A SECOND-ORDER THEORY OF THE GALILEAN SATELLITES OF JUPITER

S. FERRAZ-MELLO
University of São Paulo, Brazil

ABSTRACT. The theory of the motion of the Galilean satellites of Jupiter is developed up to the second-order terms. The disturbing forces are those due to mutual attractions, to the non-symmetrical internal mass distribution of Jupiter and to the attraction from the Sun. The mean equator of Jupiter is taken as the reference plane and its motion is considered. The integration of the equations is performed. The geometric equations are solved for the case in which the amplitude of libration is zero. The perturbation method is shortly commented on the grounds of some recent advances in non-linear mechanics.

In a previous paper (Ferraz-Mello, 1974) one perturbation theory has been constructed with special regard to the problem of the motion of the Galilean satellites of Jupiter. In this problem, the motions are nearly circular and coplanar; on the other hand the quasi-resonances lead to strong perturbations. The main characteristic of the theory is that it allows the main frequencies to be kept fixed from the earlier stages, and so, to have a purely trigonometric solution.

1. THE EQUATIONS

The equations were derived for a second-order theory; the small parameters are the satellite masses, the eccentricities and the inclinations. For each satellite the variables are the radius vector, the longitude, and two pairs of variables P,Q, and K,H, built respectively from Laplace's and the area's first integrals. These variables are close to Poincaré's variables in exponential form: e.exp-i(1-ϖ) and I.exp-i(1-Ω) and their complex conjugates.

The equations for the variables P_j and Q_j are separated; they are integro-differential linear equations;

$$DP_j + \kappa_j P_j = \frac{\lambda_j - \kappa_j^2}{\kappa_j} [1 + \frac{1}{4} \kappa_j D^{-1}(P_j+Q_j) - \frac{3}{4} \kappa_j^2 D^{-2}(P_j-Q_j) -$$

$$- \kappa_j D^{-1}(P_j-Q_j)] + \frac{1}{\kappa_j} \mathscr{R}_j + \frac{1}{16} D^{-1} \chi_j \quad 7P_j - 50Q_j + 7\kappa_j D^{-1}(P_j-Q_j) \quad +$$

V. Szebehely (ed.), Dynamics of Planets and Satellites and Theories of Their Motion, 209-236.

$$+ \frac{1}{4\kappa_j} \chi_j \; 1+ \frac{1}{2} \kappa_j D^{-1}(P_j-Q_j) + \frac{1}{4}\kappa_j D^{-1}(P_j+Q_j) - \frac{3}{2}\kappa_j^2 D^{-2}(P_j-Q_j) \;;$$

$$DQ_j - \kappa_j Q_j = \frac{\lambda_j-\kappa_j^2}{\kappa_j} [-1+ \frac{1}{4} \kappa_j D^{-1}(P_j+Q_j) - \frac{3}{4} \kappa_j^2 D^{-2}(P_j-Q_j) +$$

$$+ \kappa_j D^{-1}(P_j-Q_j)] - \frac{1}{\kappa_j} \tau_j - \frac{1}{16} D^{-1} \chi_j [7Q_j-5P_j+7\kappa_j D^{-1}(P_j-Q_j)$$

$$- \frac{1}{4\kappa_j} \chi_j [1+ \frac{1}{2} \kappa_j D^{-1}(P_j-Q_j) - \frac{1}{4}\kappa_j D^{-1}(P_j+Q_j)+ \frac{3}{2} \kappa_j^2 D^{-2}(P_j-Q_j)].$$

In these equations we have

$$\kappa_j = \frac{n_j}{\nu_3}$$

and

$$\lambda_j = \frac{k^2 m_0 (1+m_j)}{\nu_3^2 a_j^3}$$

where the mean motion ν_3 is that of the third satellite with respect to a rotating frame in which the mean motions of the three inner satellites are exactly commensurable; the mean motion n_j is that of the jth satellite with respect to a Galilean (inertial axes) frame and a_j is the mean distance of the jth satellite to the planet. D is the same differential operator as in Hill's moon theory

$$D = \zeta \frac{d}{d\zeta} \qquad (\zeta = \exp i\nu_3 t).$$

The equations for K_J and H_j are also separated; they are linear:

$$(DK_j + \kappa_j K_j) = \frac{1}{\kappa_j} \mathscr{V}_J + \frac{1}{2\kappa_j} (K_j + H_j) \mathscr{R}_j$$

$$(DH_j - \kappa_j H_j) = - \frac{1}{\kappa_j} \mathscr{V}_j - \frac{1}{2\kappa_j} (K_j + H_j) \mathscr{T}_j.$$

Still we have the geometric equations

$$D \varepsilon_j = \frac{1}{2} \kappa_j (P_j-Q_j) - \frac{1}{4} \kappa_j (P_j+Q_j) \Gamma_j + \frac{1}{4} \kappa_j (K_j+H_j)(K_j-H_j)$$

$$D\Gamma_j = - 3\kappa_j \varepsilon_j + 2\kappa_j \varepsilon_j^2 + \frac{1}{2} \kappa_j (P_j+Q_j) - \frac{3}{2} \kappa_j (P_j+Q_j)\varepsilon_j - \qquad (3.19)$$

$$- \frac{1}{4} \kappa_j (P_j-Q_j) . \Gamma_j + \frac{1}{4} \kappa_j [(K_j-H_j)^2 - \frac{1}{2} (K_j+H_j)^2]$$

The variables Γ_j and ε_j are closely related to the perturbations in longitude and vector radius as defined in Ferraz-Mello (1974). The cylindrical coordinates of each satellite in one Galilean frame of reference are easily obtained from the solutions of the above set of equations through

$$\rho_j = a_j(1 + \varepsilon_j)$$

$$\phi_j = \phi_{0j} + n_j t + \frac{1}{2} i\Gamma_j$$

$$z_j = -\frac{1}{2} a_j(K_j+H_j) - \frac{1}{4} a_j\Gamma_j(K_j-H_j) - \frac{1}{4} a_j\varepsilon_j(K_j+H_j) +$$

$$+ \frac{1}{8} a_j(K_j+H_j)(P_j=Q_j).$$

All these equations except those for K_j, H_j and z_j are exact. The equations for the space variables are approximated up to the second-order terms.

The disturbing forces enter the equations for P_j, Q_j, K_j, H_j through $\mathscr{R}, \tau$, ϑ and χ. The main disturbing forces acting on the satellites arise from their mutual attractions, form the non-symmetrical distribution of the masses inside the planet and from the Sun.

We could also consider the forces arising from other planets as well as those which give account of the corrections to Newton's equations due to the general relativity. The effects of the disturbing forces arising for other planets are not sensible in reason of their differential action; other planets may be only considered through their perturbations on the orbit of Jupiter, which leads to modulations of the solar action. The only noticeable relativistic effects are the advance of the perijoves (Ferraz-Mello, 1966); these effects do not need to be considered now and will be introduced in a later paper.

2. THE MUTUAL ATTRACTIONS

The disturbing actions due to the mutual attractions are given by

$$\mathscr{R}_j = \sum_{i\neq j} \mathscr{R}_{ji} = \sum_{i\neq j} \frac{k^2 m_i}{\nu_3^2}\left\{\frac{(1+U_j) - \alpha_{ij}(1+U_i)\xi_{ij}}{r_{ij}^3} + \frac{\alpha_{ij}(1+U_i)\xi_{ij}}{r_i^3}\right\}$$

$$\mathscr{T}_j = \sum_{i\neq j} \mathscr{T}_{ji} = \sum_{i\neq j} \frac{k^2 m_i}{\nu_3^2}\left\{\frac{(1+S_j) - \alpha_{ij}(1+S_i)\xi_{ij}^{-1}}{r_{ij}^3} + \frac{\alpha_{ij}(1+S_i)\xi_{ij}^{-1}}{r_i^3}\right\}$$

$$\vartheta_j = \sum_{i\neq j} \vartheta_{ji} = \sum_{i\neq j} \frac{k^2 m_j}{\nu_3^2}\left\{\frac{Z_j - \alpha_{ij} Z_i}{r_{ij}^3} + \frac{\zeta_{ij} Z_i}{r_i^3}\right\},$$

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where r_{ij} is the mutual distance between the satellites P_i and P_j, α_{ij} is the ratio a_i/a_j and

$$U_j = \frac{3}{4}\kappa_j D^{-1} P_j - \frac{1}{4}\kappa_j D^{-1} Q_j - \frac{3}{4}\kappa_j^2 D^{-2}(P_j - Q_j)$$

$$S_j = \frac{1}{4}\kappa_j D^{-1} P_j - \frac{3}{4}\kappa_j D^{-1} Q_j + \frac{3}{4}\kappa_j^2 D^{-2}(P_j - Q_j)$$

$$Z_j = -\frac{1}{2}(K_j + H_j).$$

These three relations are exact up to the first order only; they are always multiplied by first-order parameters and then the error will be of third order.

The ξ_{ij} are

$$\xi_{ij} = \sigma_i \sigma_j^* \xi^{g_i - g_j},$$

where $g_j = \nu_j/\nu_3$ (ratio of Eulerian mean motions), σ_j i exp $i\theta_{0j}$ (position of the jth satellite at the epoch) and σ_j^* its complex conjugate (Ferraz-Mello, 1974).

The distances r_i may be written

$$r_i = a_i(1 + U_i + S_i + U_i S_i + Z_i^2)^{1/2}$$

and then, to the first order,

$$r_i^{-3} = a_i^{-3}\,[1 - \frac{3}{2}(U_i + S_i)].$$

The mutual distances r_{ij}^{-3} may be easily calculated to the same order; we have

$$R_{ij}^{-3} = a_i^{-3}\,[1 - \alpha_{ji}(\xi_{ij} + \xi_{ij}^{-1}) + \alpha_{ji}^2]^{-3/2}$$

$$- \frac{3}{2}\, a_i^{-3}\,[1 - \alpha_{ji}(\xi_{ij} + \xi_{ij}^{-1}) + \alpha_{ji}^2]^{-5/2} \cdot [(U_i + S_i) -$$

$$- \alpha_{ji}\xi_{ij}(U_i + S_j) - \alpha_{ji}\xi_{ij}^{-1}(S_i + U_j) + \alpha_{ji}^2(U_j + S_j)].$$

For the developments we follow Sagnier (1973) and we introduce the coefficients $\gamma_s^{(k)}$ through

$$[1 - \alpha_{ji}(\xi_{ij} + \xi_{ij}^{-1}) + \alpha_{ji}^2]^{-s/2} = \sum_{-\infty}^{+\infty} \gamma_s^{(k)}(\alpha_{ji}) . \xi_{ij}^k .$$

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These coefficients are related to Laplace's $b_{s/2}^{(k)}$ (Brouwer and Clemence, 1961, p. 495) through the relations

$$\gamma_s^{(k)}(\alpha) = \frac{1}{2}\, b_{s/2}^{(k)}(\alpha) \qquad \text{if } \alpha < 1$$

$$\gamma_s^{(k)}(\alpha) = \frac{1}{2}\, \alpha^{-s} b_{s/2}^{(k)}(\alpha) \qquad \text{if } \alpha > 1.$$

For the brackets which appear in the actual equations we write

$$[1 - \alpha_{ji}(\xi_{ij} + \xi_{ij}^{-1}) + \alpha_{ji}^2]^{-3/2} = \sum_{-\infty}^{+\infty} \mathscr{G}_{ji}^{(k)} \varepsilon_{ij}^k$$

$$[1 - \alpha_{ji}(\xi_{ij} + \xi_{ij}^{-1}) + \alpha_{ji}]^{-5/2} = \sum_{-\infty}^{+\infty} \mathscr{H}_{ji}^{(k)} \xi_{ij}^k .$$

The disturbing forces arising from the mutual attractions are

$$\mathscr{R}_{ji} = \sum_{-\infty}^{+\infty} \{R_{1(ji)}^{(k)} + R_{2(ji)}^{(k)} D^{-1} P_j + R_{3(ji)}^{(k)} D^{-1} Q_j + R_{4(ji)}^{(k)} D^{-2} (P_j - Q_j) + R_{5(ji)}^{(k)} D^{-1} P_i + R_{6(ji)}^{(k)} D^{-1} Q_i + R_{7(ji)}^{(k)} D^{-2} (P_i - Q_i)\} . \xi_{ij}^k$$

$$\mathscr{T}_{ji} = \sum_{-\infty}^{+\infty} \{T_{1(ji)}^{(k)} + T_{2(ji)}^{(k)} D^{-1} P_j + T_{3(ji)}^{(k)} D^{-1} Q_j + T_{4(ji)}^{(k)} D^{-2} (P_j - Q_j) + T_{5(ji)}^{(k)} D^{-1} P_i + T_{6(ji)}^{(k)} D^{-1} Q_u + T_{7(ji)}^{(k)} D^{-2} (P_i - Q_i)\} . \xi_{ij}^k$$

$$\mathscr{V}_{ji} = \sum_{-\infty}^{+\infty} \{V_{1(ji)}^{(k)} (K_j + H_j) + V_{2(ji)}^{(k)} (K_i + H_i)\} \xi_{ij}^k ,$$

where the coefficients $R_{m(ji)}^{(k)}$, $T_{m(ji)}^{(k)}$, $V_{m(ji)}^{(k)}$ are numerical factors that depend only on the mean distances a_j. The function χ_j which appears as a factor in the integro-differential equations may be calculated after its definition

$$\chi_j = (1 - \frac{1}{2} c_j^2)^{-1} [(1 + S_j)\, \mathscr{R}_j - (1 + U_j)\, \mathscr{T}_j],$$

where

$$c_j^2 = (1 + U_j)\,(1 + S_j).$$

It follows

$$\chi_j = \sum_{i \neq j} \chi_{ji}$$

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and

$$\chi_{ji} = \sum_{-\infty}^{+\infty} \{X^{(k)}_{1(ji)} + X^{(k)}_{2(ji)}D^{-1}P_j + X^{(k)}_{3(ji)}D^{-1}Q_j + X^{(k)}_{4(ji)}D^{-2}(P_j - Q_j) +$$

$$+ X^{(k)}_{5(ji)}D^{-1}P_i + X^{(k)}_{6(ji)}D^{-1}Q_i + X^{(k)}_{7(ji)}D^{-2}(P_i - Q_i)\}.\xi^k_{ij}.$$

When calculating $D^{-1}\chi_j$ it is enough to restrict it to the first-order. Then, since the $X^{(k)}_{m(ji)}$ are of first-order in the disturbing mass, it follows

$$D^{-1}\chi_{ji} = \sum_{-\infty}^{+\infty} \frac{X^{(k)}_{1(ji)}.\xi_{ij}}{k(g_i - g_j)}.$$

This primitive function in singular when k = 0 unless $X^{(0)}_{1(ji)}$ is also equal to zero; this fact indeed happens as a consequence of $\mathcal{G}^{(-1)}_{ji} = \mathcal{G}^{(+1)}_{ji}$.

3. JUPITER's SHAPE ACTIONS

In rectangular coordinates the equations of the motion of the j-th satellite under the action of the planet writes

$$\ddot{x}_j - 2N\dot{y}_j - N^2 x_j = \frac{\partial}{\partial x_j} k^2(1 + m_j)\ (\frac{1}{r_j} + \Omega)$$

$$\ddot{y}_j + 2N\dot{x}_j - N^2 y_j = \frac{\partial}{\partial y_j} k^2(1 + m_j)\ (\frac{1}{r_j} + \partial)$$

$$\ddot{z}_j = \frac{\partial}{\partial z_j} k^2(1 + m_j)\ (\frac{1}{r_j} + \Omega),$$

where

$$\Omega = -\frac{1}{r_j} \sum_{k=2}^{4} \frac{J_k R^k}{r_j^k} P_k(\sin\beta_j).$$

and N is the angular velocity of the rotating Eulerian frame. The zonal harmonics of subscripts greater than 4 are neglected, as well as the tesseral harmonics and the harmonics related to the shape of the satellites (Ferraz-Mello, 1966). J_2 is considered as a first-order parameter while J_3 and J_4 are considered as second-order parameters. It follows

$$\ddot{x}_j - 2Ny_j - N^2 x_j = -k^2(1+m_j)\ \{\frac{x_j}{r_j^3} + \frac{3}{2}\frac{J_2R^2x_j}{r_j^5} - \frac{15}{8}\frac{J_4R^4x_j}{r_j^7}\}$$

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$$\ddot{y}_j + 2N\dot{x}_j \; N^2 y_j = -k^2(1+m_j)\left\{\frac{y_j}{r_j^3} + \frac{3}{2}\frac{J_2R^2y_j}{r_j^5} - \frac{15}{8}\frac{J_4R^4y_j}{r_j^7}\right\}$$

$$\ddot{z}_j = -k^2(1+m_j)\left\{\frac{z_ja_j}{r_j^3} + \frac{3}{2}\frac{J_2R^2z_ja_j}{r_j^5} + \frac{3J_2R^2z_p\sin\beta_j}{r_j^4} - \right.$$
$$\left. - \frac{3J_3R^3z_p}{2r_j^5}\right\}.$$

In these equations β_j is the latitude of the satellite over the planet's figure equator and z_p is the third coordinate of the planet's figure pole. If the special coordinates U_j and S_j are introduced instead of x_j, y_j we may derive the values of the force components $\tilde{\mathcal{R}}_j$, $\tilde{\mathcal{T}}_j$ and $\tilde{\mathcal{V}}_j$ due to the non-sphericity of Jupiter. We have

$$\tilde{\mathcal{R}}_j = \lambda_j a_j^3 \left[\frac{3J_2R^2}{2r_j^5} - \frac{15J_4R^4}{8r_j^7}\right] (1 + U_j)$$

$$\tilde{\mathcal{T}}_j = \lambda_j a_j^3 \left[\frac{3J_2R^2}{2r_j^5} - \frac{15J_4R^4}{8r_j^7}\right] (1 + S_j)$$

$$\tilde{\mathcal{V}}_j = \lambda_j a_j^3 \frac{3J_2R^2}{2r_j^5} z_j + \lambda_j a_j^2 \left[\frac{3J_2R^2\sin\beta_j}{r_j^4} - \frac{3J_3R^3}{2r_j^5}\right] z_p.$$

These disturbing forces may be expanded as in the previous section. It follows then:

$$\tilde{\mathcal{R}}_j = \tilde{R}_{1(j)} + \tilde{R}_{2(j)}D^{-1}P_j + \tilde{R}_{3(j)}D^{-1}Q_j + \tilde{R}_{4(j)}D^{-2}(P_j - Q_j)$$

$$\tilde{\mathcal{T}}_j = \tilde{T}_{1(j)} + \tilde{T}_{2(j)}D^{-1}P_j + \tilde{T}_{3(j)}D^{-1}Q_j + \tilde{T}_{4(j)}D^{-2}(P_j - Q_j)$$

$$\tilde{\mathcal{V}}_j = \tilde{V}_{0p(j)} + \tilde{V}_{1(j)}(K_j + H_j) + \tilde{V}_{3(j)}K_p + \tilde{V}_{4(j)}H_p.$$

The coefficients $\tilde{R}_{m(j)}$, $\tilde{T}_{m(j)}$ and $\tilde{V}_{m(j)}$ are numerical factors depending only in the mean distances. χ_j in the order of approximation of this theory is zero. In the calculation of $\tilde{\mathcal{V}}_j$ the coordinates of the pole of inertia of Jupiter intervenes also through

$$\sin\beta_j = \frac{x_jx_p + y_jy_p + z_jz_p}{r_j}.$$

If I_p and Ω_p are the inclination and the longitude of the ascending node of the figure's equator, then

$$x_p = \sin I_p \sin \Omega_p \qquad Y_p = -\sin I_p \cos \Omega_p \qquad Z_p = Z_p = 1$$

except for terms of the third-order in the equations for $x_p Y_p$ and of the second-order in the equation for z_p. We also introduce two new parameters

$$K_p = \sin I_p \exp \; i(\Omega_p + \frac{3\pi}{2})$$

$$H_p = \sin I_p \exp - i(\Omega_p + \frac{3\pi}{2}) \; .$$

4. THE MEAN EQUATOR OF JUPITER AS REFERENCE

The inclination of the equator of Jupiter over the planet's orbital plane is $3^\circ\!.07$. If the orbital plane is conserved as a reference plane, we have

$$|K_j| \sim 0.06 \qquad |K_j|^2 \sim 0.004 \qquad |K_j|^3 \sim 0.0002$$

since the satellites move very close to the plane of the equator of the planet. We may compare these figures to those for the eccentricities:

$$|P_j| \sim 10^{-2} \qquad |P_j|^2 \sim 10^{-4}.$$

These figures allow us to see that if the orbital plane of Jupiter is kept as the reference plane, we need the third powers of the K_j to get the same precision as that of the second powers of the P_j. To reestablish homogeneity it is convenient to adopt the mean equator of Jupiter as the reference plane; in this case we will have $|K_j|^2 \sim 5\text{x}10^{-5}$.

Let **x** be the matrix of the coordinates with respect to the mean equator of Jupiter and **X** the matrix of the coordinates with respect to an inertial plane close to the plane of the orbital motion of the planet. We have

$$\mathbf{x} = (1 + R) . \mathbf{X},$$

where

$$R = \begin{Bmatrix} \sin^2\theta_p(\cos I-1) & \sin\theta_p\cos\theta_p(1 - \cos I) & -\sin\theta_p\sin I \\ \sin\theta_p\cos\theta_p(1 - \cos I) & \cos^2\theta_p(\cos I - 1) & \cos\theta_p\sin I \\ \sin\theta_p\sin I & - \cos\theta_p\sin I & \cos I-1 \end{Bmatrix}$$

The equations of the motion of a satellite in reactangular coordinates writes (Section 3)

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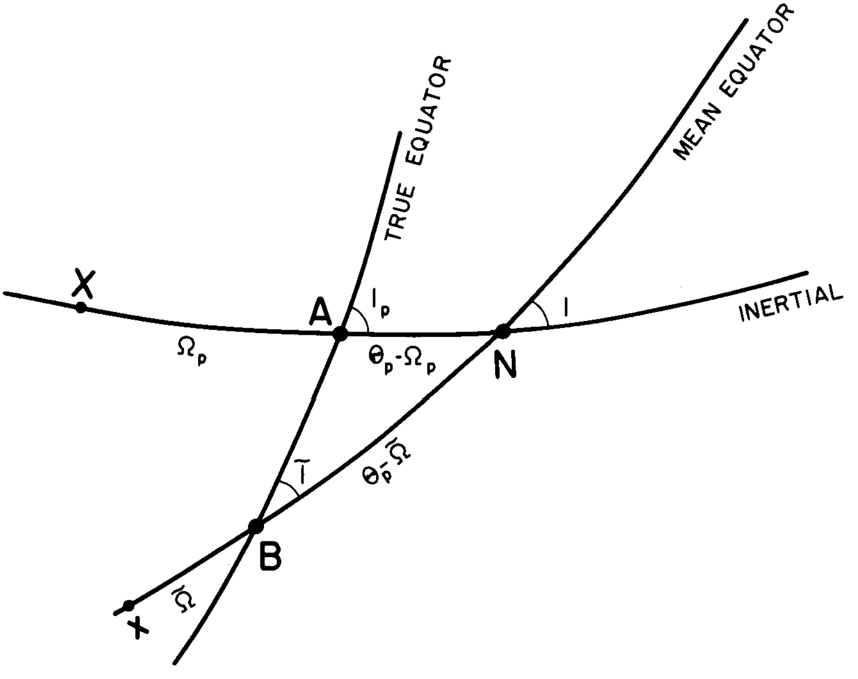

$$A\ddot{X} + B\dot{X} + Cx = \mathscr{L}(x)$$

where A is the unit-matrix,

$$B = \begin{Bmatrix} 0 & -2N & 0 \\ 2N & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}, \qquad C = \begin{Bmatrix} -N^2 & 0 & 0 \\ 0 & -N^2 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$

and $\mathscr{L}(\mathbf{X})$ is a vector function of $\mathbf{X}$. It is important to observe that $\mathscr{L}(\mathbf{X})$ depends on $\mathbf{X}$ in two different ways: directly through linear relations and indirectly through intrinsic parameters like distances, mutual distances and latitudes. Then

$$(1 + R) . \mathscr{L}(\mathbf{X}) = \mathscr{L}(\mathbf{X}) .$$

If the rotation matrix (1-R) is applied to both sides of the equations of the motion it follows that

$$A\ddot{x} + B\dot{x} + Cx = \mathscr{L}(x) + B\dot{R}(1 + R)^{-1}x + 2A\dot{R}(1 + R)^{-1}\dot{x} -$$

$$- 2A[\dot{R}(1 + R)^{-1}2]x + AR(1 + R)^{-1}x.$$

The matrix $\dot{R}$ is

$$\dot{R} = \dot{\theta}\begin{Bmatrix} -\sin 2\theta_p(1 - \cos I) & \cos 2\theta_p(1 - \cos I) & -\cos\theta_p \sin I \\ \cos 2\theta_p(1 - \cos I) & \sin 2\theta_p(1 - \cos I) & -\sin\theta_p \sin I \\ \cos \theta_p \sin I & \sin \theta_p \sin I & 0 \end{Bmatrix}$$

Since θ_p is very small, $2.185\text{x}10^{-6}$ degrees per day (Tisserand, 1896), we may neglect the term in which $\dot{R}^2$ appears; also, since $\ddot{\theta}_p = 0$ we may also neglect the term in $\ddot{R}$. The new equations are

$$A\ddot{x} + B\dot{x} + Cx = (x) + B\dot{R}(1 + R)^{-1}x + 2A\dot{R}(1 + R)^{-1}\dot{x}.$$

The effect of the adoption of the mean equator of Jupiter as the reference plane is the introduction of the disturbing force $B\dot{R}(1 + R)^{-1}x + + 2A\dot{R}(1 + R)^{-1}\dot{x}$. The magnitude of this force is at most equal to $3\text{x}10^{-6}$ AU/y^2 per unit mass. It may be compared to the magnitude of the mutual interactions between two satellites (10^{-2} to 10^{-1} in the same units). They will be neglected, and we will adopt in the new frame the same equations as in the inertially oriental frame. As a consequence, this theory will not be able to show the eventual existence of inequalities of very long peridds ($4.5\text{x}10^{r}$ yr) arising from the precession of the equator of the planet.

In the preceding section, in the expansion giving $\mathscr{V}_j$ the parameters K_p and H_p were related to the inclination and longitude of the ascending node of the true equator over the reference frame, i.e., the orbital plane. In the new reference frame we consider.

$$\tilde{K} = \tilde{I}.\exp i(\tilde{\Omega} + \frac{3}{2}\pi)$$

$$\tilde{H} = \tilde{I}.\exp -i(\tilde{\Omega} + \frac{3}{2}\pi)$$

and then

$$\mathscr{V}_j = \tilde{V}_{0p(j)} + \tilde{V}_{1(j)}(\tilde{K}_j + \tilde{H}_j) + \tilde{V}_{3(j)}\tilde{K} + \tilde{V}_{(4j)}\tilde{H}.$$

The fundamental relations of the spherical triangle NAB allow us to write

$$\sin I.\exp i(\theta_p - \tilde{\Omega}) = \sin I_p \exp i(\theta_p - \tilde{\Omega}_p) - \cos I_p.\sin I +$$

$$+ \sin I_p \cos(\theta_p - \Omega_p)\ (\cos I-1)$$

or, neglecting all terms of second or higher order with respect to $\tilde{I}$,

$$\tilde{I}.\exp(-i\tilde{\Omega}) = \sin I_p.\exp(-i\Omega_p) - \sin I.\exp - i\theta_p,$$

and then

$$\tilde{H} = H_p - \sin I.\exp - i(\theta_p + \frac{3\pi}{2})$$

$$\tilde{K} = K_p - \sin I.\exp i(\theta_p + \frac{3\pi}{2}).$$

The new expression for $\tilde{\mathcal{V}}_j$ is then

$$\tilde{\mathcal{V}}_j = \tilde{V}_{0(j)} + \tilde{V}_{1(j)}(K_j + H_j) + \tilde{V}_{3(j)}K_p + \tilde{V}_{4(j)}H_p,$$

where

$$\tilde{V}_{0(j)} = V_{0p(j)} - \frac{3}{2}\lambda_j J_2 \frac{R^2}{a_j^2} \sin I.(\xi_{jp} - \xi_{jp}^{-1}).$$

In analogy to former parameters we introduce

$$\xi_{jp} = \sigma_j \sigma_p^* \zeta^{g_j - g_p}$$

where

$$\sigma_p = i \exp i\theta_{op} \qquad g_p \nu_3 = \nu_p \qquad \text{and} \qquad \nu_p = \dot{\theta}_p - N.$$

5. MOTION OF THE POLE OF JUPITER

The researches of Souillart and Laplace (Tisserand, 1896) show that it is not possible to consider the motion of the orbital plane of the satellites and the motion of the equator of the planet separately. These motions appear in the approximation considered here, as a linear system with 5 degrees of freedom.

If the planet is supposed to have axial symmetry, we have (Ferraz-Mello, 1972)

$$\sin I_p \dot{\Omega}_p = - \sum_j \frac{1}{rC} (\frac{\partial W}{\partial Z_j}) \bar{y}_j$$

$$\dot{I}_p = - \sum_j \frac{1}{rC} (\frac{\partial W}{\partial Z_j}) \bar{x}_j$$

where C is the polar momentum of inertia of the planet, r the angular velocity of the planetary rotation, and $\bar{x}_j, \bar{y}_j$ are the rectangular coordinates of the disturbing bodies in a frame in which the true equator of Jupiter is the fundamental plane of reference, and, in which the

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x-axis is directed to point A (see Figure 1 in the preceding section) ascending node of the true equator over the inertial frame. We have

$$\bar{x}_j = x_j.\cos\theta_p + y_j.\sin\theta_p$$

$$\bar{y}_j = -x_j.\sin\theta_p + y_j.\cos\theta_p$$

the other components of the rotation, the amplitudes of which are I, $-I_p$ and $\theta_p-\Omega_p$, give rise to corrections small enough to be neglected. We stlll have

$$\frac{\partial W}{\partial z_j} + \frac{4k^2 m_j J_2 R^2}{3r_j^4}\sin\beta_j$$

which is proportional to the torque of the attraction of the satellite about the polar axis of the planet. These equations and the quantities K_p and H_p already defined allow us to write the equations of the motion of the equator of Jupiter in a suitable form. From the definition of K_p we have

$$\dot{K}_p = \cos I_p.\dot{I}_p \exp i(\Omega_p + \frac{3\pi}{2}) + i.\sin I_p.\dot{\Omega}_p.\exp i(\Omega_p + \frac{3\pi}{2})$$

and then

$$K_p = \sum_j \frac{4k^2 m_j J_2 R^2}{3rCr_j^4} a_j i\sigma_j \zeta^{g_j}\sin\beta_j.$$

If we observe that

$$\sin\beta_j = \frac{1}{2}\sigma_j\zeta^{g_j}[H_p - \sin I \exp{-i}(\theta + \frac{3\pi}{2})] +$$

$$+ \frac{1}{2}\sigma_j^*\zeta^{-g_j}[K_p - \sin I \exp i(\theta + \frac{3\pi}{2})] - \frac{1}{2}(K_j + H_j)$$

we obtain

$$DK_p + \kappa_p K_p = \tilde{V}_{0(p)} + \tilde{B}_{1(p)}H_p + \sum_j \tilde{V}_{3(p)}(K_j + H_j)$$

and in a similar way

$$DH_p - \kappa_p H_p = -\tilde{V}_{0(p)} - V^*_{1(p)}K_p - \sum_j \tilde{V}^*_{3(p)}(K_j + H_j)$$

where

$$\tilde{V}_{0(p)} = \sum_j - \frac{2k^2 m_j J_2 R^2}{3Cr\nu_3^2 a_j^3}\sin I.\exp{-i}(\theta + \frac{3\pi}{2})\sigma_j^2\zeta^{2g_j} + \kappa_p \sin I \exp i(\theta + \frac{3\pi}{2})$$

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$$V_{1(p)} = \sum_j \frac{2k^2 m_j J_2 R^2}{3Cr\nu_3^2 a_j^3} \sigma_j^2 \zeta^{2g_j}$$

$$\tilde{V}_{3(p)} = - \frac{2k^2 m_j J_2 R^2}{3Cr\nu_3^2 a_j^3} \sigma_j \zeta^{g_j}$$

and

$$\kappa_p = - \sum_j \frac{2k^2 m_j J_2 R^2}{3Cr\nu_3 a_j^3}$$

6. SOLAR PERTURBATIONS

The disturbing actions due to the Sun are given by the same relations as for the satellites when i=o (see Section 2). The difference is that in this case the α_{j0} are very small (5×10^{-4} to 3×10^{-3}). The quantity r_{0j}^{-3} may be expanded in the powers of α_{j0} without the need of introducing new Laplace coefficients. Up to the second-order we have

$$r_{0j}^{-3} = r_0^{-3} \{1 + \frac{3}{2}\alpha_{j0}(\frac{a_o}{r_0})^2(u_0 s_j + s_0 u_j) - \frac{3}{2}\alpha_{j0}^2(\frac{a_0}{r_0})^2 u_j s_j +$$

$$+ 3\alpha_{jo}(\frac{a_0}{r_0})^2 z_0 z_j + \frac{15}{8}\alpha_{j0}^2(\frac{a_0}{r_0})^4(u_0 s_j + s_0 u_j)^2\},$$

where

$$u_j = \sigma_j \zeta^{g_j}(1 + U_j); \qquad s_j = \sigma_j^* \zeta^{-g_j}(1 + S_j) \qquad j=0,1,2,3,4.$$

After the substitutions

$$\mathscr{R}_{j0} = \sum_{k\neq 0,2} R_{1(j0)}^{(k)} + R_{2(j0)}^{(k)} D^{-1} P_j + R_{3(j0)}^{(k)} D^{-1} Q_j + R_{4(j0)}^{(k)} D^{-2} (P_j - Q_j)\}\xi_{0j}^k$$

$$\mathscr{T}_{j0} = \sum_{k\neq 0,-2} \{T_{1(j0)}^{(k)} + T_{2(j0)}^{(k)} D^{-1} P_j + T_{3(j0)}^{(k)} D^{-1} Q_j + T_{4(j0)}^{(k)} D^{-2}(P_j - Q_j)\}\xi_{0j}^k$$

$$\mathscr{V}_{j0} = - \frac{3}{2} \frac{k^2 m_0 z_0}{\nu_3^2 r_0^3} (\frac{a_0}{r_0})^2 [(1+U_0)\xi_{oj} + (1+S_o)\xi_{0j}^{-1}] - \frac{1}{2} \frac{k^2 m_0}{\nu_3^2 r_0^3} (K_j + H_j)$$

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and also

$$\chi_{j0} = \frac{3}{2}(\mathscr{R}_{j0} - \mathscr{T}_{j0}) + \frac{1}{2}\kappa_j D^{-1}(P_j - Q_j)(\mathscr{R}^I_{j0} - \mathscr{T}^I_{j0}) +$$

$$+ \frac{3}{8}[\kappa_j D^{-1}P_j - 3\kappa_j D^{-1}Q_j + 3\kappa_j^2 D^{-2}(P_j - Q_j)]\mathscr{R}^I_{j0} -$$

$$- \frac{3}{8}[3\kappa_j D^{-1}P_j - \kappa_j D^{-1}Q_j - 3\kappa_j^2 D^{-2}(P_j - Q_j)]\mathscr{T}^I_{j0}$$

where $\mathscr{R}^I_{j0}$ and $\mathscr{T}^I_{j0}$ represent the first-order terms of $\mathscr{R}_{j0}$ and $\mathscr{T}_{j0}$. The $R^{(k)}_{m(j0)}$ and $T^{(k)}_{m(j0)}$ are numerical coefficients.

The introduction of the solar perturbations does not give rise to any difficulties except for the complexity of the formulae giving the relative motion of Jupiter around the Sun. In this theory U_0, S_0 and Z_0 are considered as known functions of the time through the Theory of Hill (Hill, 1890) and they are written in a form compatible with the kind of algebra adopted here. All terms which give rise to long-period inequalities of amplitude larger than 0.1 arcsecond, and especially those that give rise to secular inequalities must be kept. In this paper, for the sake of simplicity, we restrain ourselves to

$$\mathscr{R}_{j0} = R^{(0)}_{1(j0)} + R^{(2)}_{1(j0)}\ \xi^2_{0j}$$

$$\mathscr{T}_{j0} = R^{(0)}_{1(j0)} + R^{(2)}_{1(j0)}\ \xi^{-2}_{0j}$$

These resctrictions are justified because $k^2 m_0/\nu_3^2 r_0^3$ is a second-order quantity; the terms which are not considered are quantities of third-order. These restrictions do not affect the presentation of the theory itself but only the actual calculations.

7. THE INTEGRO-DIFFERENTIAL EQUATIONS. FINAL FORM

In order to obtain the final form of the integro-differential equations for P_j and Q_j we must collect the partial contributions of $\mathscr{R}_j$, $\mathscr{T}_j$ and χ_j and to introduce them in the equations of Section 1. So we obtain

$$\begin{pmatrix} D\bar{P} \\ D\bar{Q} \end{pmatrix} + \begin{pmatrix} X_0 & 0 \\ 0 & -X_0 \end{pmatrix}\begin{pmatrix} \bar{P} \\ \bar{Q} \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 7A^I & -5A^I \\ 5A^I & -7A^I \end{pmatrix}\begin{pmatrix} \bar{P} \\ \bar{Q} \end{pmatrix} + \begin{pmatrix} V_{11} \\ V_{12} \end{pmatrix} + \begin{pmatrix} V_{21} \\ V_{22} \end{pmatrix} +$$

$$+ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} D^{-1}\bar{P} \\ D^{-1}\bar{Q} \end{pmatrix} + \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}\begin{pmatrix} D^{-2}\bar{P} \\ D^{-2}\bar{Q} \end{pmatrix} = 0$$

where $\bar{P}$ and $\bar{Q}$ are 4-vectors whose components are respectively P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4; $X_0 = \mathrm{diag}\ (\kappa_j)$;

$$A^I = \mathrm{diag}\ \{ \sum_{i \neq j} \sum_{-\infty}^{+\infty} X_{0(ji)}^{(k)}\ \xi_{ij}^k \};$$

V_{11} and V_{12} are the 4-vectors whose components are

$$V_{11(j)} = -\frac{\lambda_j - \kappa_j^2}{\kappa j} - \frac{1}{\kappa_j} R_{1(j)} - \sum_{i \neq j} \sum_{-\infty}^{+\infty} \left(\frac{1}{\kappa_j} R_{1(ji)}^{(k)} + \frac{1}{4\kappa_j} X_{1(ji)}^{(k)}\right) \xi_{ij}^k$$

$$V_{21(j)} = -\frac{1}{\kappa_j} \mathscr{R}_{j0} - \frac{1}{4\kappa_j} \mathscr{T}_{j0}$$

and M_{11}, M_{12} and M_{11} are matrices which components are

$$M_{11(jj)} = \frac{3}{4}(\lambda_j - \kappa_j^2) - \frac{1}{\kappa_j} \tilde{R}_{2(j)} - \sum_{i \neq j} \sum_{-\infty}^{+\infty} \left([2;3] - \frac{7}{8}\kappa_j X_{0(ji)}^{(k)}\right) \xi_{ij}^k$$

$$M_{11(ji)} = -\sum_{-\infty}^{+\infty} \left(\frac{1}{\kappa_j} R_{5(ji)}^{(k)} + \frac{1}{4\kappa_j} X_{5(ji)}^{(k)}\right) \xi_{ij}^k \qquad (j \neq i)$$

$$M_{12(jj)} = -\frac{5}{4}(\lambda_j - \kappa_j^2) - \frac{1}{\kappa_j} \tilde{R}_{3(j)} - \sum_{i \neq j} \sum_{-\infty}^{+\infty} \left([3;-1] + \frac{7}{8}\kappa_j X_{0(ji)}^k\right) \xi_{ij}^k$$

$$M_{12(ji)} = -\sum_{-\infty}^{+\infty} \left(\frac{2}{\kappa_j} R_{6(ji)}^{(6)} + \frac{1}{4\kappa_j} X_{6(ji)}^{(k)}\right) \xi_{ij}^k \qquad (j \neq i)$$

$$N_{11(jj)} = \frac{3}{4}\kappa_j(\lambda_j - \kappa_j^2) - \frac{1}{\kappa_j} \tilde{R}_{4(j)} - \sum_{i \neq j} \sum_{-\infty}^{+\infty} [4;-3\kappa_j] \xi_{ij}^k$$

$$N_{11(ji)} = -\sum_{-\infty}^{+\infty} \left(\frac{1}{\kappa_j} R_{7(ji)}^{(k)} + \frac{1}{4\kappa_j} X_{7(ji)}^{(k)}\right) \xi_{ij}^k \qquad (j \neq i)$$

where the brackets mean

$$[a;b] = \frac{1}{\kappa_j} R_{a(ji)}^{(k)} + \frac{1}{4\kappa_j} X_{a(ji)}^{(k)} + \frac{1}{16} b X_{1(ji)}^{(k)}.$$

We still have $V_{12} = V_{11}^*$, $V_{22} = V_{21}^*$, $N_{21} = M_{11}^*$, $M_{22} = M_{12}^*$, $N_{12} = -N_{11}$, $N_{21} = N_{11}^*$ and $N_{22} = -N_{11}^*$ where the asterisks indicate complex conjugation.

This equation still may be written using vectors and matrices of rang 8:

$$XP + \hat{X}_0 P + L_1 P + V_1 + V_2 + M_1 D^{-1} P + N_1 D^{-2} P = 0$$

where the symbols introduced have obvious meanings; still, in every case the subscripts give the order of magnitude of the elements they represent.

8. THE AVERAGING METHOD OF KRASINSKY

In order to reduce the integro-differential linear equation we introduce the linear transformation (the subscripts indicate the order of the term)

$$P = \hat{P} + B_1 + B_2 + C_1 \hat{P} + E_1 D^{-1} \hat{P} + T_1 D^{-2} \hat{P}.$$

It follows the new integro-differential equation:

$$D\hat{P} + DB_1 + DB_2 + DC_1 . \hat{P} + C_1 . D\hat{P} + DE_1 . D^{-1}\hat{P} + E_1 . \hat{P} + DT_1 . D^{-2}\hat{P} +$$

$$+ T_1 D^{-1}\hat{P} + \hat{X}_0 \hat{P} + \hat{X}_0 B_1 + \hat{X}_0 B_2 + \hat{X}_0 C_1 \hat{P} + \hat{X}_0 E_1 D^{-1}\hat{P} + \hat{X}_0 T_1 D^{-2}\hat{P} +$$

$$+ L_1 \hat{P} + L_1 B_1 + V_1 + V_2 + M_1 D^{-1}\hat{P} + M_1 D^{-1} B_1 + N_1 D^{-2}\hat{P} +$$

$$+ N_1 D^{-2} B_1 = 0.$$

The optimum transformation would be obtained if we could find constant vectors b_1 and b_2, and constant matrices e_1, f_1 and g_1 such that

$$(I+C_1)^{-1}(DB_1+DB_2+\hat{X}_0 B_1+\hat{X}_0 B_2+L_1 B_1+V_1+V_2+M_1 D^{-1} B_1+N_1 D^{-2} B_1) = b_1+b_2=b_1$$

$$(I+C_1)^{-1}(DC_1+E_1+\hat{X}_0+\hat{X}_0 C_1+L_1) = \hat{X}_0+e_1$$

$$(I+C_1)^{-1}(DE_1+T_1+\hat{X}_0 E_1+M_1)=f_1=0$$

$$(I+C_1)^{-1}(DT_1+\hat{X}_0 T_1+N_1)=g_1=0$$

if we restrict these equations to the second order and if we equate separately the terms of different orders, it follows that

$$DB_1+\hat{X}_0 B_1+V_1=b_1$$

$$DT_1+\hat{X}_0 T_2+N_1=g_1=0$$

$$DE_1+T_1+\hat{X}_0 E_1+M_1=f_1=0$$

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$$DC_1+\hat{X}_0C_1-C_1\hat{X}_0+E_1+L_1=e_1$$

$$DB_2+\hat{X}_0B_2+L_1B_1+M_1D^{-1}B_1+N_1D^{-2}B_1+V_2-C_1b_1=b_2=0$$

These equations have been written following their increasing difficulty. If they are solved adequately the integro-differential equation becomes

$$D\hat{P} + \hat{X}_0\hat{P} + e_1\hat{P} + b_1 = 0,$$

and it may be solved by elementary methods.

The fact that g_1, f_1 and b_2 are made equal to zero will become clear in the following section.

8. DISCUSSION OF THE EQUATIONS

To discuss the first equation let it be decomposed into its two parts of rang 4:

$$DB_{11}+X_0B_{11}+V_{11}=b_{11}$$

$$DB_{12}-X_0B_{12}+V_{12}=b_{12};$$

the complex conjugate of the first part is

$$DB^*_{11}-X_0B^*_{11}-V^*_{11}=-b_{11}$$

($b_{11}=b^*_{11}$ since this vector is constant). The comparison lead to

$$B_{12} = B^*_{11} \qquad \text{and} \qquad b_{12} = -b_{11}.$$

The problem is reduced to the solution of only one of the two rang-4 equations. Equating the elements for the first of them we have

$$DB_{11(j)} + \kappa_j B_{11(j)} + V_{11(j)} = b_{11(j)}$$

and we adopt the solution

$$B_{11(j)} = \sum_{i\neq j} \sum_{\substack{-\infty \\ k\neq 0}}^{+\infty} \frac{1}{\kappa_j+k(g_i-g_j)} \left(\frac{1}{\kappa_j} R^{(k)}_{1(ji)} + \frac{1}{4\kappa_j} X^{(k)}_{1(ji)}\right)\xi^k_{ij}$$

$$b_{11(j)} = -\frac{1}{\kappa_j}(\lambda_j-\kappa_j^2+\tilde{R}_{1(j)}) - \sum_{i\neq j} \frac{1}{4\kappa_j}(4R^{(0)}_{1(ji)}+X^{(0)}_{1(ji)})$$

i.e., we put the secular terms in b_1 and the periodic terms in B_1.
Let the second equation be decomposed:

$$DT_{11}+X_0T_{11}+N_{11}=g_{11}$$

$$DT_{21}-X_0T_{21}+N^*_{11}=g_{21}$$

$$DT_{12}+X_0T_{12}-N_{11}=g_{12}$$

$$DT_{22}-X_0T_{22}-N^*_{11}=g_{22}.$$

As for the first equation we obtain

$$T_{21} = - T^*_{11} \qquad T_{22} = - T^*_{12} \qquad T_{12} = - T_{11}$$

$$g_{21} = g_{11} \qquad g_{22} = g_{12} \qquad g_{12} = - g_{11}$$

The question is then reduced to the solution of only one equation

$$DT_{11} + X_0T_{11} + N_{11} = g_{11}$$

and we adopt the solution

$$T_{11(jj)} = -\frac{3}{4}(\lambda_j-\kappa_j^2) + \frac{1}{\kappa_j^2}R_{4(j)} - \sum_{i\neq j}\sum_{-\infty}^{+\infty}\frac{[4;-3\kappa_j]\xi_{ij}^k}{k(g_i-g_j)+\kappa_j}$$

$$T_{11(ji)} = \sum_{-\infty}^{+\infty}\frac{4R_{7(ji)}^{(k)}+X_{7(ji)}^{(k)}}{4\kappa_j[k(g_i-g_j)+\kappa_j]}\xi_{ij}^k \qquad (j\neq i)$$

$$g_{11(jj)} = g_{11(ji)} = 0.$$

The condition of validity of the above solution is

$$k(g_i-g_j) + \kappa_j \neq 0 \qquad (i\neq j)$$

or, since $\kappa_j = g_j + m$ $(m=-0.01448)$:

$$k(g_i-g_j) + g_j + m \neq 0. \qquad (i\neq j)$$

For the three inner satellites g_j are integers (4,2, and 1); the condition of validity reduces then to $m\notin Z$. For the fourth satellite $g_4=0.437$. It is easy to see that we do not have a small divisor. In the problem of the motion of an asteroid disturbed by Jupiter this leads to exclude the commensurabilities of the kind

$$a\kappa_i + b\kappa_j = 0; \qquad |a + b| = 1. \qquad (a,b\in Z).$$

The third equation may be discussed exactly in the same manner as the former two. The fourth equation shows special features and will be explicitly discussed. After decomposition we have for this equation two independent relations:

$$DC_{11}+X_0C_{11}-C_{11}X_0+E_{11}+L_{11}=e_{11}$$

$$DC_{12}+X_0C_{12}+C_{12}X_0+E_{12}+L_{12}=e_{12}.$$

The other two relations do not need to be considered since we can prove that $C_{22}=C^*_{11}$, $C_{21}=C^*_{12}$, $e_{22}=-e^*_{11}$ and $e_{21}=-e^*_{12}$.

The basic difference for the above equation is that the method of Krasinsky does not allow us to eliminate all periodic terms. For instance, for each (j,i) from the first equation we have

$$DC_{11(ji)} + (\kappa_j-\kappa_i)C_{11(ji)} + E_{11(ji)} + L_{11(ji)} = e_{11(ji)}$$

therefore, if terms factored by ξ_{ij} exist in the non-homogeneous part of the equation, since their motion is exactly $(\kappa_j-\kappa_i)$, they lead to null divisors. These terms must then be excluded from the equation. We adopt

$$e_{11(jj)} = -\frac{1}{\kappa_j^3}\tilde{R}_{4(j)} + \frac{1}{\kappa_j^2}\tilde{R}_{2(j)} + \sum_{i\neq j}\{\frac{1}{\kappa_j^2}[4;-\frac{3}{\kappa_j}]+\frac{1}{\kappa_j^3}[2;\frac{3}{\kappa_j}]-$$

$$-\frac{7}{8}X^{(0)}_{0(ji)}\} = \tilde{e}_{jj}$$

$$e_{11(ji)} = -\frac{4R^{(1)}_{7(ji)}+X^{(1)}_{7(ji)}}{4\kappa_j\kappa_i^2}\xi_{ij} - \frac{4R^{(1)}_{5(ji)}+X^{(1)}_{5(ji)}}{4\kappa_j\kappa_i}\xi_{ij} = \tilde{e}_{ji}\xi_{ij}$$

$$e_{12(jj)} = e_{12(ji)} = 0.$$

Now the system giving the components of the matrices C_{11} and C_{12} may be solved. The condition for the validity of the solutions are

$$k(g_i-g_j) + \kappa_i + \kappa_j \neq 0$$

$$k(g_i-g_j) + 2\kappa_j \neq 0$$

$$g_i-g_j \neq 0.$$

These relations exclude two well-known families of commensurabilities:
(a) the trojan commensurabilities

$$a\kappa_i + b\kappa_j; \qquad a + b = 0 \qquad (a,b \in Z)$$

(b) the commensurabilities

$$a\kappa_i + b\kappa_j; \qquad |a + b| = 2 \qquad (a,b \in Z).$$

9. THE GALILEAN CRITICAL TERMS

The fifth equation arising from Krasinsky's method may be decomposed intwo two 4-vector equations which are complex conjugates. Let one of them be considered.

$$DB_{21} + X_0B_{21} + \frac{7}{8}A^IB_{11} - \frac{5}{8}A^IB_{12} + M_{11}D^{-1}B_{11} + M_{12}D^{-1}B_{12} +$$

$$+ N_{11}D^{-2}B_{11} - N_{11}D^{-2}B_{12} + V_{21} - C_{11}b_{11} - C_{12}b_{12} = b_{21}$$

The solution which is adopted is

$$b_{21} = 0$$

$$B_{21(j)} = \sum_s \delta_s(B_{21(j)})$$

(sum of parts arising from different terms of the non-homogeneous side of the equation). The parts which arise respectively from $(\frac{7}{8}A^iB_{11})_{(j)}$ and $(M_{11}D^{-1}B_{11})_{(j)}$ have the form

$$\delta_1(B_{21(j)}) = \sum_{i\neq j}\ \sum_{l\neq j}\ \sum_{\substack{-\infty \\ (\neq 0)}}^{+\infty}\ \sum_{-\infty}^{+\infty} \frac{A^{kk'}_{jil}\xi^k_{ij}\xi^{k'}_{lj}}{\kappa_j + k(g_i-g_j) + k'(g_l-g_j)}$$

and

$$\delta_2(B_{21(j)}) = \sum_{i\neq j}\ \sum_{l\neq j}\ \sum_{-\infty}^{+\infty}\ \sum_{+\infty}^{+\infty} \frac{B^{kk'}_{jil}\xi^k_{ij}\xi^{k'}_{li}}{\kappa_j + k(g_i-g_j) + k'(g_l-g_i)}$$

They introduce as new conditions of validity

$$\kappa_j + k(g_i-g_j) + k'(g_l-g_j) \neq 0$$

$$\kappa_j + k(g_i-g_j) + k'(g_l-g_i) \neq 0$$

$$\delta_9(B_{21(j)}) = (D + \kappa_j)^{-1} \{\frac{1}{\kappa_j} \mathscr{R}_{j0} + \frac{1}{4\kappa_j} \chi_{j0}\}.$$

10. SOLUTION OF THE EQUATIONS FOR P_j and Q_j

The 'averaged' equation is

$$D\hat{P}_j + \kappa_j \hat{P}_j + \sum_{i=1}^{4} e_{11(ij)} \hat{P}_i + b_{1j} = 0,$$

this equation is not actually averaged since

$$e_{11(ij)} = \tilde{e}_{ij} \xi_{ji}.$$

If the variables $\hat{p}_j = \hat{P}_j \sigma_j \zeta^{\kappa_j}$ are introduced, we have the linear differential equation with constant coefficients

$$D\hat{p}_j + \sum_i e_{ij} \hat{p}_i + \sigma_j b_{ij} \xi^{\kappa_j} = 0$$

whose general solution is

$$\hat{p}_j = \sum_{i=1}^{4} A_{ji} \zeta^{-\beta_i};$$

the constants A_{ji} may be known if four among them are supposed as known integration constants. The motions β_i are the roots of the characteristic equation

$$\det(\tilde{e}_{ij} - \beta \delta_{ij}) = 0.$$

These characteristic roots are the motions of the proper perijoves of the satellites in unities of the Eulerian mean motion ν_3.

The particular solution of the complete equations is easily obtained; it gives rise to terms having the same frequencies as those of the satellites. The mean feature of the solution obtained in this way is the Laplace's result after which it is not possible to separate the proper oscillations. The system oscillates as a whole: every satellite shows in its motion the four proper oscillations.

11. THE EQUATIONS FOR K_j AND H_j

In order to obtain the final form of the differential equations for K_j and H_j we must collect the disturbing terms and substitute them in the space equations given in Section 1. It follows

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which are satisfied in the problem of the motion of the Galilean satellites because $m \notin Z$.

The Galilean critical terms are those for which

$$k(g_i - g_j) + k'(g_1 - g_j) = 0$$

$$k(g_i - g_j) + k'(1 - g_i) = 0.$$

They may arise in two different ways: (a) the trojan resonance, which happens in the case i = 1, when $g_i = g_j$, i.e. in the case

$$a\kappa_i + b\kappa_j = 0; \qquad a + b = 0 \qquad (a,b \in Z),$$

(b) the Galilean resonance, which happens in the case $i \neq 1$, when

$$a\kappa_i + b\kappa_j + c\kappa_1 = 0; \qquad a + b + c = 0 \qquad (a,b,c \in Z).$$

In this case it is possible to find a system of rotating axes for which $\kappa_i, \kappa_j, \kappa_1$ reduce to three integers g_i, g_j, g_1 (as it has been made in this theory). Indeed, if $\kappa_i = g_i + m$, since $a+b+c = 0$, we have

$$ag_i + bg_j + ch_1 = 0$$

or

$$a(g_i - g_1) + b(g_j - g_1) = 0;$$

the last equation has always integer solutions. We may still choose the rotating axes so that

$$ag_i - bg_1 = ag_1 - bg_j = 0$$

i.e.

$$g_1^2 = g_i \cdot g_j$$

which gives rise to the following situation

$$g_i = 1, \qquad g_1 = a, \qquad g_j = a^2 \qquad (a \in Z);$$

the Galilean satellites are such that m=0.01448 and a = 2.

The Galilean critical terms do not disturb the integration of the integro-differential equation; they will actually become critical when the geometric equations are considered.

The other parts of the vector equation will introduce terms exactly like the two discussed above. The only part which introduces terms which are different, is that which contains the solar perturbations:

$$DK_j+\kappa_j K_j = \frac{1}{\kappa_j}\sum_{i\neq j}\sum_{-\infty}^{+\infty}\{(V^{(k)}_{1(ji)}+\frac{1}{2}R^{(k)}_{1(ji)})\xi^k_{ij}+\frac{2}{3\kappa_j}\tilde{V}_{1(j)}\}(K_j+H_j)+$$

$$+\frac{1}{\kappa_j}\sum_{i\neq j}\sum_{-\infty}^{+\infty} V^{(k)}_{2(ji)}\xi^k_{ij}(K_i+H_i)+$$

$$+\frac{1}{\kappa_j}\{\tilde{V}_{3(j)}K_p+\tilde{V}_{4(j)}H_p+\tilde{V}_{0(j)}+\mathscr{V}_{jo}\}$$

and its complex conjugate for $-DH_j+\kappa_j H_j$. To these equations we must add the differential equations giving the motion of the pole of Jupiter:

$$DK_p+\kappa_p K_p=\tilde{V}_{0(p)}+\tilde{V}_{1(p)}H_p+\sum_i \tilde{V}_{3(p)}(K_i+H_i)$$

$$DH_p-\kappa_p H_p=-\tilde{V}_{0(p)}-\tilde{V}^*_{1p}K_p-\sum_i \tilde{V}^*_{3(p)}(K_i+H_i).$$

All these equations may be written using vectors and matrices of rang 10

$$DX+\hat{X}'_0K+L'_1K+V'_1+V'_2 = 0,$$

where the new symbols have evident meanings. This vector equation may be averaged by means of Krasinsky's method, by using the transformation

$$K=\hat{K}+B'_1+B'_2+C'_1\hat{K}.$$

The averaged equation will have the form

$$D\hat{K}+\hat{X}'_0\hat{K}+e_1\hat{K}'= 0$$

and may be solved by elementary methods like those used in the preceding section. The main feature of the solution thus obtained is Laplace's result after which it is not possible to separate the proper oscillations in latitude as well as the proper oscillation of the planetary equator. The systems oscillates as a whole and the plane of the motion of each satellite and the plane of Jupiter's equator show the five proper oscillations.

The details of the actual calculations are nothing but similar to those for P_j and Q_j.

12. THE GEOMETRIC EQUATIONS

At least we need to consider the geometric equations

$$D\varepsilon_j = \frac{1}{2}\kappa_j(P_j-Q_j) - \frac{1}{4}\kappa_j(P_j+Q_j)\Gamma_j+\frac{1}{4}\kappa_j(K_j+H_j)(K_j-H_j)$$

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$$D\Gamma_j = -3\kappa_j\varepsilon_j + \frac{1}{2}\kappa_j(P_j+Q_j) - \frac{3}{2}\kappa_j(P_j+Q_j)\varepsilon_j -$$

$$- \frac{1}{4}\kappa_j(P_j-Q_j).\Gamma_j + \frac{1}{4}\kappa_j[(K_j-H_j)^2 - \frac{1}{2}(H_j+H_j)^2] + 2\kappa_j\varepsilon_j^2.$$

Now P_j, Q_j, K_j, H_j are known functions since these variables have been supposed to be integrated in the preceding sections. These functions contain the four circulatory frequencies g_j, the four oscillatory frequencies $\dot{\varpi}_j = \beta_j\nu_3$ and the five oscillatory frequencies $\dot{\Omega}_j = \beta_j'\nu_3$. The geometric equations may be considered separately for each satellite. For this reason in what follows subscripts will be omitted. We have, after substitutions

$$D\varepsilon = S_1 + S_2\Gamma$$

$$D\Gamma = S_4 + (S_5 - 3\kappa)\varepsilon + S_6\Gamma + 2\kappa\varepsilon^2$$

where

$$S_i = \sum_k S_{ik}\zeta^{(k)};$$

$\zeta^{(k)}$ means ζ to the power arising in the k-th term of the series. Two particular cases are of importance: $\zeta^{(k)} = \zeta^0$ and $\zeta^{(k)} = \zeta^G$ where G is the critical Galilean frequency

$$G = g_1 - 3g_2 + 2g_3.$$

All series are first-order and the series S_1 and S_4 contain also second-order parts. The galilean critical frequency may appear only in these second-order parts. The averaging method of Krasinsky is used notwithstanding the fact that the geometric equations are not linear. Such use is possible since the coefficient of the non-linear term is a constant. We consider the transformation

$$\varepsilon = Y_1 + (1 + Y_2)\,\hat{\varepsilon} + Y_3\hat{\Gamma}$$

$$\Gamma = Y_4 + (1 + Y_5)\hat{\Gamma} + Y_6\hat{\varepsilon},$$

where the Y_j are all first-order and Y_1 and Y_4 have also second-order parts. After substitution we have

$$D\hat{\varepsilon} = S_1-DY_1+S_2\hat{\Gamma}+S_2Y_4-\hat{\varepsilon}DY_2-\hat{\Gamma}DY_3-Y_2(S_1-DY_1)-Y_3(S_4-DY_4)$$

$$+ 3\kappa Y_3(Y_1+\hat{\varepsilon})$$

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$$D\hat{\Gamma}=S_4-DY_4+S_5\hat{\varepsilon}=S_6\hat{\Gamma}+S_5Y_1+S_6Y_4+2\kappa\hat{\varepsilon}^2+2\kappa Y_1^2 +$$

$$+4\kappa\hat{\varepsilon}Y_1-\hat{\Gamma}DY_5-\hat{\varepsilon}DY_6-3\kappa\hat{\varepsilon}-Y_5(S_4-DY_4)-Y_6(S_1-DY_1) +$$

$$+3\kappa Y_5(Y_1+\hat{\varepsilon})-3\kappa Y_1+3\kappa Y_2\hat{\varepsilon}-3\kappa Y_3\hat{\Gamma},$$

to reduce these equations to

$$D\hat{\varepsilon}=W_1+W_2\hat{\Gamma}+W_3\hat{\varepsilon}$$

$$D\hat{\Gamma}=W_4-(3\kappa+W_5)\hat{\varepsilon}+W_6\hat{\Gamma}+2\kappa\hat{\varepsilon}^2,$$

where the W_j are free from non-libratory periodic oscillations, it is enough to solve

$$S_{11}-DY_{11}=W_{11}$$

$$S_{12}-DY_{12}+S_2Y_{41}-Y_2W_{11}-Y_3W_{41}=W_{12}$$

$$S_2-DY_3=W_2$$

$$3\kappa Y_3-DY_2=W_3$$

$$S_{41}-DY_{41}-3\kappa Y_{11}=W_{41}$$

$$S_{42}-DY_{42}+S_5Y_{11}+2\kappa Y_{11}^2-Y_5W_{41}-Y_6W_{11}-3\kappa Y_{12}+S_6Y_{41}=W_{42}$$

$$S_5-DY_6+4\kappa Y_{11}+3\kappa Y_5+3\kappa Y_2=-W_5$$

$$S_6-3\kappa Y_3-DY_5=W_6$$

in which the series where the indices were 1 and 4 have been split in first-order (indices 11 and 41) and second-order (indices 12 and 42) parts. The integration is easy. The first equation, for example, leads to

$$W_{11}=\langle S_{11}\rangle$$

$$Y_{11}=D^{-1}(S_{11}-W_{11}),$$

where $\langle S_{11}\rangle$ means the constant term in S_{11}. In the equations for the second-order parts we need to include in the average not only the constant but also the critical terms.

The difficulties in the integration of the averaged equations, in reason of the critical terms, is the same which appear in Laplace's equations for the inequalities of the mean longitudes. As we know, the critical terms lead to libration in the mean longitudes whose period is close to 6 years and whose amplitude is an integration constant to be determined from the observations. This determination is a very difficult problem since the amplitude is very small. In 1907 de Sitter found $0\overset{\circ}{.}158 \pm 0\overset{\circ}{.}033$. In 1928, after a new discussion (de Sitter, 1931) he found $0\overset{\circ}{.}0247 \pm 0\overset{\circ}{.}0075$. In a discussion using only the first satellite he found an amplitude 4 times greater, and using only the second satellite the amplitude remained unchanged but the phase shifted by 100°. These determinations are very uncertain and they did not affect the mean residuals of the observations. At the same epoch Brouwer (1928), from a different set of observations found $0\overset{\circ}{.}0309 \pm 0\overset{\circ}{.}0058$ for the amplitude of the libration. Also, when using separately the first or the second satellite he obtained contradictory results. He concluded that the libration is much too small to be determined from the observations.

This fact justifies omitting the libration and integrating the averaged equations using G=0 identically. The solution in this case may be easily obtained.

(a) $W_{11} = 0$; indeed $W_{11} = \langle S_{11}\rangle = \frac{1}{2}\kappa_j\langle P_j - Q_j\rangle = 0$, since P_j and Q_j are complex conjugates and so $P_j - Q_j$ is a sinus series.

(b) $W_3 = 0$; the solution Y_3 of the equation $S_2 - DY_3 = W_2$ is an odd function because the parity of the series is changed by the operator D^{-1}. The result then follows as in (a).

(c) $W_{12} = 0$; the proof of this statement is longer. By Krasinsky's method we have $W_{12} = \langle S_{12} + S_2Y_{41} - Y_2W_{11} - Y_3W_{41}\rangle$; the analysis of the parities must be made with details but it is algebraically elementary. W_{12} will have the form $A(\zeta^G - \zeta^{-G})$ and its average is zero under the hypotheses previously adopted.

(d) $W_6 = 0$; this result follows as in (b).

Thus, the remaining system is

$$D\hat{\varepsilon} = W_2\hat{\Gamma}$$

$$D\hat{\Gamma} = W_4 - (3\kappa - W_5)\hat{\varepsilon} + 2\kappa\hat{\varepsilon}^2$$

and we have

$$D^2\hat{\varepsilon} = -\nu_3^2\,\frac{d^2\hat{\varepsilon}}{dt^2} = W_2W_4 - (3\kappa - W_5)W_2\hat{\varepsilon} + 2\kappa W_2\hat{\varepsilon}^2.$$

This equation is the equation of a non-linear oscillator. Its stationary solution is

$$\hat{\varepsilon} = \hat{\varepsilon}_E = \frac{W_4}{3\kappa}\left[1 + \frac{W_5}{3\kappa} + \frac{2W_4}{9\kappa}\right].$$

If this stationary solution is substituted in the equation for $\hat{\Gamma}$ it follows that $D\hat{\Gamma}$ is equal to a non-zero constant: $-2W_4^2/9$. Then $\hat{\Gamma}(t)$ and $\phi(t)$, will have one term which is linear with respect to time. The existence of such linear term contradicts the working hypothesis that the frequencies n_i are exactly the observed mean motions. Thus, this linear term may not exist. We impose $W_4=0$, in order to have the solutions $\hat{\Gamma}=0$, $\hat{\varepsilon}=0$. The equations $W_4=0$ determine the normalization parameters a_j, mean distances from the satellites to the planet.

This technique may be compared to the technique recently suggested by Eminhizer et al. (1976). Indeed the variational equations built with respect to a given set of circular orbits (intermediate orbits) allow us to introduce a fixed set of frequencies; these equations are a 'relocated' version of the equations of motion. Note that the 'relocation' in this case is restricted only to the mean motions. The Eminhizer et al.'s 'forward scheme' would consist in the improvement of the introduced frequencies while the mean distances a_j were kept fixed (frequencies normalization). The technique adopted in this theory is similar to Eminizer et al.'s 'backward scheme'; the frequencies are kept fixed and the value of the amplitudes a_j is improved (amplitudes renormalization). In both cases the non-zero solutions of the averaged geometric equations correspond to orbits in the neighbourhood of one central orbit. This theory allows us to obtain solutions without secular terms. It is hoped that the rate of formal convergence of the solution is better than in solutions founded on the osculating frequencies

CONCLUSION

The theory outlined in a preceding paper has been developed up to the second order. The integration has been performed and the possibility of doing it completely has been demonstrated. The techniques used are able to show the main known features of the motion. They may also show new features especially when the perturbations arising from the Sun are completely considered. New features may also arise when the coupling of the oscillations in longitude and in latitude are considered - indeed, on several occasions very weak coupling terms arose and have been neglected since they are of higher orders.

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