad\Delta^{(3)}_m=2Tr(\Omega_{m,0})$$

in the notation of [Ma1, Ma2] for degree 2 (for weight (m,n) the latter assertion extends to the modified Laplacian $\widetilde{\Delta}^{(3)}_{m,n}$ presented in Section 3). The operator $\Delta^h_{K_{\mathbb{C}}}$ is also a constant multiple of the operator $\mathbb{D}$ considered, for example, in [CE, Ch].

§2. Images of Theta Lifts under $R^{(b_-)}_m$ and $L^{(b_-)}$

For natural r, s, t, and l we define the polynomials

$$P_{r,s,t}(\mu,Z)=\frac{(\mu,Z_V)^r(\mu,\overline{Z_V})^t}{(Y^2)^s}\qquad\text{and}\qquad P^{(l)}_{r,s,t}(\mu,Z)=P_{r,s,t}(\mu,Z)(\mu^2_-)^l.$$

As a function of $\mu\in V$, the polynomial $P_{r,s,t}(\mu,Z)$, considered, for example, in [Ze2], is homogeneous of degree $(r+t,0)$ with respect to the element of $G(V)$ represented by Z, while $P^{(l)}_{r,s,t}(\mu,Z)$ has homogeneity degree

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$(r+t, 2l)$. Equation (5) of [Ze2] extends from $P_{r,s,t} = P^{(0)}_{r,s,t}$ to the more general polynomials $P^{(l)}_{r,s,t}$: The equality

$$P^{(l)}_{r,s,t}(MZ, \mu) = J(M, Z)^{s-r}\overline{J(M, Z)}^{s-t} P^{(l)}_{r,s,t}(Z, M^{-1}\mu) \tag{3}$$

holds for every $\mu \in V$, $Z \in K_{\mathbb{R}} + iC$, $M \in O^{+}(V)$, and r, s, t, and l from $\mathbb{N}$.

We shall assume that $V = L_{\mathbb{R}}$ for some fixed even lattice L (of signature $(2, b_{-})$), and consider the theta function of L which is based on the polynomial $P^{(l)}_{r,s,t}$. This is a (vector-valued) function of $\tau = x + iy \in \mathcal{H}$ and $Z \in K_{\mathbb{R}} + iC$, which is a sum of expressions of the form

$$F^{(l)}_{r,s,t}(\tau, Z, \mu) = e^{-\Delta_v/8\pi y}(P^{(l)}_{r,s,t})(\mu, Z)\mathbf{e}\left(\tau\frac{\mu_+^2}{2} + \overline{\tau}\frac{\mu_-^2}{2}\right). \tag{4}$$

Here $\mu_{\pm}$ are the parts of $\mu \in V$ which lie in the spaces $v_{\pm}$ according to the element of $G(V)$ corresponding to Z, Δ_v is the Laplacian on V which corresponds to the *majorant* associated with that element (i.e., to the bilinear form in which the sign on the pairing on v_- is inverted to be positive as well), and $\mathbf{e}(w) = e^{2\pi i w}$ for every complex w. A simple and direct calculation proves

LEMMA 2.1. (i) *We have the equality $\mu_+^2 = P_{1,1,1}(\mu, Z)$. In addition, the following equalities hold:*

$$\text{(ii)} \quad \Delta_{v_+} P_{r,s,t} = 4rtP_{r-1,s-1,t-1}.$$
$$\text{(iii)} \quad \Delta_{v_-}(\mu_-^2)^l = 2l(2l + b_- - 2)(\mu_-^2)^{l-1}.$$

Part (i) of Lemma 2.1 shows that we can write the exponent in Equation (4) as the constant $\mathbf{e}\big(\overline{\tau}\frac{\mu^2}{2}\big)$ (independent of Z) times $e^{-2\pi y P_{1,1,1}}$. Since the differences in the indices in part (ii) of Lemma 2.1 remain the same, l does not affect the weight of modularity of $P^{(l)}_{r,s,t}$, and $P_{1,1,1}$ is invariant (by Equation (3)), we find that replacing P by F in Equation (3) still yields a valid equation. Let $L^* = \mathrm{Hom}(L, \mathbb{Z})$ be the dual lattice of L and L^*/L the (finite) *discriminant group* of L. Then the theta function $\Theta^{(l)}_{L,r,s,t}$ is the $\mathbb{C}[L^*/L]$-valued function defined by

$$\Theta^{(l)}_{L,r,s,t}(\tau, Z) = \sum_{\gamma \in L^*/L} \theta^{(l)}_{\gamma+L,r,s,t}(\tau, Z)e_{\gamma},$$
$$\theta^{(l)}_{\gamma+L,r,s,t}(\tau, Z) = \sum_{\mu \in \gamma+L} F^{(l)}_{r,s,t}(\tau, Z, \mu)$$

(this function is $\Theta_L(\tau,0,0;v,P^{(l)}_{r,s,t})$ in the notation of [B], where $v\in G(L_{\mathbb{R}})$ corresponds to $Z\in K_{\mathbb{R}}+iC$). The extension of Equation (3) to $\Theta^{(l)}_{L,r,s,t}$ shows that $\Theta^{(l)}_{L,r,s,t}$ is automorphic of weight $(s-r,s-t)$ as a function of $Z\in K_{\mathbb{R}}+iC$, and [B, Theorem 4.1] shows that as a function of $\tau\in\mathcal{H}$ it is a vector-valued modular form of weight $\left(1+r+t,2l+\frac{b_-}{2}\right)$ and the *Weil representation* ρ_L. The latter is a representation of the metaplectic double cover $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$, which is defined by sending the generators T and S of $\mathrm{Mp}_2(\mathbb{Z})$ lying over the elements $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$, respectively, to

$$\rho_L(T)(e_\gamma)=\mathbf{e}(\gamma^2/2)e_\gamma,$$
$$\rho_L(S)(e_\gamma)=\frac{\zeta_8^{b_--b_+}}{\sqrt{\Delta_L}}\sum_{\delta\in L^*/L}\mathbf{e}(-(\gamma,\delta))e_\delta,$$

respectively. For the properties of ρ_L see [Ze1], as well as the references cited there. The space $\mathbb{C}[L^*/L]$ comes with a Hermitian pairing $\langle\cdot,\cdot\rangle_{\rho_L}$ in which the e_γ are orthonormal, and ρ_L is a unitary representation with respect to this pairing. The operation of complex conjugation on $\Theta^{(l)}_{L,r,s,t}$ interchanges r and t and sends τ to $-\overline{\tau}$ (this is equivalent to multiplying the bilinear form on V by -1, but as we rather stay in the signature $(2,b_-)$ setting, we prefer this anti-holomorphic operation on τ). It also replaces ρ_L by its dual representation, but we shall consider the effect of complex conjugation only for the automorphy in the Z variable.

We are interested in the action of the operators $R_m^{(b_-)}$ and $L^{(b_-)}$ on theta kernels, and the resulting differential properties of the associated theta lifts. Several proofs will involve comparisons of these actions on theta kernels with the actions of the operators δ_k and $y^2\partial_{\overline{\tau}}$ on these theta kernels (multiplied by the appropriate powers of y). The latter are given (in a more general context) in [Ze2, equations (6a) and (6b)]. As $P^{(l)}_{r,s,t}=P_{r,s,t}(\mu_-^2)^l$, Lemma 2.1 shows that in our case these equations take the form

(5)
$$\delta_k y^{\frac{b_-}{2}+2l}\Theta^{(l)}_{L,r,s,t}=\pi i y^{\frac{b_-}{2}+2l}\Theta^{(l)}_{L,r+1,s+1,t+1}+\frac{il(2l+b_--2)}{8\pi}y^{\frac{b_-}{2}+2l-2}\Theta^{(l-1)}_{L,r,s,t}$$

(where $k=1-\frac{b_-}{2}+r+t-2l$) and

$$(6)\quad y^2\partial_{\overline{\tau}}y^{\frac{b_-}{2}+2l}\Theta^{(l)}_{L,r,s,t}=\pi i y^{\frac{b_-}{2}+2l+2}\Theta^{(l+1)}_{L,r,s,t}+\frac{irt}{4\pi}y^{\frac{b_-}{2}+2l}\Theta^{(l)}_{L,r-1,s-1,t-1}$$

(note again the different normalization of these operators).

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Recall that given a modular form F of weight $1+r+t-\frac{b_-}{2}-2l$ and representation ρ_L, possibly with exponential growth at the cusps, its *theta lift with respect to the polynomial* $P_{r,s,t}^{(l)}$ is defined in [B, Ze2], and others as follows. For $w>1$ let

$$D_w=\{\tau\in\mathcal{H}\big||\Re\tau|\leqslant 1/2, |\tau|\geqslant 1, \Im\tau\leqslant w\},$$

and assume that

$$\lim_{w\to\infty}\int_{D_w}y^{1+r+t-\sigma}\langle F(\tau),\Theta_L(\tau,v,p_v)\rangle_{\rho_L}\frac{dxdy}{y^2}$$

exists for $\Re\sigma\gg 0$ and defines a holomorphic function of σ on some right half-plane, which may be extended to a meromorphic function of σ for all $\sigma\in\mathbb{C}$. Then the theta lift $\Phi_{L,r,s,t}^{(l)}(F,Z)$ is the constant term of the expansion of this meromorphic function at $\sigma=0$. Now, the modular form F has a Fourier expansion of the sort

$$F(\tau)=\sum_{\gamma\in L^*/L}\sum_{n\in\mathbb{Q}}c_{n,\gamma}(y)q^n e_\gamma, \tag{7}$$

where q^n denotes $\mathbf{e}(n\tau)$ and the $c_{n,\gamma}$ are smooth functions of $y=\Im\tau$, which vanish unless $n\in\frac{\gamma^2}{2}+\mathbb{Z}$. The modular forms which are usually considered also satisfy the condition that $c_{n,\gamma}=0$ unless $n\gg-\infty$ (for F which is holomorphic on $\mathcal{H}$ this means at most a pole at the cusp, and not an essential singularity).

The relations between the action of the (classical) Shimura–Maaß operators on the lifted modular form and the action of these operators on the theta kernel used for the theta lift are given in the following

LEMMA 2.2. *Let $F_\pm$ be a modular form of weight $k\pm 2=1+r+t-\frac{b_-}{2}-2l\pm 2$ and representation ρ_L, with Fourier expansion as in Equation (7), and assume that the regularized theta lifts $\Phi_{L,r,s,t}^{(l)}(Z,\delta_k F_-)$ and $\Phi_{L,r,s,t}^{(l)}(Z,y^2\partial_{\overline{\tau}}F_+)$ are well defined. Assume that the growth condition $c_{\gamma,n}(y)=o(e^{\varepsilon y})$ as $y\to\infty$ holds for every γ, n, and $\varepsilon>0$, and that $c_{0,0}(y)$ is $o(y^T)$ as $y\to\infty$ for some T. Then the theta lift $\Phi_{L,r,s,t}^{(l)}(Z,\delta_k F_-)$ coincides, up to an additive constant which may appear only if $r=t$, with the value at Z of the theta lift of F_- with respect to $-y^2\partial_{\overline{\tau}}\Theta_{L,r,s,t}^{(l)}$. The same assertion holds for $\Phi_{L,r,s,t}^{(l)}(Z,y^2\partial_{\overline{\tau}}F_+)$ and the theta lift of F_+ with respect to $-\delta_k\Theta_{L,r,s,t}^{(l)}$.*

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Proof. See [Ze2, Lemmas 3.4 and 3.6] as well as the argument proving Lemma 2.7 of that reference. Note the factors of $2i$ distinguishing our operators here from those of [Ze2], and observe that the theta function is conjugated in the integral defining the theta lift. □

The complex conjugation of $\Theta_{L,r,s,t}^{(l)}$ in the definition of the theta lift implies that $\Phi_{L,r,s,t}^{(l)}(Z,F)$ is automorphic of weight $(s-t,s-r)$. We shall thus consider only the case $r=s$, where the automorphy in (the corresponding) Equation (3) involves only $J(M,Z)$ and not its complex conjugate. As with $P_{r,s,t}$, we may omit the superscript (l) in case $l=0$. In the same manner as in Section 1, we shall postpone most of the (calculational) proofs to Section 4. Only the assertions about theta lifts will be proved here.

The first assertion we are interested in is

PROPOSITION 2.3. *The action of $R_m^{(b_-)}$ takes $y^{\frac{b_-}{2}}\overline{\Theta_{L,m,m,0}}$ to $4\pi i$ times the complex conjugate of $y^2\partial_{\overline{\tau}}\big(y^{\frac{b_-}{2}}\Theta_{L,m+2,m+2,0}\big)$.*

We remark that Proposition 2.3 may be formulated in terms of comparing the actions of elements from the universal enveloping algebras of $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(V)\cong\mathfrak{so}_{2,b_-}$ on the theta kernel. However, unlike [Ze2, Proposition 2.3] (and [Bru, Proposition 4.5]), which compares the action of order 2 elements of both universal enveloping algebras, here the one from the algebra of $\mathfrak{so}_{2,b_-}$ has order 2 while the element from $\mathfrak{sl}_2$ has order 1.

We can now establish the first property of the theta lift from [Ze2].

THEOREM 2.4. *Assume that b_- is even, and let f be a weakly holomorphic modular form of weight $1-\frac{b_-}{2}-m$ and representation ρ_L. Consider the modular form $F=\frac{1}{(2\pi i)^m}\delta^m_{1-\frac{b_-}{2}-m}f$, of weight $k=1-\frac{b_-}{2}+m$, and its theta lift $\Phi_{L,m,m,0}(Z,F)$ considered in [Ze2, Theorem 3.9]. The image of the latter automorphic form under $\frac{1}{(8\pi^2)^{b_-/2}}(R_m^{(b_-)})^{b_-/2}$ is a meromorphic automorphic form of weight $m+b_-$ on $K_{\mathbb{R}}+iC$, whose singularities are poles of order $m+b_-$ along special divisors.*

Proof. Proposition 2.3 yields the equality

$$\frac{1}{8\pi^2}R_m^{(b_-)}y^{\frac{b_-}{2}}\overline{\Theta_{L,m,m,0}}(\tau,Z)=\frac{i}{2\pi}\overline{y^2\partial_{\overline{\tau}}y^{\frac{b_-}{2}}\Theta_{L,m+2,m+2,0}(\tau,Z)}$$

for every $\tau\in\mathcal{H}$ and $Z\in K_{\mathbb{R}}+iC$. As F (as well as its images under any power of δ_k) satisfies the conditions of Lemma 2.2, we establish

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the equality

$$\frac{1}{8\pi^2}R_m^{(b_-)}\Phi_{L,m,m,0}(Z,F)=\Phi_{L,m+2,m+2,0}\left(Z,\frac{1}{2\pi i}\delta_k F\right).$$

Repeating this argument, we get

$$\frac{1}{(8\pi^2)^l}(R_m^{(b_-)})^l\Phi_{L,m,m,0}(Z,F)=\Phi_{L,m+2l,m+2l,0}\left(Z,\frac{1}{(2\pi i)^l}\delta_k^l F\right)$$

for any $l\in\mathbb{N}$. Consider now the case $l=\frac{b_-}{2}$. Then $\widetilde{F}=\frac{1}{(2\pi i)^{b_-/2}}\delta_k^{b_-/2}F$ is $\frac{1}{(2\pi i)^{m+b_-/2}}\delta_{k-2m}^{m+b_-/2}f$ with f weakly holomorphic of weight $1-\frac{b_-}{2}-m$ (which is integral since b_- is even). But then $\frac{1}{(2\pi i)^{m+b_-/2}}\delta_{k-2m}^{m+b_-/2}$ is just the operator $\left(\frac{\partial_\tau}{2\pi i}\right)^{m+b_-/2}$ (which takes q^n from a Fourier expansion to $n^{m+b_-/2}q^n$—this is the reason for our normalization), so that the weight $1+\frac{b_-}{2}+m$ modular form $\widetilde{F}$ is again weakly holomorphic. Theorem 14.3 of [B] now shows that our automorphic form of weight $m+b_-$, which we write as $\Phi_{L,m+b_-,m+b_-,0}(Z,\widetilde{F})$, is meromorphic on $K_{\mathbb{R}}+iC$, with poles of order $m+b_-$ along rational quadratic divisors associated with negative norm vectors in L^* whose corresponding coefficients in Equation (7) do not vanish. This completes the proof of the theorem. □

We remark that in case the modular form f is a harmonic weak Maaß form the modular form $\widetilde{F}$ from the proof of Theorem 2.4 is again weakly holomorphic. Moreover, in case the image of f under the operator ξ_{k-2m} of [BF] does not have a pole at the cusp, the theta lift has no additional singularities, and the result of Theorem 2.4 extends to this case. However, in the theta lift $\Phi_{L,m,m,0}(Z,F)$ itself one can still distinguish the case where f is weakly holomorphic from the one where F is such a harmonic weak Maaß form.

For the weight lowering operator $L^{(b_-)}$, we do not have a nice equivalent to Proposition 2.3. However, we do have an interesting result concerning its mth power. We begin with

LEMMA 2.5. *The image of* $\Theta_{L,k,n,n}^{(l)}(-\overline{\tau},Z)$ *under* $L^{(b_-)}$ *is*

$$4\pi^2y^2\Theta_{L,k+2,n,n}^{(l+1)}(-\overline{\tau},Z)+n\left(2l+\frac{b_-}{2}\right)\Theta_{L,k+1,n-1,n-1}^{(l)}(-\overline{\tau},Z)$$
$$+\frac{n(n-1)l\left(l-1+\frac{b_-}{2}\right)}{4\pi^2y^2}\Theta_{L,k,n-2,n-2}^{(l-1)}(-\overline{\tau},Z).$$

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Lemma 2.5 allows us to establish the following

PROPOSITION 2.6. *For any $s \in \mathbb{N}$, the image of $\overline{\Theta_{L,m,m,0}}$ under $(L^{(b_-)})^s$ attains, on τ and Z, the value*

$$\sum_h \binom{s}{h} \frac{\Gamma\left(s+\frac{b_-}{2}\right)}{\Gamma\left(h+\frac{b_-}{2}\right)} \frac{m!(4\pi^2 y^2)^h}{(m-s+h)!} \Theta^{(h)}_{L,s+h,m-s+h,m-s+h}(-\overline{\tau}, Z).$$

The case $s = m$ in Proposition 2.6 is of particular importance, as is shown in the following

PROPOSITION 2.7. *The expression $(L^{(b_-)})^m y^{\frac{b_-}{2}} \overline{\Theta_{L,m,m,0}}$ equals the complex conjugate of $(-4\pi i)^m \delta^m_{1-\frac{b_-}{2}-m,\tau} y^{\frac{b_-}{2}+2m} \Theta^{(m)}_{L,0,0,m}(\tau, Z)$.*

Automorphic forms of nonzero weight can never be real-valued, because complex conjugation yields an automorphic form with a different weight. However, multiplying the complex conjugate automorphic form by a power of Y^2 leads to an object which is comparable with the image of our automorphic form under the appropriate power of a weight changing operator, as these two functions do have the same weight. We shall thus say that an automorphic form Φ, of positive weight m, is *m-real* if its image under the mth power of the weight lowering operators $L^{(b_-)}$ coincides with its complex conjugate multiplied by a positive multiple of $(Y^2)^m$. We now show that the theta lifts from [Ze2, Theorem 3.9] are m-real, or more generally:

THEOREM 2.8. *Let F be as in Theorem 2.4 (but without the restriction on the parity of b_-), and assume that F is an eigenfunction with respect to (minus) the Laplacian of weight $1-\frac{b_-}{2}+m$, with eigenvalue $\lambda = -\frac{mb_-}{2}$. Assume further that the Fourier coefficients $c_{\gamma,n}$ of F appearing in Equation (7) are real. Then applying the operator $(L^{(b_-)})^m$ to $\frac{i^m}{2}\Phi_{L,m,m,0}(Z, F)$ yields $m!\Gamma\left(m+\frac{b_-}{2}\right)(Y^2)^m/\Gamma\left(\frac{b_-}{2}\right)$ times the complex conjugate of $\frac{i^m}{2}\Phi_{L,m,m,0}(Z, F)$.*

Proof. By Proposition 2.7, the image of $\frac{i^m}{2}\Phi_{L,m,m,0}$ under $(L^{(b_-)})^m$ coincides with $\frac{i^m}{2}$ times the regularized integral of F paired with the function

$$(-4\pi i)^m \delta^m_{1-\frac{b_-}{2}-m,\tau} y^{\frac{b_-}{2}+2m} \Theta^{(m)}_{L,0,0,m}(\tau, Z).$$

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On the other hand, the fact that the first index in $P_{0,0,m}$ vanishes allows us to use Equation (6) successively m times and write

$$(-\pi i)^m y^{\frac{b_-}{2}+2m}\Theta^{(m)}_{L,0,0,m}(\tau,Z) \quad \text{as just } (-y^2\partial_{\overline{\tau}})^m y^{\frac{b_-}{2}}\Theta_{L,0,0,m}(\tau,Z).$$

As in the proof of Theorem 2.4, we can write $(L^{(b_-)})^m\Phi_{L,m,m,0}(Z,F)$, using Lemma 2.2, as the theta lift $\frac{i^m}{2}\Phi_{L,0,0,m}\big(Z,4^m\delta^m_{1-\frac{b_-}{2}-m,\tau}(-y^2\partial_{\overline{\tau}})^m F\big)$. Now, as F is an eigenfunction and $y^2\partial_{\overline{\tau}}$ takes eigenfunctions to eigenfunctions, we can replace each combination $-4\delta_l y^2\partial_{\overline{\tau}}$, starting from the innermost pair, by the appropriate eigenvalue. As after applying $(-y^2\partial_{\overline{\tau}})^r$ the eigenvalue becomes $\lambda - r\big(m-r-\frac{b_-}{2}\big)$, the modular form we plug inside the latter lift is just F multiplied by the scalar $\prod_{r=0}^{m-1}\big[\lambda - r\big(m-r-\frac{b_-}{2}\big)\big]$. Substituting the value of λ, the rth multiplier becomes just $(r-m)\big(r+\frac{b_-}{2}\big)$, and the product is $(-1)^m m!\Gamma\big(m+\frac{b_-}{2}\big)/\Gamma\big(\frac{b_-}{2}\big)$. Division by $m!\Gamma\big(m+\frac{b_-}{2}\big)(Y^2)^m/\Gamma\big(\frac{b_-}{2}\big)$ thus gives $\frac{(-i)^m}{2}\Phi_{L,0,m,m}(Z,F)$, so that we need to show why $\Phi_{L,0,m,m}(Z,F)$ is the complex conjugate of $\Phi_{L,m,m,0}(Z,F)$. As the Fourier coefficients of F are real, we obtain $\overline{F(\tau)} = F(-\overline{\tau})$. On the other hand, we have seen that complex conjugation on our theta function interchanges the indices r and t and replaces the variable τ by $-\overline{\tau}$. The required assertion now follows from the fact that powers of y and the measure $\frac{dxdy}{y^2}$ are both preserved by the change of variable $\tau \mapsto -\overline{\tau}$. This completes the proof of the theorem. □

We remark that the choice of $\lambda = -\frac{mb_-}{2}$ in Theorem 2.8 is not crucial. Any choice of λ for which the number $\prod_{r=0}^{m-1}\big[r\big(m-r-\frac{b_-}{2}\big)-\lambda\big]$ is positive will be sufficient for Theorem 2.8 to hold (with the same proof). However, we chose this eigenvalue as it is the eigenvalue of the theta lifts from [Ze2].

§3. Proofs of the Properties of $R_m^{(b_-)}$ and $L^{(b_-)}$

In this section, we include the proofs of the properties of the weight raising and weight lowering operators appearing in Section 1.

We first introduce (following [Na]) a convenient set of generators for $O^+(V)$. For $\xi \in K_{\mathbb{R}}$ we define the element $p_\xi \in \mathrm{SO}^+(V)$ whose action is

$$\big[\mu \in K_{\mathbb{R}} = \{z,\zeta\}^{\perp}\big] \mapsto \mu - (\mu,\xi)z, \quad \zeta \mapsto \zeta + \xi - \frac{\xi^2}{2}z, z \mapsto z.$$

Furthermore, given an element $A \in O(K_{\mathbb{R}})$ and a scalar $a \in \mathbb{R}^*$ such that $a > 0$ if $A \in O^+(K_{\mathbb{R}})$ and $a < 0$ otherwise, we let $k_{a,A} \in O^+(V)$ be the

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element acting as

$$\left[\mu \in K_{\mathbb{R}} = \{z, \zeta\}^{\perp}\right] \mapsto A\mu, \quad \zeta - \frac{\zeta^2}{2}z \mapsto \frac{1}{a}\left(\zeta - \frac{\zeta^2}{2}z\right), z \mapsto az.$$

For any $Z \in K_{\mathbb{R}} + iC$ we have

$$p_\xi Z = Z + \xi, \quad J(p_\xi, Z) = 1, \qquad k_{a,A}Z = aAZ, \qquad \text{and} \quad J(k_{a,A}, Z) = \frac{1}{a}.$$

Note that the relation between A and the sign of a is equivalent to preserving C rather than mapping Z into $K_{\mathbb{R}} - iC$—it appears that [Na] ignored this point. Choose now an element of $G(K_{\mathbb{R}})$ in which the positive definite space is generated by the norm 1 vector u_1, and consider the involution $w \in \mathrm{SO}^+(K_{\mathbb{R}})$ defined by

$$\left[\mu \in K_{\mathbb{R}} = \{z, \zeta\}^{\perp}\right] \mapsto \mu - 2(\mu, u_1)u_1, \quad \zeta - \frac{\zeta^2}{2}z \mapsto -z, \quad z \mapsto -\left(\zeta - \frac{\zeta^2}{2}z\right)$$

(w inverts the positive definite space $\mathbb{R}u_1$). Its action on $K_{\mathbb{R}} + iC$ is through

$$wZ = \frac{2}{Z^2}\left[Z - 2(Z, u_1)u_1\right] \quad \text{with } J(w, Z) = \frac{Z^2}{2}.$$

The elements $k_{a,A}$ with (a, A) in the index 2 subgroup of $R^* \times O(K_{\mathbb{R}})$ thus defined and p_ξ for $\xi \in K_{\mathbb{R}}$ generate the stabilizer $St_{O^+(V)}(\mathbb{R}z)$ of the isotropic space $\mathbb{R}z$ in $O^+(V)$ as the semi-direct product of these groups. The fact that adding w to $St_{O^+(V)}(\mathbb{R}z)$ generates $O^+(V)$ is now easily verified by considering the action on isotropic 1-dimensional subspaces of V.

Some useful relations are derived in the following

LEMMA 3.1. *Let $K_{\mathbb{R}}$ be a nondegenerate vector space of dimension b_-, fix $\alpha \in \mathbb{C}$, and let F be a $\mathcal{C}^2$ function that is defined on a neighborhood of a point $Z = X + iY \in K_{\mathbb{C}}$ with $Y^2 > 0$. Then the following relations hold:*

$$(Y^2)^{-\alpha}\Delta^h_{K_{\mathbb{C}}}\big((Y^2)^\alpha F\big)(Z) = \Delta^h_{K_{\mathbb{C}}}F(Z) - \frac{2i\alpha}{Y^2}D^*F(Z) - \frac{\alpha(\alpha - 1 + \frac{b_-}{2})}{Y^2}F(Z)$$

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and

$$(Y^2)^{-\alpha}\Delta^{\overline{h}}_{K_{\mathbb{C}}}\big((Y^2)^{\alpha}F\big)(Z) = \Delta^{\overline{h}}_{K_{\mathbb{C}}}F(Z) + \frac{2i\alpha}{Y^2}\overline{D^*}F(Z) - \frac{\alpha(\alpha-1+\frac{b_-}{2})}{Y^2}F(Z).$$

We remark that Lemma 3.1 holds for $K_{\mathbb{R}}$ of arbitrary signature (not necessarily Lorentzian), but not negative definite (for $Y^2 > 0$ to be possible).

Proof. The proof is obtained by a straightforward calculation, using an orthonormal basis for $K_{\mathbb{R}}$ and the action of ∂_k and $\partial_{\overline{k}}$ on functions of Y alone. □

We remark that the third operator $\Delta^{\mathbb{R}}_{K_{\mathbb{C}}}$ bears a property similar to Lemma 3.1, which is used implicitly in [Ze2, Section 3] in order to prove Equation (2).

We can now present the

Proof of Theorem 1.1. Multiply both sides of the desired assertion for $R_m^{(b-)}$, as well as the function F there, by $(Y^2)^m$. Lemma 3.1, the first definition of $R_m^{(b-)}$, and Equation (1) show that this yields the equivalent equality

$$(R_0^{(b-)}F)[M]_{2,-m} = R_0^{(b-)}\big(F[M]_{0,-m}\big).$$

Observe that conjugating the latter equation and multiplying by $(Y^2)^2$ yields the required equality for $L^{(b-)}$. Hence we are reduced to proving only this equality. Moreover, $R_0^{(b-)}$ involves only holomorphic differentiations, which means that it commutes with the power of $\overline{J(M,Z)}$ coming from the anti-holomorphic weights. Hence we can take $m = 0$, which implies that proving the equation

$$(R_0^{(b-)}F)[M]_2 = R_0^{(b-)}\big(F[M]_0\big)$$

(which the assertion for $R_0^{(b-)}$ in the formulation of the theorem) suffices for proving the theorem. Writing the arguments as $M^{-1}(Z)$ in both sides and using the cocycle condition brings the latter equation to the form

$$(R_0^{(b-)}F)(Z)J(M^{-1},Z)^2 = (R_0^{(b-)})^{M^{-1}}F(Z). \tag{8}$$

By a standard argument it suffices to verify Equation (8) for M^{-1} being one of the generators of $O^+(V)$ considered above. Equation (8) with $M^{-1} =$

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p_ξ follows from the invariance of both $\Delta^h_{K_\mathbb{C}}$ and D^* under translations of $X = \Re Z$ and the fact that $J(p_\xi, Z) = 1$. The action of $M^{-1} = k_{a,A}$ divides $\Delta^h_{K_\mathbb{C}}$ by a^2, leaves D^* invariant, and divides Y^2 by a^2 (since $A \in O(K_\mathbb{R})$), which proves Equation (8) since $J(k_{a,A}, Z) = \frac{1}{a}$. Finally, for $M^{-1} = w$ we have the equalities

$$(\Delta^h_{K_\mathbb{C}})^w = \left(\frac{Z^2}{2}\right)^2 \Delta^h_{K_\mathbb{C}} - (b_- - 2)\frac{Z^2}{2} D, \qquad (D^*)^w = \frac{Z^2}{\overline{Z}^2} D^* - \frac{2iY^2}{\overline{Z}^2} D$$

with $D = \sum_k z_k \partial_k$ from [Na] (the corresponding operator from [Na] is $\frac{1}{2}\Delta^h_{K_\mathbb{C}}$ rather than $\Delta^h_{K_\mathbb{C}}$, while $\delta = \frac{Z^2}{2}$, $\overline{\delta} = \frac{\overline{Z}^2}{2}$, and $d = \frac{Y^2}{2}$ there). Using Equation (1) we thus find that applying $M^{-1} = w$ to the sum of $\Delta^h_{K_\mathbb{C}}$ and $\frac{i(b_- - 2)}{Y^2} D^*$ (which is $R_0^{(b_-)}$) multiplies it by $\left(\frac{Z^2}{2}\right)^2$ (as the coefficients in front of D cancel), which establishes Equation (8) also for this case using the value of $J(w, Z)$. This completes the proof of the theorem. $\square$

In order to indicate what is the Lie-theoretic interpretation of the operators $R_m^{(b_-)}$ and $L^{(b_-)}$, we recall the vector u_1 we used for defining w above, and take a vector $\tilde{u} \in K_\mathbb{R}$ of norm -1 which is orthogonal to u_1 (we assume here $b_- > 1$, but for $b_- = 1$ our operators are squares of the order 1 operators δ_{2m} and $y^2 \partial_{\overline{\tau}}$, whose Lie-theoretic interpretation is given, for example, in [Ve]). These choices determine the parabolic subgroup of $\mathrm{SO}^+(V)$ appearing in the following

PROPOSITION 3.2. *Let $H_{K_\mathbb{R}}$ be the subgroup of $\mathrm{SO}^+(K_\mathbb{R})$ consisting of those matrices which preserve the isotropic subspace $\mathbb{R}(u_1 + \tilde{u})$ and whose action on the quotient $(u_1 + \tilde{u})^\perp / \mathbb{R}(u_1 + \tilde{u})$ is trivial. Define H to be the group generated by all the elements p_ξ with $\xi \in K_\mathbb{R}$ and by the elements $k_{a,A}$ with $a > 0$ and $A \in H_{K_\mathbb{R}}$. Then the group H operates freely and transitively on $K_\mathbb{R} + iC$.*

Let $K \cong \mathrm{SO}(2) \times \mathrm{SO}(b_-)$ be the stabilizer, in $\mathrm{SO}^+(V)$, of the element of $G(V)$ represented by $Z = iu_1$, and let $\mathfrak{k}$ be its Lie algebra. The action of a normalized generator of $\mathfrak{so}(2) \subseteq \mathfrak{k}$ on $\mathfrak{so}(V)_\mathbb{C}$ decomposes the latter space into the eigenspaces with eigenvalue 0 (this is precisely $\mathfrak{k}$) and $\pm i$ (complex conjugate spaces of dimension b_- each). Hence the action on the space of products of two elements of $\mathfrak{so}(V)$ (inside its universal enveloping algebra, say) decomposes into eigenspaces with eigenvalues 0, $\pm i$, and $\pm 2i$. One verifies that in each of the $\pm 2i$-eigenspaces, precisely one combination commutes with the part $\mathfrak{so}(b_-)$ of $\mathfrak{k}$. As our automorphic forms correspond

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to functions on $\mathrm{SO}(V)$ on which $\mathrm{SO}(2) \subseteq K$ operates according to a specific character and $\mathrm{SO}(b_-)$ operate trivially (normalized suitably), these elements (of order 2) of the universal enveloping algebra of $\mathfrak{so}(V)$ lead to weight raising and weight lowering operators. One may then evaluate, using the interplay between the operations of $\mathfrak{k}$ and the Lie algebra of the group H from Proposition 3.2, the action of these operators, and find that they lead to our $R_m^{(b_-)}$ and $L^{(b_-)}$. However, the change of coordinates between $H_{K_\mathbb{R}}$ and $K_\mathbb{R} + iC$ in this evaluation is more tedious than one might believe.

We also indicate briefly the connection between our operators and those of [Sh1]. That reference defines, for every representation ρ of $\mathbb{C}^\times \times GL_{b_-}(\mathbb{C})$ (a subgroup of which we identify with the complexification of the compact subgroup K, which is isomorphic to the product $\mathbb{C}^\times \times \mathrm{SO}(b_-, \mathbb{C})$), a differential operator that roughly sends (vector-valued) automorphic forms with weight (i.e., representation) ρ to automorphic forms having representation $\rho \otimes \omega$, where ω is the standard representation of that product on $\mathbb{C}^{b_-}$. This representation space is considered as the holomorphic cotangent space of $G(V)$, and the operator is, in fact, just the holomorphic differential map d, twisted by the image of a scalar η and a matrix ξ (both defined explicitly in [Sh1]) via the representation ρ. Starting with the 1-dimensional representation which is the mth power of $\mathbb{C}^\times$ (this is the representation associated with our automorphic forms of weight m) and repeating this operation twice, we obtain an automorphic form with representation involving $\omega^{\otimes 2}$. The idea is expressing the resulting automorphic form when ω is identified with $K_\mathbb{C}$, and using the bilinear form on the latter space in order to replace the $\omega^{\otimes 2}$-valued automorphic forms by scalar-valued ones.

Now, we replace the coordinate denoted z in [Sh1] by $u = \sqrt{2}z$, considering it as lying in the complexified space $(v_-)_\mathbb{C}$ associated with some base point for $G(V)$, and decompose it as some multiple u_z of z_{v_-} plus a vector $u_\perp$ which is perpendicular to z_{v_-}. Here z is again the isotropic vector we used for defining $K_\mathbb{R}$. Choosing the positive part of z appropriately (recall that the vector denoted $p(z)$ in [Sh1] is not presented in the canonical form), we obtain that our norm 0 vector has pairing $1 + u^t u - 2u^t z_{v_-}$ with z and its positive and negative $K_\mathbb{C}$ coordinates are $i(1 - u^t u)$ and $2u_\perp$, respectively. It follows that the associated element Z of $K_\mathbb{C}$ (which can be shown to be in $K_\mathbb{R} + iC$) satisfies $(Z + ie_+)^2 = \frac{-4}{1+u^t u - 2u^t z_-}$ (where e_+ is the generator of the positive part of $K_\mathbb{R}$), so that the inverse map sends Z to the vector obtained by multiplying the positive part of $-2\frac{Z+ie_+}{(Z+ie_+)^2}$ by i, and adding z_{v_-} to the result. Given an automorphic form F of weight m on $G(V)$, a very lengthy, tedious, and involved calculation gives us the expression for the $\omega^{\otimes 2}$-valued automorphic form obtained from F under the operator mentioned in the previous paragraph, and after applying the pairing we obtain an expression closely related to $(Z+ie_+)^{2m}R_m^{(b_-)}[(Z+ie_+)^{-2m}F]$. Indeed, the expression denoted by η in [Sh1] becomes $\frac{16Y^2}{|(Z+ie_+)^2|^2}$ using our variable, so that multiplying by η^m before applying the operator and by η^{-m} afterward corresponds to the operation involving Y^{2m} appearing in the definition of $R_m^{b_-}$, as well as the additional operation with $(Z+ie_+)^{2m}$. However, the details of this calculation are very long as well, and therefore we have chosen to state and prove Theorem 1.1 more directly.

For calculational purposes it turns out convenient to introduce the operator

$$\widetilde{\Delta}_{m,n}^{(b_-)} = \Delta_{m,n}^{(b_-)} - 2n(2m-b_-),$$

on which complex conjugation interchanges the indices m and n. The operator

$$(D^*)^2 - \frac{D^*}{2i} = \sum_{k,l} y_k y_l \partial_k \partial_l$$

will also show up, so we denote it $\widetilde{(D^*)^2}$. We now turn to the

Proof of Proposition 1.3. Conjugating the desired equality for $R_m^{(b_-)}$ by $(Y^2)^m$, applying Equation (2), and taking the differences between the operators $\widetilde{\Delta}_{m,n}^{(b_-)}$ and $\Delta_{m,n}^{(b_-)}$ into consideration, we see that the asserted equality for $R_m^{(b_-)}$ is equivalent to

$$\widetilde{\Delta}_{2,-m}^{(b_-)} R_0^{(b_-)} - R_0^{(b_-)} \widetilde{\Delta}_{0,m}^{(b_-)} = (2b_- + 4m - 4) R_0^{(b_-)}.$$

Moreover, multiplying the complex conjugate of the latter equation by $(Y^2)^2$ and comparing $\widetilde{\Delta}_{2,-m}^{(b_-)}$ with $\Delta_{2,-m}^{(b_-)}$ yields the required property for $L^{(b_-)}$ (with the index m replaced by $-m$). Hence, as in the proof of Theorem 1.1, we are reduced to proving this single equation. In addition, the dependence on m of the left hand side enters only through the difference $-4imD^*$ between the operators $\widetilde{\Delta}_{l,-m}^{(b_-)}$ and $\Delta_l^{(b_-)}$ with $l \in \{0,2\}$. As a simple calculation yields

$$\left[D^*, \Delta_{K_{\mathbb{C}}}^h\right] = i\Delta_{K_{\mathbb{C}}}^h \qquad \text{and} \qquad \left[D^*, \frac{D^*}{Y^2}\right] = \frac{iD^*}{Y^2},$$

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it suffices to prove the equality for $m=0$ (i.e., the original assertion for $R_0^{(b_-)}$):

$$\Delta_2^{(b_-)}R_0^{(b_-)}-R_0^{(b_-)}\Delta_0^{(b_-)}=(2b_--4)R_0^{(b_-)}.$$

The commutator of $\Delta_0^{(b_-)}$ and $R_0^{(b_-)}$ is evaluated using the equalities

$$\left[|D^*|^2,\Delta_{K_\mathbb{C}}^h\right]=i\overline{D^*}\Delta_{K_\mathbb{C}}^h+iD^*\Delta_{K_\mathbb{C}}^\mathbb{R}+\frac{\Delta_{K_\mathbb{C}}^\mathbb{R}}{2},$$

$$\left[|D^*|^2,\frac{D^*}{Y^2}\right]=\frac{3i|D^*|^2-i\widetilde{(D^*)^2}+D^*}{2Y^2},$$

$$\left[Y^2\Delta_{K_\mathbb{C}}^\mathbb{R},\Delta_{K_\mathbb{C}}^h\right]=2iD^*\Delta_{K_\mathbb{C}}^\mathbb{R}+\frac{b_-}{2}\Delta_{K_\mathbb{C}}^\mathbb{R},\qquad\text{and}$$

$$\left[Y^2\Delta_{K_\mathbb{C}}^\mathbb{R},\frac{D^*}{Y^2}\right]=\frac{2i|D^*|^2-2i\widetilde{(D^*)^2}+i\Delta_{K_\mathbb{C}}^h+i\Delta_{K_\mathbb{C}}^\mathbb{R}+(2-b_-)D^*}{2Y^2}$$

(all of which follow from straightforward calculations). Applying the equalities

$$\Delta_2^{(b_-)}=\Delta_0^{(b_-)}-8i\overline{D^*}\qquad\text{and}\qquad\overline{D^*}\circ\left(\frac{D^*}{Y^2}\right)=\frac{2|D^*|^2-iD^*}{2Y^2}$$

and putting in the appropriate scalars now establishes the proposition. □

Our next task is the

Proof of Proposition 1.5. We begin by evaluating $R_{m-2}^{(b_-)}L^{(b_-)}$ written as

$$R_{m-2}^{(b_-)}(Y^2)^2\Delta_{K_\mathbb{C}}^{\overline{h}}+R_{m-2}^{(b_-)}i(2-b_-)Y^2\overline{D^*}=(Y^2)^2R_m^{(b_-)}\Delta_{K_\mathbb{C}}^{\overline{h}}+i(2-b_-)Y^2R_{m-1}^{(b_-)}\overline{D^*}.$$

Using the equalities

$$\left[\Delta_{K_\mathbb{C}}^h,\overline{D^*}\right]=-i\Delta_{K_\mathbb{C}}^\mathbb{R}\qquad\text{and}\qquad D^*\overline{D^*}=|D^*|^2+\frac{\overline{D^*}}{2i}$$

we establish the equation

$$R_{m-2}^{(b_-)}L^{(b_-)}=\Xi_m^{(b_-)}+\frac{(2-b_-)(2m-b_-)}{8}\Delta_m^{(b_-)},$$

where $\Xi_m^{(b_-)}$ is defined in the formulation of the proposition. We now decompose $R_m^{(b_-)}$ in $L^{(b_-)}R_m^{(b_-)}$ (which is $(Y^2)^2\overline{R_0^{(b_-)}}R_m^{(b_-)}$), yielding

$$(Y^2)^2\overline{R_0^{(b_-)}}\Delta_{K_\mathbb{C}}^h - i(2m+2-b_-)Y^2\overline{R_{-1}^{(b_-)}}D^* - \frac{m(2m+2-b_-)}{2}Y^2\overline{R_{-1}^{(b_-)}}.$$

The formulas

$$\left[\Delta_{K_\mathbb{C}}^{\overline{h}}, D^*\right] = i\Delta_{K_\mathbb{C}}^{\mathbb{R}} \quad \text{and} \quad \overline{D^*}D^* = |D^*|^2 - \frac{D^*}{2i}$$

now show that

$$L^{(b_-)}R_m^{(b_-)} = \Xi_m^{(b_-)} - \frac{b_-(2m+2-b_-)}{8}\Delta_m^{(b_-)} + \frac{mb_-(2m+2-b_-)}{4}.$$

The required commutation relation follows. As Theorem 1.1 shows that the compositions $R_{m-2}^{(b_-)}L^{(b_-)}$ and $L^{(b_-)}R_m^{(b_-)}$ commute with all the slash operators of weight m, and Proposition 1.3 implies that these operators commute with Δ_m, the assertion about $\Xi_m^{(b_-)}$ is also established. This proves the proposition. ∎

Finally, we come to the

Proof of parts (iii) and (iv) of Proposition 1.6. We prove part (iii) by induction (the case $l=0$ being trivial). If $(R_m^{(b_-)})^l$ is presented by the asserted formula then $(R_m^{(b_-)})^{l+1}$, which is $R_{m+2l}^{(b_-)}(R_m^{(b_-)})^l$, equals

$$R_{m+2l}^{(b_-)}\sum_{c=0}^{l}\sum_{s=0}^{c}A_{s,c}^{(l)}\frac{(iD^*)^{c-s}(\Delta_{K_\mathbb{C}}^h)^{l-c}}{(-Y^2)^c} = \sum_{s,c}A_{s,c}^{(l)}\frac{R_{m+2l-c}^{(b_-)}(iD^*)^{c-s}(\Delta_{K_\mathbb{C}}^h)^{l-c}}{(-Y^2)^c}.$$

For each c, the term involving $\frac{D^*}{Y^2}$ (resp. $\frac{1}{Y^2}$) in $R_{m+2l-c}^{(b_-)}$ takes the term with indices c and s (for l) to a multiple of the term corresponding to $c+1$ and s (resp. $c+1$ and $s+1$) for $l+1$. For $\Delta_{K_\mathbb{C}}^h$ we have

$$\left[\Delta_{K_\mathbb{C}}^h, iD^*\right] = \Delta_{K_\mathbb{C}}^h \quad \text{hence } \Delta_{K_\mathbb{C}}^h(iD^*)^{c-s} = \sum_{a=s}^{c}\binom{c-s}{a-s}(iD^*)^{c-a}\Delta_{K_\mathbb{C}}^h,$$

and we multiply the latter sum by $\frac{(\Delta_{K_\mathbb{C}}^h)^{l-c}}{(-Y^2)^c}$. This shows that $(R_m^{(b_-)})^{l+1}$ can be expressed by the asserted formula. Putting in the multipliers $A_{s,c}^{(l)}$ from $(R_m^{(b_-)})^l$ and the coefficients of $\frac{D^*}{Y^2}$ and $\frac{1}{Y^2}$ in $R_{m+2l-c}^{(b_-)}$, summing over c and

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s, and taking the coefficient in front of the term with indices c and a (and $l+1$) in the result, we obtain the recursive relation asserted in part (iii). We now observe that for $a=0$ the recursive formula reduces to

$$A_{0,c}^{(l+1)} = A_{0,c}^{(l)} + (2m+4l-2c+4-b_-)A_{0,c-1}^{(l)}.$$

Denote the asserted value of $A_{0,c}^{(l)}$ by $B_{0,c}^{(l)}$. As $A_{0,0}^{(0)}=1=B_{0,0}^{(0)}$, it suffices to show that the numbers $B_{0,c}^{(l)}$ satisfy the latter recursive formula. But the equality

$$\begin{aligned}2(l-c+1)\left(m+l-c-\frac{b_-}{2}+1\right)+c(2m+4l-2c+4-b_-)\\ =2(l+1)\left(m+l-\frac{b_-}{2}+1\right)\end{aligned}$$

holds for every l and c (and m and b_-), and multiplication by $\frac{l!\cdot 2^{c-1}}{c(l+1-c)!}$ and by the binomial coefficient $\binom{m+l-\frac{b_-}{2}}{c-1}$ yields the required recursive relation for the numbers $B_{0,c}^{(l)}$. This completes the proof of the proposition. □

§4. Actions on theta kernels—proofs

The main technical lemma, which will be required for the evaluations in most of the following proofs, is based on

LEMMA 4.1. *Given $\mu \in L_{\mathbb{R}}$, the operators $R_0^{(b_-)}$ and $L^{(b_-)}$ take the function $P_{1,1,1}$ of $Z \in K_{\mathbb{R}}+iC$ to $-\frac{b_-}{2}P_{0,2,2}$ and $-\frac{b_-}{2}P_{2,0,0}$ respectively.*

Proof. The commutation relation between powers of Y^2 and the operators $R_m^{(b_-)}$ obtained from the first definition of the latter operators in Theorem 1.1 and the fact that the latter operators involve only holomorphic differentiation allows us to write $R_0^{(b_-)}P_{1,1,1}$ as $P_{0,1,1}R_{-1}^{(b_-)}(\mu, Z_{V,Z})$. Hence we must evaluate the operation of $\Delta_{K_{\mathbb{C}}}^h$ and D^* on $(\mu, Z_{V,Z})$. For the latter operator a simple calculation yields

$$\begin{aligned}2iD^*(\mu, Z_{V,Z}) &= 2i(\mu, Y_{V,Z}) + 2Y^2(\mu, z)\\ &= (\mu, Z_{V,Z}) - (\mu, \overline{Z_{V,Z}}) + 2Y^2(\mu, z).\end{aligned}$$

The former operator is pure of weight 2, hence its action gives a nonzero result only on the part $-\frac{Z^2}{2}(\mu, z)$, and using an orthonormal basis one finds that this result is just $-b_-(\mu, z)$. Combining these results, we find that

$$\left[R_{-1}^{(m)} = \Delta_{K_{\mathbb{C}}}^h + \frac{ib_-}{Y^2}D^* - \frac{b_-}{2Y^2}\right](\mu, Z_{V,Z}) = -\frac{b_-}{2Y^2}(\mu, \overline{Z_{V,Z}}),$$

from which the value of $R_0^{(b_-)}P_{1,1,1}$ follows. The assertion about $L^{(b_-)}P_{1,1,1}$ is a consequence of the value of $R_0^{(b_-)}P_{1,1,1}$, since $P_{1,1,1}$ is a real function and $L^{(b_-)}$ is the operator which is complex conjugate to $R_0^{(b_-)}$, multiplied by $(Y^2)^2$. This proves the lemma. □

Another useful evaluation appears in the following

LEMMA 4.2. *The holomorphic and anti-holomorphic Z-gradients of $P_{1,1,1}$ have, as vectors in $K_{\mathbb{C}}$, the norms $P_{0,2,2}\mu_-^2$ and $P_{2,2,0}\mu_-^2$ respectively.*

Proof. $(\mu, \overline{Z_{V,Z}})$ is anti-holomorphic, and the holomorphic gradients of $(\mu, Z_{V,Z})$ and Y^2 are $\mu_{K_{\mathbb{R}}} - (\mu, z)Z$ and $-iY$ respectively, where $\mu_{K_{\mathbb{R}}}$ is the orthogonal projection of $\mu \in L_{\mathbb{R}}$ onto $K_{\mathbb{R}} = \{z, \zeta\}^\perp$. It follows that $P_{1,1,1}$ has holomorphic gradient

$$P_{0,2,1}\big[Y^2(\mu_{K_{\mathbb{R}}} - (\mu, z)Z) + i(\mu, Z_{V,Z})Y\big].$$

Now, the (easily evaluated) equalities

$$\big(\mu_{K_{\mathbb{R}}} - (\mu, z)Z, Y\big) = (\mu, Y_{V,Z}) - iY^2(\mu, z)$$

and

$$(\mu, z)^2Z^2 - 2(\mu, z)(\mu_{K_{\mathbb{R}}}, Z) + 2(\mu, z)(\mu, Z_{V,Z}) = 2(\mu, z)(\mu, \zeta) - \zeta^2(\mu, z)^2$$

reduce to the norm of the latter gradient to

$$P_{0,2,2}\big[\mu_{K_{\mathbb{R}}}^2 + 2(\mu, \zeta)(\mu, z) - \zeta^2(\mu, z)^2 - P_{1,1,1}\big].$$

But μ is $\big(\mu_{K_{\mathbb{R}}}, \mu_z, (\mu, \zeta) - \zeta^2\mu_z\big)$ in the $K_{\mathbb{R}} \times \mathbb{R} \times \mathbb{R}$ coordinates, so that the sum of the first three terms in the brackets is just μ^2. Subtracting $P_{1,1,1} = \mu_+^2$ completes the proof of the first assertion, and the second assertion follows from complex conjugation since the function $P_{1,1,1}$ is real-valued. This proves the lemma. □

For $\mu \in L_{\mathbb{R}}$ and $\tau = x + iy \in \mathcal{H}$ we denote the vector $\sqrt{2\pi y}\mu$ by $\tilde{\mu}$. Its norm is $2\pi y\mu^2$, and after choosing an element of $G(L_{\mathbb{R}})$, it decomposes into $\tilde{\mu}_+$ (of norm $2\pi y\mu_+^2$) and $\tilde{\mu}_-$ (whose norm is $2\pi y\mu_-^2$). We now prove

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PROPOSITION 4.3. *Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth function. Then the images of the function $f(\tilde{\mu}_+)$ under $R_0^{(b_-)}$ and $L^{(b_-)}$ are $2\pi y P_{0,2,2}\big[\tilde{\mu}_-^2 f''(\tilde{\mu}_+) - \frac{b_-}{2} f'(\tilde{\mu}_+)\big]$ and $2\pi y P_{2,0,0}\big[\tilde{\mu}_-^2 f''(\tilde{\mu}_+) - \frac{b_-}{2} f'(\tilde{\mu}_+)\big]$ respectively.*

Proof. Both operators consist of a first order operator D (a multiple of D^* or of $\overline{D^*}$) and a second order operator Δ (which equals $\Delta_{K_{\mathbb{C}}}^h$ or $\Delta_{K_{\mathbb{C}}}^{\overline{h}}$). Then $D\big(f(T)\big) = DT \cdot f'(T)$, and $\Delta\big(f(T)\big)$ is the sum of $\Delta T \cdot f'(T)$ and an expression involving $f''(T)$. In our case $T = \tilde{\mu}_+ = 2\pi y \mu_+^2 = 2\pi y P_{1,1,1}$, so that the coefficient of $f'(T)$ is just $2\pi y$ times $R_0^{(b_-)} P_{1,1,1}$ and $L^{(b_-)} P_{1,1,1}$, and the latter expressions are evaluated using Lemma 4.1. The coefficients of $f''(T)$ coming from Δ being $\Delta_{K_{\mathbb{C}}}^h$ or $(Y^2)^2 \Delta_{K_{\mathbb{C}}}^{\overline{h}}$ are the norms (in $K_{\mathbb{C}}$) of the holomorphic and anti-holomorphic gradients of T, the latter being multiplied by $(Y^2)^2$. For $T = 2\pi y P_{1,1,1}$ these norms take the values given in Lemma 4.2, multiplied by $(2\pi y)^2$. Gathering these results together and substituting the value of $\tilde{\mu}_-^2$ completes the proof of the proposition. □

We now turn to proving assertions concerning the images of theta lifts (or complex conjugates of theta functions), having only holomorphic weights of automorphy, under the operators $R_m^{(b_-)}$ and $L^{(b_-)}$. This was seen to boil down to the operation on the function $F_{r,s,t}^{(l)}$ from Equation (4), with τ replaced by $-\overline{\tau}$, under the additional assumption $s = t$. The exponent was seen, using part (i) of Lemma 2.1, to be $\mathbf{e}\big(-\tau\frac{\mu^2}{2}\big)e^{-\tilde{\mu}_+^2}$, where the first multiplier is a constant (i.e., independent of Z). The polynomial part is evaluated in

LEMMA 4.4.

(i) *For any natural numbers k and n we have*

$$(2\pi y)^n P_{n-k,0,0} e^{-\Delta_{v_+}/8\pi y}(P_{k,n,n}) e^{-2\pi y P_{1,1,1}} = (-1)^k \frac{d^k}{dT^k}(T^n e^{-T})\bigg|_{T=\tilde{\mu}_+^2}.$$

(ii) *Applying $e^{\Delta_{v_-}/8\pi y}$ to $(2\pi y)^l (\mu_-^2)^l$ yields*

$$\sum_p \binom{l}{p} \left[\Gamma\left(l + \frac{b_-}{2}\right) \Big/ \Gamma\left(p + \frac{b_-}{2}\right)\right] (\tilde{\mu}_-^2)^p.$$

We allow the index $n-k$ appearing in Part (i) here to be negative, with the natural extension of the definition of $P_{r,s,t}$ to negative r. We remark that the expressions obtained in this part are just the generalized Laguerre polynomials $L_k^{(n-k)}$, multiplied by the exponents, and normalized appropriately.

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Proof. Multiple applications of part (ii) of Lemma 2.1 show that

$$\frac{\Delta_{v_+}^h}{h!(-8\pi y)^h} P_{k,n,n} = \frac{k!n!P_{k-h,n-h,n-h}}{(k-h)!(n-h)!h!(-2\pi y)^h}.$$

Multiplying by $(2\pi y)^n P_{n-k,0,0}$ and summing over h, the left hand side of the equation in part (i) becomes just

$$\sum_h \binom{k}{h} \frac{n!}{(n-h)!}(-1)^h (2\pi y P_{1,1,1})^{n-h} e^{-2\pi y P_{1,1,1}}.$$

On the other hand, differentiating the product $T^n e^{-T}$ k times with respect to T yields

$$\sum_{h=0}^k \binom{k}{h} \left(\frac{d}{dT}\right)^h T^n \cdot \left(\frac{d}{dT}\right)^{k-h} e^{-T} = \sum_{h=0}^k \binom{k}{h} \frac{n!T^{n-h}}{(n-h)!}(-1)^{k-h} e^{-T},$$

and substituting $T = \tilde{\mu}_+^2 = 2\pi y P_{1,1,1}$ yields the same expression multiplied by $(-1)^k$. This establishes part (i). For part (ii), applying part (iii) of Lemma 2.1 successively evaluates

$$\Delta_{v_-}^{l-p}(\mu_-^2)^l = \frac{4^{l-p}l!}{p!} \cdot \frac{\Gamma\left(l+\frac{b_-}{2}\right)}{\Gamma\left(p+\frac{b_-}{2}\right)}(\mu_-^2)^p.$$

Dividing this term by $(8\pi y)^{l-p}(l-p)!$, multiplying everything by $(2\pi y)^l$, and substituting $\tilde{\mu}_-^2 = 2\pi y \mu_-^2$ gives the asserted expression. This completes the proof of the lemma. □

As $\tilde{\mu}_-^2 = \tilde{\mu}^2 - \tilde{\mu}_+^2$, Lemma 4.4 implies that the dependence of the expression $(2\pi y)^{n+l} P_{n-k,0,0} F_{k,n,n}^{(l)}(-\overline{\tau}, Z, \mu)$ (or the corresponding theta function) on the variable Z is only through the quantity $\tilde{\mu}_+^2$. For convenience, we gather these results in the following

COROLLARY 4.5. *Define the functions*

$$f_{k,n,p}^{(w)}(T) = (-1)^k \frac{d^k}{dT^k}(T^n e^{-T}) \cdot (w-T)^p,$$

where k, p, and n are natural numbers and $w \in \mathbb{R}$. Then the theta function $\Theta_{L,k,n,n}^{(l)}(-\overline{\tau}, Z)$ equals

$$\sum_{\mu \in L^*} \sum_p \binom{l}{p} \frac{\Gamma\left(l+\frac{b_-}{2}\right)}{\Gamma\left(p+\frac{b_-}{2}\right)} \frac{f_{k,n,p}^{(\tilde{\mu}^2)}(\tilde{\mu}_+^2)}{(2\pi y)^{n+l} P_{n-k,0,0}} \mathbf{e}\left(-\tau \frac{\mu^2}{2}\right) e_{\mu+L}.$$

Proof. Just substitute the value of $e^{-\Delta_v/8\pi y}(P^{(l)}_{k,n,n})$, which equals the product of $e^{-\Delta_{v_+}/8\pi y}(P_{k,n,n})$ and $e^{-\Delta_{v_+}/8\pi y}\big((\mu_-^2)^l\big)$, from Lemma 4.4 into the expression defining the theta function. ∎

We can now present the

Proof of Proposition 2.3. As seen above, it suffices to consider the action of $R_m^{(b_-)}$ on the expression $P_{0,m,m}(\mu,Z)e^{-2\pi y P_{1,1,1}}$ with fixed μ (recall that $P_{0,m,m}$ is harmonic). The holomorphicity of the differentiation in $R_m^{(b_-)}$ shows that the result is the same as $P_{0,m,m}R_0^{(b_-)}e^{-\tilde{\mu}_+^2}$. By putting $f(T)=e^{-T}$, Proposition 4.3 evaluates $R_0^{(b_-)}e^{-\tilde{\mu}_+^2}$ as $2\pi y P_{0,2,2}(\tilde{\mu}_- + \frac{b_-}{2})e^{-\tilde{\mu}_+}$, and multiplying by $P_{0,m,m}$ yields

$$R_m^{(b_-)}P_{0,m,m}e^{-\tilde{\mu}_+^2} = 4\pi^2y^2P_{0,m+2,m+2}\left[\mu_-^2 + \frac{b_-}{4\pi y}\right]e^{-2\pi y P_{1,1,1}}.$$

But the expression in parentheses is $e^{\Delta_{v_-}/8\pi y}(\mu_-^2)$ by part (ii) of Lemma 4.4, and the harmonicity of $P_{0,m+2,m+2}$ allows us to put it also into the action of $e^{-\Delta_v/8\pi y}$ without affecting the resulting expression. Putting in the missing constant $y^{\frac{b_-}{2}}\mathbf{e}\big(-\tau\frac{\mu^2}{2}\big)e_{\mu+L}$ and summing over $\mu\in L^*$ we establish the equality

$$R_m^{(b_-)}y^{\frac{b_-}{2}}\Theta_{L,0,m,m}(-\overline{\tau},Z) = 4\pi^2y^{2+\frac{b_-}{2}}\Theta^{(1)}_{L,0,m+2,m+2}(-\overline{\tau},Z).$$

But as $P_{m+2,m+2,0}$ is harmonic, Equation (6) shows that applying the operator $-4\pi i y^2\partial_{\overline{\tau}}$ to $y^{\frac{b_-}{2}}\Theta_{L,m+2,m+2,0}(\tau,Z)$ yields the complex conjugate of the latter expression, and complex conjugation inverts the sign of $4\pi i$. This proves the proposition. ∎

We now turn to the

Proof of Lemma 2.5. Write the theta function $\Theta^{(l)}_{L,k,n,n}(-\overline{\tau},Z)$ as in Corollary 4.5. It suffices to fix $\mu\in L^*$ and compare the coefficients of $\mathbf{e}\big(-\tau\frac{\mu^2}{2}\big)e_{\mu+L}$ in both sides. Take some $0\leqslant p\leqslant l$, and apply Proposition 4.3 with the function $f=f^{(\tilde{\mu}^2)}_{k,n,p}$. The powers of $2\pi y$ and $P_{1,0,0}$ from Corollary 4.5 and Proposition 4.3 merge to $(2\pi y)^{n+l-1}P_{n-2-k,0,0}$ in the denominator, and the remaining part of $L^{(b_-)}f^{(\tilde{\mu}^2)}_{k,n,p}$ is

$$\binom{l}{p}\frac{\Gamma\big(l+\frac{b_-}{2}\big)}{\Gamma\big(p+\frac{b_-}{2}\big)}\left[\tilde{\mu}_-^2\big(f^{(\tilde{\mu}^2)}_{k,n,p}\big)''(\tilde{\mu}_+^2) - \frac{b_-}{2}\big(f^{(\tilde{\mu}^2)}_{k,n,p}\big)'(\tilde{\mu}_+^2)\right].$$

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As $\tilde{\mu}_-^2 = \tilde{\mu}^2 - \tilde{\mu}_+^2$, and as one easily evaluates

$$(f_{k,n,p}^w)'(T) = -pf_{k,n,p-1}^w(T) - f_{k+1,n,p}^w(T),$$

the part in brackets in the latter expression equals

(9)
$$f_{k+2,n,p+1}^{(\tilde{\mu}^2)}(\tilde{\mu}_+^2) + \left(2p + \frac{b_-}{2}\right) f_{k+1,n,p}^{(\tilde{\mu}^2)}(\tilde{\mu}_+^2) + p\left(p + \frac{b_-}{2} - 1\right) f_{k,n,p-1}^{(\tilde{\mu}^2)}(\tilde{\mu}_+^2).$$

We now write the denominator in the preceding constant as

$$\frac{(p + \frac{b_-}{2})}{\Gamma(p + 1 + \frac{b_-}{2})}, \quad \frac{1}{\Gamma(p + \frac{b_-}{2})}, \quad \text{and} \quad \frac{1}{(p - 1 + \frac{b_-}{2})\Gamma(p - 1 + \frac{b_-}{2})}$$

in front of the three terms in Equation (9) respectively, and after taking the sum over p and gathering the functions with the same index p together, we see that the quotient $\Gamma(l + \frac{b_-}{2})/\Gamma(p + \frac{b_-}{2})$ multiplies

$$\left(p - 1 + \frac{b_-}{2}\right)\binom{l}{p-1} f_{k+2,n,p}^{(\tilde{\mu}^2)} + \left(2p + \frac{b_-}{2}\right)\binom{l}{p} f_{k+1,n,p}^{(\tilde{\mu}^2)} + (p+1)\binom{l}{p+1} f_{k,n,p}^{(\tilde{\mu}^2)}$$

(where we have omitted the variable $\tilde{\mu}_+^2$). Using the identity $b\binom{a}{b} = a\binom{a-1}{b-1}$ we can write the latter expression as

$$l\left[\binom{l-1}{p-2} f_{k+2,n,p}^{(\tilde{\mu}^2)}(\tilde{\mu}_+^2) + 2\binom{l-1}{p-1} f_{k+1,n,p}^{(\tilde{\mu}^2)}(\tilde{\mu}_+^2) + \binom{l-1}{p} f_{k,n,p}^{(\tilde{\mu}^2)}(\tilde{\mu}_+^2)\right]$$
(10)
$$+ \frac{b_-}{2}\left[\binom{l}{p-1} f_{k+2,n,p}^{(\tilde{\mu}^2)}(\tilde{\mu}_+^2) + \binom{l}{p} f_{k+1,n,p}^{(\tilde{\mu}^2)}(\tilde{\mu}_+^2)\right].$$

Now, differentiating k times and multiplying by $(w - T)^p$ takes the equality

$$(T^n e^{-T})' = (nT^{n-1} - T^n)e^{-T} \quad \text{to } f_{k,n,p}^{(w)}(T) - f_{k+1,n,p}^{(w)}(T) = nf_{k,n-1,p}^{(w)}(T).$$

One application of this relation replaces $f_{k+1,n,p}^{(\tilde{\mu}^2)}$ by $f_{k+2,n,p}^{(\tilde{\mu}^2)} + nf_{k+1,n-1,p}^{(\tilde{\mu}^2)}$, and we also obtain

$$f_{k,n,p}^{(\tilde{\mu}^2)} = f_{k+2,n,p}^{(\tilde{\mu}^2)} + 2nf_{k+1,n-1,p}^{(\tilde{\mu}^2)} + n(n-1)f_{k,n-2,p}^{(\tilde{\mu}^2)}.$$

Each of the terms in Equation (10) thus contributes to the total coefficient in front of $f_{k+2,n,p}^{(\tilde{\mu}^2)}$, which using the classical properties of the binomial coefficients reduces to $\left(l+\frac{b_-}{2}\right)\binom{l+1}{p}$. Using the recursive property of the gamma function again, we obtain the coefficient $\binom{l+1}{p}\Gamma\left(l+1+\frac{b_-}{2}\right)/\Gamma\left(p+\frac{b_-}{2}\right)$, which together with

$$\frac{1}{P_{n-2-k,0,0}(2\pi y)^{n+l-1}}=\frac{4\pi^2y^2}{P_{n-2-k,0,0}(2\pi y)^{n+l+1}}$$

yields the coefficient appearing in front of $f_{k+2,n,p}^{(\tilde{\mu}^2)}(\tilde{\mu}_+^2)$ in the expansion of $4\pi^2y^2\Theta_{L,k+2,n,n}^{(l+1)}(-\overline{\tau},Z)$ in Corollary 4.5. The total coefficient in front of the function $f_{k+1,n-1,p}^{(\tilde{\mu}^2)}$ in Equation (10) becomes (again, using binomial identities) just $\left(2l+\frac{b_-}{2}\right)\binom{l}{p}$, and the gamma quotient and the powers of $2\pi y$ and $P_{1,0,0}$ complete the formula for the second asserted term. For the remaining term $n(n-1)l\binom{l-1}{p}f_{k,n-2,p}^{(\tilde{\mu}^2)}$ from Equation (10) we use the functional equation of the gamma function again to write $\Gamma\left(l+\frac{b_-}{2}\right)$ as $\left(l-1+\frac{b_-}{2}\right)\Gamma\left(l-1+\frac{b_-}{2}\right)$, and we also decompose

$$P_{n-2-k,0,0}(2\pi y)^{n+l-1}=4\pi^2y^2P_{n-2-k,0,0}(2\pi y)^{n-2+l-1}.$$

Corollary 4.5 then establishes the remaining asserted term in a similar manner. This completes the proof of the lemma. □

We go on to the

Proof of Proposition 2.6. We prove the assertion by induction on s. The case $s=0$ is trivial. Denote the asserted coefficient corresponding to the hth term in the expression for the image under $(L^{(b_-)})^s$ by $a_{s,h}(y)$. We need to evaluate

$$\sum_h a_{s,h}(y)L^{(b_-)}\Theta_{L,s+h,m+s-h,m+s-h}^{(h)},$$

and compare it with the asserted expression for $s+1$. Lemma 2.5 shows that for each h the $L^{(b_-)}$-image of the corresponding theta function is a linear combination of three theta functions, which correspond to the index $s+1$ and the indices $h-1$, h, and $h+1$. After applying the appropriate summation index changes, the coefficient which we get in front of $\Theta_{L,s+1+h,m-s-1+h,m-s-1+h}^{(h)}$ in $(L^{(b_-)})^{s+1}\overline{\Theta_{L,m,m,0}}$ is

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$$4\pi^2y^2a_{s,h-1}(y)+(m-s+h)\left(2h+\frac{b_-}{2}\right)a_{s,h}(y)$$
$$+\frac{(m-s+h)(m-s+h+1)(h+1)\big(h+\frac{b_-}{2}\big)}{4\pi^2y^2}a_{s,h+1}(y).$$

Substituting the values of $a_{s,t}$ for t being $h-1$, h, and $h+1$, one easily sees that all three terms yield the same multiplier $\frac{m!(4\pi^2y^2)^h}{(m-s-1+h)!}$. Applying the functional equation for the gamma function in the first and third term, we obtain that the remaining expression equals

$$\frac{\Gamma\big(s+\frac{b_-}{2}\big)}{\Gamma\big(h+\frac{b_-}{2}\big)}\left[\left(h-1+\frac{b_-}{2}\right)\binom{s}{h-1}\right.$$
$$\left.+\left(2h+\frac{b_-}{2}\right)\binom{s}{h}+(h+1)\binom{s}{h+1}\right].$$

The same considerations we applied for evaluating the coefficient of $f_{k+2,n,p}^{(\tilde{\mu}^2)}$ in Lemma 2.5 show that the expression in brackets equals $\big(s+\frac{b_-}{2}\big)\binom{s+1}{h}$. Applying the functional equation of the gamma function once more, this yields the asserted value of $a_{s+1,h}$. This completes the proof of the proposition. □

Finally, we come to the

Proof of Proposition 2.7. We begin by proving that for any $q\in\mathbb{N}$, the action of the operator $(-4\pi i)^q\delta^q_{1-\frac{b_-}{2}+r+t-2l,\tau}$ sends $y^{\frac{b_-}{2}+2l}\Theta^{(l)}_{L,r,s,t}(\tau,Z)$ to

$$\sum_{h=0}^{q}\binom{q}{h}(4\pi^2)^h y^{\frac{b_-}{2}+2l-2q+2h}\frac{l!\Gamma\big(l+\frac{b_-}{2}\big)}{(l-q+h)!\Gamma\big(l-q+h+\frac{b_-}{2}\big)}$$
$$\times\,\Theta^{(l-q+h)}_{L,r+h,s+h,t+h}(\tau,Z).$$

For $q=0$ the assertion is trivially true. We write the asserted function of y preceding the theta function in the term corresponding to h in the sum arising from the index q as $\binom{q}{h}(4\pi^2)^h b_{l-q+h}(y)$. Given that this assertion holds for q, we apply Equation (5) for the operator $-4\pi i\delta_{1-\frac{b_-}{2}+r+t-2l+2q}$ acting on each term, and observe that the resulting theta functions correspond to the index $q+1$ and to the summation indices $h+1$ and h. Moreover, after the usual index change manipulations one sees that the total coefficient in

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front of the theta function with indices $q+1$ and h is

$$(4\pi^2)^h\left[\binom{q}{h-1}b_{l-q+h-1}(y)+\binom{q}{h}(l-q+h)\right.$$
$$\left.\times\left(l-q+h+\frac{b_-}{2}-1\right)\frac{b_{l-q+h}(y)}{y^2}\right].$$

As the second term here is easily seen to be just $\binom{q}{h-1}b_{l-q+h-1}(y)$, the inductive assertion follows from the classical property of the binomial coefficients. With $r=s=0$ and $t=l=q=m$ the general formula from above becomes

$$\sum_h \binom{m}{h}\frac{\Gamma\left(m+\frac{b_-}{2}\right)}{\Gamma\left(h+\frac{b_-}{2}\right)}\frac{m!(4\pi^2y^2)^h}{h!}y^{\frac{b_-}{2}}\Theta^{(h)}_{L,h,h,m+h}(\tau,Z).$$

On the other hand, putting $m=s$ in Proposition 2.6, multiplying by $y^{\frac{b_-}{2}}$ (which commutes with differential operators in the variable Z), and taking the complex conjugate of the result, yields precisely the same expression. This proves the proposition. □

Acknowledgment. I would like to thank J. Bruinier for numerous suggestions and intriguing discussions regarding the results of this paper. I also wish to thank the anonymous referee for his/her helpful comments.

References

[B] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), 491–562.

[Bru] J. H. Bruinier, *Borcherds Products on $O(2,l)$ and Chern Classes of Heegner Divisors*, Lecture Notes in Mathematics **1780**, Springer, 2002.

[BF] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math J. **125**(1) (2004), 45–90.

[Ch] Y. J. Choie, *Multilinear differential operators on Siegel modular forms of genus 1 and 2*, J. Math. Anal. Appl. **232** (1999), 34–44.

[CE] Y. J. Choie and W. Eholzer, *Rankin–Cohen operators of Jacobi and Siegel forms*, J. Number Theory **68**(2) (1998), 160–177.

[DN] K. Doi and H. Naganuma, *On the functional equation of certain Dirichlet Series*, Invent. Math. **9** (1969), 1–14.

[G] V. A. Gritsenko, *Jacobi Functions of n Variables*, Zap. Nauch. Seminars of LOMI **168**, 1988, 32–54. (in Russian); English translation in J. Soviet Math., **53** (1991) 243–252.

[Ma1] H. Maaß, *Die Differenzialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann. **126** (1953), 44–68.

https://doi.org/10.1017/nmj.2016.57 Published online by Cambridge University Press

[Ma2] H. Maaß, *Siegel's Modular Forms and Dirichlet Series*, Lecture Notes in Mathematics **216**, Springer, 1971.

[Na] S. Nakajima, *On invariant differential operators on bounded symmetric domains of type IV*, Proc. Japan Acad. **58**(A) (1982), 235–238.

[Ng] H. Naganuma, *On the coincidence of two Dirichlet series associated with cusp forms of Hecke's "Neben"-type and Hilbert modular forms over a real quadratic field*, J. Math. Soc. Jpn. **25** (1973), 547–555.

[Sh1] G. Shimura, *The arithmetic of certain zeta functions and automorphic forms on orthogonal groups*, Ann. of Math. (2) **111** (1980), 313–375.

[Sh2] G. Shimura, *Arithmetic of differential operators on symmetric domains*, Duke Math. J. **48**(4) (1981), 813–843.

[Sh3] G. Shimura, *Differential operators and the singular values of Eisenstein series*, Duke Math. J. **51**(2) (1984), 261–329.

[Sh4] G. Shimura, *On differential operators attached to certain representations of classical groups*, Invent. Math. **77** (1984), 463–488.

[Sh5] G. Shimura, *Invariant differential operators on Hermitian symmetric spaces*, Ann. of Math. (2) **132**(2) (1990), 237–272.

[Ve] J. L. Verdier, Sur les intégrales attachées aux formes automorphes (d'après Goro SHIMURA), Sém. BOURBAKI 216, **13** (1961).

[Za] D. B. Zagier, J. H. Bruinier, G. van der Geer and G. Harder, *The 1-2-3 of Modular Forms*, Universitext, 2008.

[Ze1] S. Zemel, *A p-adic approach to the Weil representation of discriminant forms arising from even lattices*, Math. Ann. Quebec **39**(1) (2015), 61–89.

[Ze2] S. Zemel, *A Gross–Kohnen–Zagier type theorem for higher-codimensional Heegner cycles*, Research Number Theory **1** (2015), 1–44.

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