13
D-branes and geometry I

In previous chapters we became increasingly aware of the intimate relation of D-branes to both spacetime geometry and to gauge theory, via the collective description of their low energy dynamics. In fact, we have already seen that we can reinterpret many aspects of the spacetime geometry in which the brane moves by reference to the vevs of scalars in the world-volume gauge theory. In this chapter we explore this in much more detail, by using D-branes to probe a number of string theory backgrounds, and find that they allow us to get a new handle on quite detailed properties of the geometry. In addition, we will find that D-branes can take on the properties of a variety of familiar objects, such as monopoles and instantons, depending upon the situation.

13.1 D-branes as probes of ALE spaces

One of the beautiful results which we uncovered soon after constructing the type II strings was that we can 'blow-up' the 16 fixed points of the $T^4/\mathbb{Z}_2$ 'orbifold compactification' to recover string propagation on the smooth hyper-Kähler manifold K3. (We had a lot of fun with this in section 7.6.) Strictly speaking, we only recovered the algebraic data of the K3 manifold this way, and it seemed plausible that the full metric geometry of the space is recovered, but how can we see this directly?

We can recover the metric data by using a brane as a short distance 'probe' of the geometry. This is a powerful technique, which has many useful applications as we shall see in numerous examples as we proceed.

13.1.1 Basic setup and a quiver gauge theory

Let us focus on a single orbifold fixed point, and the type IIB theory. The full string theory is propagating on $\mathbb{R}^6 \times (\mathbb{R}^4/\mathbb{Z}_2)$, which arises

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from imposing a symmetry under the reflection $\mathbf{R}: (x^6, x^7, x^8, x^9) \to (-x^6, -x^7, -x^8, -x^9)$, which we used before in section 7.6. Now we can place a D1-brane in this plane at $x^2, \dots, x^9 = 0$. Let's draw a little table to help keep track of where everything is.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D1	—	—	•	•	•	•	•	•	•	•
ALE	—	—	—	—	—	—	•	•	•	•

(We have represented the $\mathbb{R}^4/\mathbb{Z}_2$ (ALE) space as a sort of five dimensional extended object in the table, since it only has structure in the directions x^6, x^7, x^8, x^9.)

The D1-brane can quite trivially sit at the origin and respect the symmetry $\mathbf{R}$, but if it moves off the fixed point, it will break the $\mathbb{Z}_2$ symmetry. In order for it to be able to move off the fixed point there also needs to be an image brane moving to the mirror image position. We therefore need two Chan–Paton indices: one for the D1-brane and the other for its $\mathbb{Z}_2$ image. So (to begin with) the gauge group carried by our D1-brane system living at the origin appears to be $U(2)$, but this will be modified by the following considerations. Since $\mathbf{R}$ exchanges the D1-brane with its image, it can be chosen to act on an open string state as the exchange $\gamma = \sigma^1$, and we shall use the Pauli matrices

$$\sigma^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13.1}$$

So we can write the representation of the action of $\mathbf{R}$ as:

$$\begin{aligned} \mathbf{R}|\psi, ij\rangle &= \gamma_{ii'}\, |\mathbf{R}\psi, i'j'\rangle \gamma^{-1}_{j'j}, \quad \text{that is,} \\ \mathbf{R}|\psi, ij\rangle &= \sigma_{ii'} |\mathbf{R}\psi, i'j'\rangle \sigma^{-1}_{j'j}. \end{aligned} \tag{13.2}$$

So it acts on the oscillators in the usual way but also switches the Chan–Paton factors for the brane and its image. The idea[132] is that we must choose an action of the string theory orbifold symmetry on the Chan–Paton factors when there are branes present and make sure that the string theory is consistent in that sector too. Note that the action on the Chan–Paton factors is again chosen to respect the manner in which they appear in amplitudes, just as in section 2.5.

We can therefore compute what happens. In the NS sector, the massless $\mathbf{R}$-invariant states are, in terms of vertex operators:

$$\begin{aligned} &\partial_{\rm t} X^\mu \sigma^{0,1}, \qquad \mu = 0, 1 \\ &\partial_{\rm n} X^i \sigma^{0,1}, \qquad i = 2, 3, 4, 5 \\ &\partial_{\rm n} X^m \sigma^{2,3}, \qquad m = 6, 7, 8, 9. \end{aligned} \tag{13.3}$$

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The first row is the vertex operator describing a gauge field with $U(1) \times U(1)$ as the gauge symmetry. The next row constitutes four scalars in the adjoint of the gauge group, parametrising the position of the D1-brane within the six-plane $\mathbb{R}^6$, and the last row is four scalars in the 'bifundamental' charges $(\pm 1, \mp 1)$ of the gauge group the transverse position on x^6, x^7, x^8, x^9. Let us denote the corresponding D-string fields A^μ, X^i, X^m, all 2×2 matrices. We may draw a 'quiver diagram'[188] displaying this gauge and matter content (see figure 13.1).

Such diagrams have in general an integer m inside each node, representing a factor $U(m)$ in the gauge group. An arrowed edge of the diagram represents a hypermultiplet transforming as the fundamental (for the sharp end) and antifundamental (for the blunt end) of the two gauge groups corresponding to the connected nodes. The diagram is simply a decorated version of the extended Dynkin diagram associated to A_1. This will make even more sense shortly, since there is geometric meaning to this. Finally, note that one of the $U(1)$s, (the σ_0 one) is trivial: nothing transforms under it, and it simply represents the overall centre of mass of the brane system.

The bosonic action for the fields is the $D = 10$ $U(2)$ Yang–Mills action, dimensionally reduced and $\mathbb{Z}_2$-projected (which breaks the gauge symmetry to $U(1) \times U(1)$). This dimensional reduction is easy to do. There are kinetic terms:

$$T = -\frac{1}{4g_{\mathrm{YM}}^2}\left(F^{\mu\nu}F_{\mu\nu} + \sum_i \mathcal{D}_\mu X^i \mathcal{D}^\mu X^i + \sum_m \mathcal{D}_\mu X^m \mathcal{D}^\mu X^m\right), \quad (13.4)$$

and potential terms:

$$U = -\frac{1}{4g_{\mathrm{YM}}^2}\left(2\sum_{i,m} \mathrm{Tr}\,[X^i, X^m]^2 + \sum_{m,n} \mathrm{Tr}\,[X^m, X^n]^2\right), \quad (13.5)$$

where by using (8.13), we have $g_{\mathrm{YM}}^2 = (2\pi)^{-1}\alpha'^{-1/2}g_{\mathrm{s}}$. (Another potentially non-trivial term disappears since the gauge group is Abelian.)

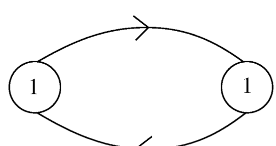

Fig. 13.1. A diagram showing the content of the probe gauge theory. The nodes give information about the gauge groups, while the links give the amount and charges of the mattter hypermultiplets.

The resulting theory has $\mathcal{N} = (4, 4)$ supersymmetry in $D = 2$, which has an $SU(2)$ R-symmetry, and can be thought of as the $SU(2)_\mathrm{R}$ left over from parametrising the $\mathbb{Z}_2$ as an action in the $SU(2)_\mathrm{L}$ of the natural $SO(4)$. See insert 7.4.

13.1.2 The moduli space of vacua

The important thing to realise is that there are large *families* of vacua (here, $U = 0$) of the theory. The space of such vacua is called the 'moduli space' of vacua, and they shall have an interesting interpretation. The moduli space has two branches.

On one, the 'Coulomb Branch', $X^m = 0$ and $X^i = u^i\sigma^0 + v^i\sigma^1$. This corresponds to two D-branes moving independently in the $\mathbb{R}^6$, with positions $u^i \pm v^i$, but staying at the origin of the $\mathbb{R}^4$. The gauge symmetry is unbroken, giving independent $U(1)$s on each D-brane.

On the other, the 'Higgs Branch', X^m is non-zero and $X^i = u^i\sigma^0$. The σ^1 gauge invariance is broken and so we can make the gauge choice $X^m = w^m\sigma^3$. This corresponds to the D1-brane moving off the fixed plane, the string and its image being at $(u^i, \pm w^m)$. We see that this branch has the geometry of the $\mathbb{R}^6 \times \mathbb{R}^4/\mathbb{Z}_2$ which we built in.

Now let us turn on twisted-sector fields which we uncovered in section 7.6, where we learned that they give the blow-up of the geometry. They will appear as parameters in our D-brane gauge theory. Define complex q^m by $X^m = \sigma^3\mathrm{Re}(q^m) + \sigma^2\mathrm{Im}(q^m)$, and define two doublets of the $SU(2)_\mathrm{R}$:

$$\Phi_0 = \begin{pmatrix} q^6 + iq^7 \\ q^8 + iq^9 \end{pmatrix}, \qquad \Phi_1 = \begin{pmatrix} \bar{q}^6 + i\bar{q}^7 \\ \bar{q}^8 + i\bar{q}^9 \end{pmatrix}. \tag{13.6}$$

These have charges ± 1 respectively under the σ^1 $U(1)$. The three NS–NS moduli can be written as a vector $\boldsymbol{\xi}$ of the $SU(2)_\mathrm{R}$, and the potential is proportional to

$$(\boldsymbol{\xi} - \boldsymbol{\mu})^2 \equiv (\Phi_0^\dagger \boldsymbol{\tau} \Phi_0 - \Phi_1^\dagger \boldsymbol{\tau} \Phi_1 + \boldsymbol{\xi})^2, \tag{13.7}$$

where the Pauli matrices are now denoted τ^I to emphasise that they act in a different space. They are assembled into a vector $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$. (The vector $\boldsymbol{\mu}$ is called a 'moment map' in the mathematical understanding of this construction, which we shall discuss later.) Its form is determined by supersymmetry, and it should be checked that it reduces to the second term of the earlier potential (13.5) when $\boldsymbol{\xi} = 0$. The entire potential arises in supersymmetric constructions using superfields as a 'D-term', and its vanishing to find the vacua is the 'D-flatness condition'. The vector $\boldsymbol{\xi}$ enters as a 'Fayet–Iliopoulos' term[222] in the D-term, and is allowed

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whenever there is an Abelian factor in the gauge group. The $SU(2)_{\rm R}$ symmetry requires that the FI term and the entire D-term come as a vector. These are all of the facts we will need about such supersymmetry techniques. Unfortunately, a fuller discussion of these matters will take us too far afield, and we refer to reader to the literature[223].

Notice that equation (13.7) implies a coupling between the open string sector and the twisted sector fields. This can be checked directly by a disc computation, where a twist field is in the interior of the disc and the open string fields are on the edge[184].

For $\boldsymbol{\xi} \neq 0$ the orbifold point is blown up. The moduli space of the gauge theory is simply the set of possible locations of the probe i.e., the blown up ALE space. (Note that the branch of the moduli space with $v^i \neq 0$ is no longer present.)

Let us count parameters and constants. The X^m contain eight scalar fields. Three of them are removed by the '$\boldsymbol{\xi}$-flatness' condition that the potential vanishes, and a fourth is a gauge degree of freedom, leaving the expected four moduli. In terms of supermultiplets, the system has the equivalent of $D = 6$ $N = 1$ supersymmetry. The D-string has two hypermultiplets and two vector multiplets, which are Higgsed down to one hypermultiplet and one vector multiplet.

13.1.3 ALE space as metric on moduli space

The idea[184] is that the metric on this moduli space, as seen in the kinetic term for the D-string fields, should be the smoothed ALE metric. Given the fact that we have eight supercharges, it should be a hyper-Kähler manifold[185], and the ALE space has this property. Let us explore this[187].

Three coordinates on our moduli space are conveniently defined as (there are dimensionful constants missing from this normalisation which we shall ignore for now):

$$\mathbf{y} = \Phi_0^\dagger \boldsymbol{\tau} \Phi_0. \tag{13.8}$$

The fourth coordinate, z, can be defined

$$z = 2\arg(\Phi_{0,1}\Phi_{1,1}). \tag{13.9}$$

The $\boldsymbol{\xi}$-flatness condition implies that

$$\Phi_1^\dagger \boldsymbol{\tau} \Phi_1 = \mathbf{y} + \boldsymbol{\xi}, \tag{13.10}$$

and Φ_0 and Φ_1 are determined in terms of $\mathbf{y}$ and z, up to a gauge choice.

The original metric on the space of hypermultiplet vevs is just the flat metric $ds^2 = d\Phi_0^\dagger d\Phi_0 + d\Phi_1^\dagger d\Phi_1$. We must project this onto the space orthogonal to the $U(1)$ gauge transformation. This is performed (for example) by coupling the Φ_0, Φ_1 for two dimensional gauge fields according to their charges, and integrating out the gauge field. The result is

$$ds^2 = d\Phi_0^\dagger d\Phi_0 + d\Phi_1^\dagger d\Phi_1 - \frac{(\omega_0 + \omega_1)^2}{4(\Phi_0^\dagger \Phi_0 + \Phi_1^\dagger \Phi_1)}, \tag{13.11}$$

with

$$\omega_i = i(\Phi_i^\dagger d\Phi_i - d\Phi_i^\dagger \Phi_i). \tag{13.12}$$

We can express the metric in terms of $\mathbf{y}$ and t using the identity:

$$(\alpha^\dagger \tau^a \beta)(\gamma^\dagger \tau^a \delta) = 2(\alpha^\dagger \delta)(\gamma^\dagger \beta) - (\alpha^\dagger \beta)(\gamma^\dagger \delta)$$

for $SU(2)$ arbitrary doublets $\alpha, \beta, \gamma, \delta$. This gives:

$$\begin{aligned} &\Phi_0^\dagger \Phi_0 = |\mathbf{y}|, \qquad \Phi_1^\dagger \Phi_1 = |\mathbf{y} + \boldsymbol{\xi}|, \\ &d\mathbf{y} \cdot d\mathbf{y} \;=\; |\mathbf{y}| d\Phi_0^\dagger d\Phi_0 - \omega_0^2 \;=\; |\mathbf{y} + \boldsymbol{\xi}| d\Phi_1^\dagger d\Phi_1 - \omega_1^2, \end{aligned} \tag{13.13}$$

and we find that our metric can be written as the $N = 2$ case of the Gibbons–Hawking metric:

$$\begin{aligned} &ds^2 = V^{-1}(dz - \mathbf{A} \cdot d\mathbf{y})^2 + V d\mathbf{y} \cdot d\mathbf{y} \\ &V = \sum_{i=0}^{N-1} \frac{1}{|\mathbf{y} - \mathbf{y}_i|}, \qquad \boldsymbol{\nabla} V = \boldsymbol{\nabla} \times \mathbf{A}, \end{aligned} \tag{13.14}$$

which is in fact a '*hyper-Kähler*' metric, as we shall see.

Up to an overall normalisation (which we will fix later), we have $\mathbf{y}_0 = 0$, $\mathbf{y}_1 = \boldsymbol{\xi}$, and the vector potential is

$$\mathbf{A}(\mathbf{y}) \cdot d\mathbf{y} = |\mathbf{y}|^{-1} \omega_0 + |\mathbf{y} + \boldsymbol{\xi}|^{-1} \omega_1 + dz, \tag{13.15}$$

and the field strength is readily obtained by taking the exterior derivative and using the identity

$$\epsilon_{abc}(\alpha^\dagger \tau^b \beta)(\gamma^\dagger \tau^c \delta) = i(\alpha^\dagger \tau^a \delta)(\gamma^\dagger \beta) - i(\alpha^\dagger \delta)(\gamma^\dagger \tau^a \beta).$$

Under a change of variables[92], this metric (for $N = 2$) becomes the Eguchi–Hanson metric, (7.53) which we first identified as the blow-up of the orbifold point. The three parameters in the vector $\mathbf{y}_1 = \boldsymbol{\xi}$ are the NS–NS fields representing the size and orientation of the blown up $\mathbb{CP}^1$.

It is easy to carry out the generalisation to the full A_{N-1} series, and get the metric (13.14) on the moduli space for a D1-brane probing a $\mathbb{Z}_N$ orbifold. The gauge theory is just the obvious generalisation derived

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from the extended Dynkin diagram: $U(1)^N$, with $N+1$ bifundamental hypermultiplets with charges $(1,-1)$ under the neighbouring $U(1)$s. (See figure 13.2.)

There will be $3(N-1)$ NS–NS moduli which will become the $N-1$ differences $\mathbf{y}_i - \mathbf{y}_0$ in the resulting Gibbons–Hawking metric (13.14). Geometrically, these correspond to the size and orientation of $N-1$ separate $\mathbb{CP}^1$'s which can be blown up. In fact, we see that the there is another meaning to be ascribed to the Dynkin diagram: each node

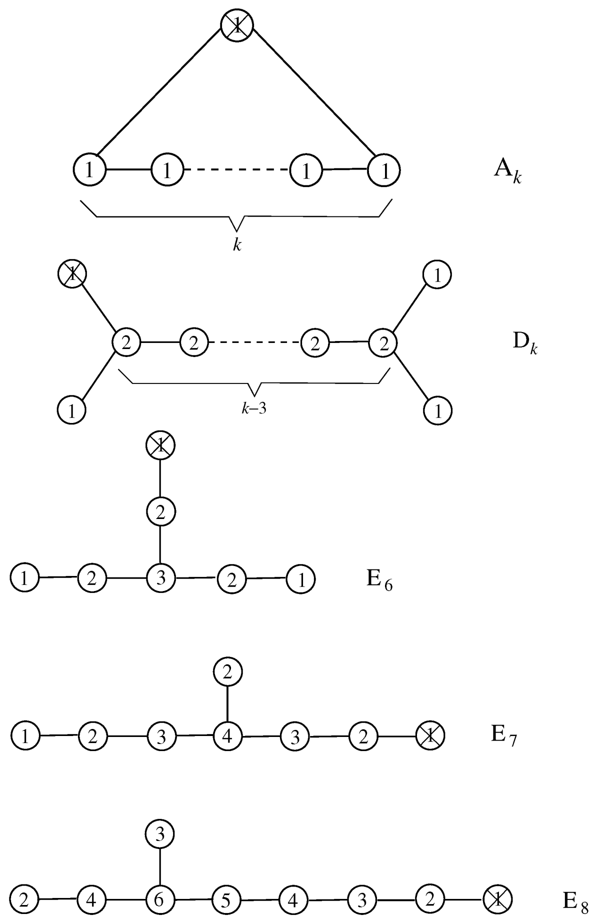

Fig. 13.2. The extended Dynkin diagrams for the A–D–E series. As quiver diagrams, they give the gauge and matter content for the probe gauge theories which compute the resolved geometry of an ALE space. At the same time they also denote the actual underlying geometry of the ALE space, as each node denotes a $\mathbb{CP}^1$, with the connecting edge representing a non-zero intersection.

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(except the trivial one) represents a $\mathbb{CP}^1$ in the spacetime geometry that the probe sees on the Higgs branch. We shall expand upon this intriguing picture in section 13.2.

13.1.4 D-branes and the hyper-Kähler quotient

This entire construction which we have just described is a '*hyper-Kähler quotient*', a powerful technique[189] for describing hyper-Kähler metrics of various types, and which has been used to prove the existence of the full family of ALE metrics[190]. Hyper-Kähler spaces are complex $4k$-dimensional manifolds ($k \in \mathbb{Z}$) which admit not just one complex structure, but three, and they transform under an $SU(2)$ symmetry which, for us, will often become and $SU(2)$ R-symmetry of some system with eight supercharges. In fact, the complex structure becomes a quaternionic structure for such manifolds. Flat $\mathbb{R}^4$, presented in the manner done in insert 7.4, is a simple example, and the $SU(2)$ is either of the $SU(2)$ isometries manifest there. The ALE spaces are non-trivial examples, as is the K3 manifold. Two other important four dimensional examples we shall encounter later are the Taub-NUT space and the Atiyah-Hitchin space. Multi-instanton and BPS multi-monopole moduli spaces[218] are higher dimensional cases which we shall also meet.

The hyper-Kähler quotient technique is a powerful mathematical method for showing the existence of (and sometimes exhibiting explicitly) such spaces. It is remarkable that using D-branes we can encounter this construction, physically realised in terms of supersymmetric gauge theory variables-precisely the same variables which appear in the mathematical description of the construction. We shall see this connection arising a number of other times in these pages[191]. Just as we got a $U(1)^2$ gauge theory from the A_1 example, and $U(1)^N$ for the A_{N-1}, the rest of the A–D–E series gives a family of associated gauge theories on the brane as well. These families, and the correspondence to the A–D–E classification arises as follows[81] (see figure 13.2). We start with D-branes on $\mathbb{R}^4/\Gamma$, where Γ is any discrete subgroup of $SU(2)$ (the cover of the $SO(3)$ which acts as rotations at fixed radii). It turns out that the Γ are classified in an 'A–D–E classification', as shown by McKay[87]. The $\mathbb{Z}_N$ are the A_{N-1} series. For the D_N and $E_{6,7,8}$ series, we have the binary dihedral ($\mathcal{D}_{N-2}$), tetrahedral ($\mathcal{T}$), octahedral ($\mathcal{O}$) and icosahedral ($\mathcal{I}$) groups. Let us list them.

• The A_k series ($k{\geq}1$). This is the set of cyclic groups of order $k{+}1$, denoted $\mathbb{Z}_{k+1}$. Their action on the z^i is generated by

$$g = \begin{pmatrix} e^{\frac{2i\pi}{k+1}} & 0 \\ 0 & e^{-\frac{2i\pi}{k+1}} \end{pmatrix}. \tag{13.16}$$

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• The D_k series ($k\geq 4$). This is the binary extension of the dihedral group, of order $4(k-2)$, denoted $\mathcal{D}_{k-2}$. Their action on the z^i is generated by

$$\mathcal{A}=\begin{pmatrix} e^{\frac{i\pi}{k-2}} & 0 \\ 0 & e^{-\frac{i\pi}{k-2}} \end{pmatrix} \quad \text{and} \quad \mathcal{B}=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{13.17}$$

In this representation the central element is $\mathcal{Z}{=}-1({=}\mathcal{A}^2{=}\mathcal{B}^2{=}(\mathcal{AB})^2)$. Note that the generators $\mathcal{A}$ form a cyclic subgroup $\mathbb{Z}_{2k-4}$.

• The $E_{6,7,8}$ series. These are the binary tetrahedral ($\mathcal{T}$), octahedral ($\mathcal{O}$) and icosahedral ($\mathcal{I}$) groups of order 24, 48 and 120, respectively.

The group $\mathcal{T}$ is generated by taking the elements of $\mathcal{D}_2$ and combining them with

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix} \tag{13.18}$$

where ε is an eighth root of unity.

The group $\mathcal{O}$ is generated by taking the elements of $\mathcal{T}$ and combining them with

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix}. \tag{13.19}$$

Finally $\mathcal{I}$ is generated by

$$-\begin{pmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{pmatrix} \quad \text{and} \quad \frac{1}{\eta^2-\eta^3}\begin{pmatrix} \eta+\eta^4 & 1 \\ 1 & -\eta-\eta^4 \end{pmatrix}, \tag{13.20}$$

where η is a fifth root of unity.

Given the action of these groups, in order to have the D-branes form a faithful representation on the covering space of the quotient, we need to start with a number equal to the order $|\Gamma|$ of the discrete group. This was two previously, and we started with $U(2)$. So we now start with a gauge group $U(|\Gamma|)$, and then project, as before.

After projecting $U(|\Gamma|)$, the gauge group turns out to be[191]:

$$F=\prod_i U(n_i),$$

where i labels the irreducible representations R_i of dimension n_i. Pictorially (see figure 13.2), the gauge group associated with a D-string on a ALE singularity is simply a product of unitary groups associated to the extended A–D–E Dynkin diagram, with a unitary group coming from each vertex[190]. In the Dynkin diagrams, each vertex represents an irreducible representation of Γ. The integer in the vertex denotes its dimension. The special vertex with the '$\times$' sign is the trivial representation, the one dimensional conjugacy class containing only the identity. The specific

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connectivity of each graph encodes the information about the following decomposition:

$$Q \otimes R_i = \bigoplus_j a_{ij} R_j, \tag{13.21}$$

where R_i is the ith irreducible representation and Q is the defining two dimensional representation. Here, the a_{ij} are the elements of the adjacency matrix A of the simply laced extended Dynkin diagrams.

Turning to the hypermultiplets, as stated before, we trivially have $\dim(F)$ hypermultiplets transforming in the adjoint of F. These come from the x^2, x^3, x^4, x^5 sector. They are simply the internal components of the six dimensional vectors after dimensional reduction.

More interestingly, we have hypermultiplets coming from the x^6, x^7, x^8, x^9 sector. These hypermultiplets transform in the fundamentals of the unitary groups, according to the representations

$$\bigoplus_i a_{ij}(\mathbf{n}_i, \bar{\mathbf{n}}_j). \tag{13.22}$$

Pictorially, the hypermultiplets are simply the links of the extended Dynkin diagrams. These hypermultiplets together with the D–flatness conditions, etc., are precisely the variables and algebraic condition which appear in Kronheimer's constructive proof of the existence of the smooth ALE metrics[190, 191]. The hyper-Kähler quotient is a more general method for constructing manifolds, and this is a well-known example. Another is the construction of moduli spaces of instantons, and we shall see that D-branes capture that rather explicitly in chapter 15.

For example, the simplest model in the D-series is $\mathcal{D}_4$, which would require eight D1-branes on the covering space. The final probe gauge theory after projecting is $F = U(2)\times U(1)^4$, with four copies of a hypermultipet in the $(\mathbf{2}, \mathbf{1})$ of this group.

Unfortunately, it is a difficult and unsolved problem to obtain explicit metrics for the resolved spaces in the D and E cases. This is in a certain sense closely related to the problem of finding an explicit metric on K3, a long-standing goal which the D-brane technique described here implicitly gives a recipe to tackle. We leave it as an exercise to the reader to apply these methods, and suggest that they publish the result if successful.

13.2 Fractional D-branes and wrapped D-branes

13.2.1 Fractional branes

Let us pause to consider the following. In the previous section, we noted that in order for the probe brane to move off the fixed point, we needed to

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make sure that there were enough copies of it (on the covering space) to furnish a representation of the discrete symmetry Γ that we were going to orbifold by. After the orbifold, we saw that the Higgs branch corresponds to a *single* D-brane moving off the fixed point to non-zero position in x^6, x^7, x^8, x^9. It is made up of the $|\Gamma|$ D-branes we started with on the cover, which are now images of each other under Γ. We can blow up the fixed point to a smooth surface by setting the three NS–NS fields $\boldsymbol{\xi}$ non-zero.

When $\boldsymbol{\xi} = 0$, there is a Coulomb branch. There, the brane is at the fixed point $x^6, x^7, x^8, x^9 = 0$. The $|\Gamma|$ D-branes are free to move apart, independently, as they are no longer constrained by Γ projection. So in fact, we have (as many as *) $|\Gamma|$ independent branes, which therefore have the interpretation as a fraction of the full brane. None of these individual fractional branes can move off. They have charges under the twisted sector R–R fields. Twisted sector strings have no zero mode, as we have seen, and so cannot propagate.

For an arbitrary number of these fractional branes (and there is no reason not to consider any number that we want) a full $|\Gamma|$ of them must come together to form a closed orbit of Γ, in order for them to move off onto the Higgs branch as one single brane. This fits with the pattern of hypermultiplets and subsequent Higgs-ing which can take place. There simply are not the hypermultiplets in the model corresponding to the movement of an individual fractional brane off the fixed point, and so they are 'frozen' there, while they can move within it[182], in the x^2, x^3, x^4, x^5 directions.

13.2.2 Wrapped branes

Notice that when the ALE space is blown-up, we don't see the fractional branes. The fancy language often used at this point is that the Coulomb branch is 'lifted', which is to say it is no longer a branch of degenerate vacua whose existence is protected by supersymmetry. While it is possible to blow-up the point with the separated fractional branes, it is not a supersymmetric operation. We shall see why presently. First, let us set up the geometry of this description.

As we have already mentioned, each node (except for the extended one) in a Dynkin diagram corresponds to a $\mathbb{CP}^1$ which can be blown-up in the smooth geometry. This is in fact a cycle on which a D3-brane can be wrapped in order to make a D1-brane on $\mathbb{R}^6$. For the A_{N-1}-series, where

* In the D and E cases, some of the branes are in clumps of size n (according to the nodes in figure 13.2) and carry non-Abelian $U(n)$.

things are simple, there are $N-1$ such cycles, giving that many different species of D1-brane. This matches with the picture of the previous section, and extends to the whole A–D–E case, since Γ is the number of $\mathbb{CP}^1$s.

Where exactly is this $\mathbb{CP}^1$ in the metric (13.14)? Notice that the 4π periodic variable z, while actually a circle, has a radius that depends upon the prefactor V^{-1}, which varies with $\mathbf{y}$ in a way that is set by the parameters ('centres') $\mathbf{y}_i$. When $\mathbf{y} = \mathbf{y}_i$, the z-circle shrinks to zero size. There is a $\mathbb{CP}^1$ between successive $\mathbf{y}_i$s, which is the minimal surface made up of the locus of z-circles which start out at zero size, grow to some maximum value, and then shrink again to zero size, where a $\mathbb{CP}^1$ then begins again as the neighbouring cycle, having intersected with the previous one in a point. The straight line connecting this will give the smallest cycle, and so the area is $4\pi|\mathbf{y}_i - \mathbf{y}_j|$ for the $\mathbb{CP}^1$ connecting centres $\mathbf{y}_{i,j}$. See figure 13.3. This is just like the case of wrapping a closed string on a circle, as we saw in chapter 4. Winding number is conserved. We saw that even if the circle shrinks away to zero size, the string cannot be pulled off. We worked in T-dual variables and saw that the winding survives as a conserved momentum. Similarly, a closed brane wrapped on a cycle is stuck there, even if the cycle shrinks away. If we don't use some sort of dual description using a large cycle, we need to find a remnant of the wrapped brane after the cycle has shrunk away completely.

Perhaps this is responsible for the fractional brane description. Let us get it to work for a single cycle[187, 201] (crucially, we need to get rid of the

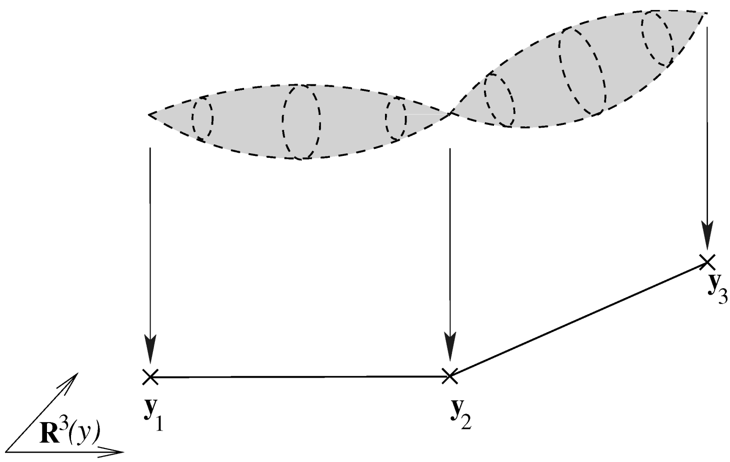

Fig. 13.3. The circles fibred above the $\mathbb{R}^3$ in which lies the centres of the ALE space metric. The collapsing of the circles above the centres results in a network of $\mathbb{CP}^1$ cycles. Their possible intersections are isomorphic to the A–D–E Dynkin diagrams.

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total D3-brane charge), and the entire A–D–E series of ALE spaces will follow from what we've already said.

Imagine[202] a D3-brane with some non-zero amount of $B+2\pi\alpha' F$ on its world-volume. Recall that this corresponds to some D1-brane dissolved into the worldvolume. We deduced this from T-duality in sections 5.2.1 and 9.1. (We did it with pure F, but we can always gauge in some B.) Since we need a total D3-brane charge of zero in our final solution, let us also consider a D3-brane with opposite charge, and with some non-zero $B+2\pi\alpha'\widetilde{F}$ on its world-volume. We write $\widetilde{F}$ to distinguish it from the F on the other brane's worldvolume, but the Bs are the same, since this is a spacetime background field. So we have a worldvolume interaction:

$$\mu_3 \int C^{(2)} \wedge \left\{(B+2\pi\alpha' F) - (B+2\pi\alpha'\widetilde{F})\right\}, \tag{13.23}$$

where we are keeping the terms separate for clarity. Our net D3-brane charge is zero. Now let us choose $2\pi\alpha'(\int_\Sigma F - \widetilde{F}) = \mu_1/\mu_3$, and $\Phi_B \equiv (\mu_3/\mu_1)\int_\Sigma B = 1/2$ for some two dimensional spatial subspace Σ of the three-volume. (Note that $\Phi_B \sim \Phi_B + 1$.) This gives a net D1-brane charge of $1/2 + 1/2 = 1$. The two halves shall be our fractional branes. Right now, they are totally delocalised in the world-volume of the D3-anti D3 system. We can make the D1s more localised by identifying Σ (the parts of the three-volume where B and F are non-zero) with the $\mathbb{CP}^1$ of the ALE space. The smaller the $\mathbb{CP}^1$ is, the more localised the D1s are. In the limit where it shrinks away we have the orbifold fixed point geometry. (Note that we still have $\Phi_B = 1/2$ on the shrunken cycle. Happily, this is just the value needed to be present for a sensible conformal field theory description of the orbifold sector[89], described for example in section 7.6).

Once the D1s are completely localised in x^6, x^7, x^8, x^9 from the shrinking away of the $\mathbb{CP}^1$, then they are free to move supersymmetrically in the x^2, x^3, x^4, x^5 directions. This should be familiar as the general facts we uncovered in chapter 11 about the Dp-D$(p+2)$ bound state system: if the D$(p+2)$ is extended, the Dp cannot move out of it and preserve supersymmetry. This is also T-dual to a single brane at an angle and we shall see this next.

13.3 Wrapped, fractional and stretched branes

There is yet another useful way of thinking of all the of the above physics, and even more aspects of it will become manifest here. It requires exploring a duality to another picture altogether. This duality is a T-duality, although since it is a non-trivial background that is involved, we should be careful. It is best trusted at low energy, as we cannot be

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sure that the string theories are completely dual at all mass levels without further analysis. We will study only the massless fields, so we should probably claim only that the backgrounds give the same low energy physics. Nevertheless, once we arrive at our dual, we can forget about where it came from and study it directly in its own right. Recent work, using extensions of the techniques of this chapter, has directly proven the duality [340].

13.3.1 NS5-branes from ALE spaces

Up to a change of variables, in the supergravity background (13.14), $\mathbf{y}$ can be taken to be the vector $\mathbf{y} = (x^7, x^8, x^9)$ while we will take x^6 to be our periodic coordinate z. (There are some dimensionful parameters which were left out of the derivation of (13.14), for clarity, and we shall put them in by hand, and try to fix the pure numbers with T-duality.)

Then, using the T-duality rules (5.4) we can arrive at another background (note that we have adjoined the flat transverse spacetime $\mathbb{R}^6$ to make a ten dimensional solution, and restored an α' for dimensions):

$$
\begin{aligned}
ds^2 &= -dt^2 + \sum_{m=1}^{5} dx^m dx^m + V(y)(dx^6 dx^6 + d\mathbf{y}\cdot d\mathbf{y}) \\
e^{2\Phi} &= V(y) = \sum_{i=0}^{N-1} \frac{\sqrt{\alpha'}}{|\mathbf{y}-\mathbf{y}_i|},
\end{aligned}
\tag{13.24}
$$

which is also a ten dimensional solution if taken with a non-trivial background field[203, 204] $H_{mns} = \epsilon_{mns}{}^{r}\partial_r\Phi$, which defines the potential B_{6i} ($i = 7, 8, 9$) as a vector A_i that satisfies $\nabla V = \nabla\times\mathbf{A}$. Non-zero B_{6i} arose because the T-dual solution had non-zero G_{6i}.

In fact, this is not quite the solution we are looking for. What we have arrived at is a solution which is independent of the x^6 direction. This is necessary if we are to use the operation (5.4). In fact, we expect that the full solution we seek has some structure in x^6, since translation invariance is certainly broken there. This is because the x^6-circle of the ALE space has N places where something special happens to the winding states, since the circle shrinks away there. So we expect that the same must be true for momentum in the dual situation[205]. A simple guess for a solution which is localised completely in the x^6, x^7, x^8, x^9 directions is to simply ask that it be harmonic there. We simply take $\mathbf{x} = (x^6, \mathbf{y})$ to mean a position in the full $\mathbb{R}^4$, and replace $V(y)$ by:

$$
V(x) = 1 + \sum_{i=0}^{N-1} \frac{\alpha'}{(\mathbf{x}-\mathbf{x}_i)^2}
\tag{13.25}
$$

We have done a bit more than just delocalised. By adding the 1 we have endowed the solution with an asymptotically flat region. However, adding

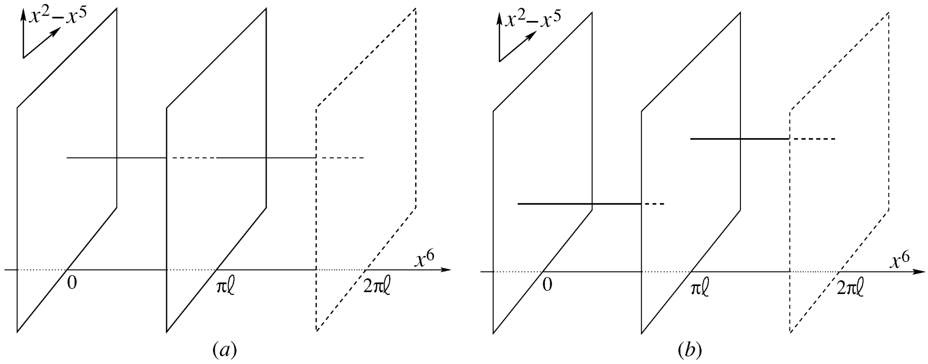

Fig. 13.4. (*a*) This configuration of two NS5-branes on a circle with D2-branes stretched between them is dual to a D1-brane probing an A_1 ALE space. (*b*) The Coulomb branch where the D2-brane splits into two 'fractional branes'.

the 1 is consistent with $V(\mathbf{x})$ being harmonic in x^6, x^7, x^8, x^9, and so it is still a solution.

The solution we have just uncovered is made up of a chain of N objects which are pointlike in $\mathbb{R}^4$ and magnetic sources of the NS–NS potential $B_{\mu\nu}$. They are in fact the 'NS5-branes' we discovered by various arguments in chapter 12, with the result (12.8). Here, the NS5-branes are arranged in a circle on x^6, and distributed on the rest of $\mathbb{R}^4$ according to the centres $\mathbf{x}_i$, $i = 0, \ldots, N-1$.

13.3.2 Dual realisations of quivers

Recall that we had a D1-brane lying along the x^1 direction, probing the ALE space. By the rules of T-duality on a D-brane, it becomes a D2-brane probing the space, with the extra leg of the D2-brane extended along the compact x^6 direction. The D2-brane penetrates the two NS5-branes as it winds around once. The point at which it passes through an NS5-brane is given by four numbers $\mathbf{x}_i$ for the ith brane. The intersection point can be located anywhere within the fivebrane's worldvolume in the directions x^2, x^3, x^4, x^5. (See figure 13.4 (*a*).)

In the table below, we show the extension of the D2 in the x^6 direction as a $|-|$ to indicate that it may be of finite extent, if it were ending on an NS5-brane.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D2	—	—	•	•	•	•	$\vert - \vert$	•	•	•
NS5	—	—	—	—	—	—	•	•	•	•

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This arrangement, with the branes lying in the directions which we have described, preserves the same eight supercharges we discussed before. Starting with the 32 supercharges of the type IIA supersymmetry, the NS5-branes break a half, and the D2-brane breaks half again. The infinite part of the probe, an effective one-brane (string), has a $U(1)$ on its worldvolume, and its tension is $\mu = 2\pi\ell\mu_2$, where ℓ is the as yet unspecified length of the new x^6 direction. However, just as in the discussion in section 10.4, we may consider different values of ℓ if we allow ourselves to consider different densities of branes in the dual picture. Let us focus on $N = 2$. If the two fivebranes (with positions $\mathbf{x}_1, \mathbf{x}_2$; we can set $\mathbf{x}_0$ to zero) are located at the same $\mathbf{y} = (x^7, x^8, x^9)$ position, then the D2-brane can break into two segments, giving a $U(1) \times U(1)$ (one from each segment) on the one-brane part stretched in the infinite x^1 direction. The two segments can move independently within the NS5-brane worldvolume, while still remaining parallel, preserving supersymmetry.

N.B. It makes sense that the D2-brane can end on an NS5-brane, as already discussed in section 12.6.2. There is a 2-form potential in the world-volume for which the string-like end can act as an electric source.

This is the precise analogue of the Coulomb branch of the D1-brane probing the ALE space that we saw earlier. The hypermultiplets of the $U(1) \times U(1)$ theory are made here by stretching fundamental strings across the NS5-branes in x^6 to make a connection between the D-brane segments[206]. The three differences $\mathbf{y}_1 - \mathbf{y}_2$ are the T-dual of the NS–NS parameters representing the size and orientation of the ALE space's $\mathbb{CP}^1$. The x^6 separation of the NS5-branes is dual to the flux $2\pi\ell\Phi_B$. This is the length of one segment while $2\pi\ell(1 - \Phi_B)$ is the length of the other. (Note that the symmetry $\Phi_B \sim \Phi_B + 1$ is preserved, as it just swaps the segments.) Notice also that there is an interesting duality between the quiver diagram and the arrangement of branes in the dual picture. (See figure 13.5.)

The original setup had the lengths equal, but we can change them at will, and this is dual to changing Φ_B. There is the possibility of one of the lengths becoming zero. The NS-branes become coincident, and at the same time a fractional brane becomes a tensionless string, and we get an A_1 enhancement of the gauge symmetry carried by the two-form potential which lives on the type IIA NS5-brane[160]. If we had D1-branes stretched between NS5s in type IIB instead, we would get massless particles, and an enhanced $SU(2)$ gauge symmetry.

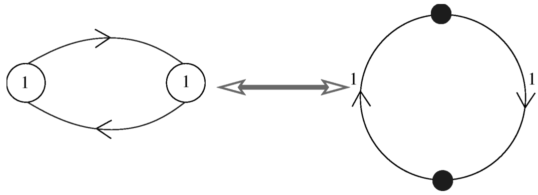

Fig. 13.5. There is a duality between the extended Dynkin diagram which gives the probe gauge theory and the diagram representing D-branes stretched between NS5-branes. The nodes in one are replaced by links in the other. In particular, the number inside the Dynkin nodes become the number of D-branes in the links in the dual diagram. The hypermultiplets associated with links in the Dynkin diagram arise from strings connecting the D-brane fragment ending on one side of an NS5-brane with the fragment on the other.

N.B. This fits nicely with the discovery we made in insert 7.5 that the type IIA string on K3 was dual to the heterotic string on T^4. There are indeed enhanced gauge symmetries on the type IIA side as well. They are not visible in the usual conformal field theory approach because there the flux Φ_B is non-zero and fixed. But now we see that if it is tuned to zero, we can then get the enhanced A–D–E symmetries[161], corresponding to wrapped D2-branes becoming shrunk to zero size with *no* remaining flux, or D1-brane segments shrinking to zero length.

If the segments are separated, and thus attached to the NS5-branes, then when we move the NS5-branes out to different x^{789} positions, the segments must tilt in order to remain stretched between the two branes. They will therefore be oriented differently from each other and will break supersymmetry. This is how the Coulomb branch is 'lifted' in this language. (See figure 13.6(c).) A segment at an orientation gives a contribution $\sqrt{(2\pi\ell\Phi_B)^2+(\mathbf{y}_1-\mathbf{y}_2)^2}$ to the D1-brane's tension. This formula should be familiar: it is of the form for the more general formula for a D1–D3 bound state (see section 11.2), to which this tilted D2-brane segment is T-dual.

For supersymmetric vacua to be recovered when the NS5-brane are moved to different positions (the dual of smoothing the ALE space) the branes segments must first rejoin with the other (Higgs-ing), giving the single D-brane. Then it need not move with the NS5-branes as they separate in $\mathbf{y}$, and can preserve supersymmetry by remaining stretched as

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a single component. (See figure 13.6(*d*).) Its $\mathbf{y}$ position and an x^6 Wilson line constitute the Higgs branch parameters. Evidently the metric on these Higgs branch parameters is that of an ALE space, since the 1+1 dimensional gauge theory is the same as the discussion in section 13.1, and hence the moduli spaces match. It is worth sharpening this into a field theory proof of the low energy validity of the T-duality, but we will not do that here.

It is worth noting here that once we have uncovered the existence of fractional D-branes with a modulus for their separation, there is no reason why we cannot separate them infinitely far from each other and consider them in their own right. We also have the right to take a limit where we focus on just one segment with a finite separation between two NS5-branes, but with a *non-compact* x^6 direction. This is achievable from what we started with here by sending $\Phi_B \to 0$, but changing to scaled variables in which there is still a finite separation, and hence a finite gauge coupling on the brane segment in question. (U-duality will then give us various species of branes ending on branes which we will discuss later.)

Fractional branes, and their duals the stretched brane segments, are useful objects since they are less mobile than a complete D-brane, in that

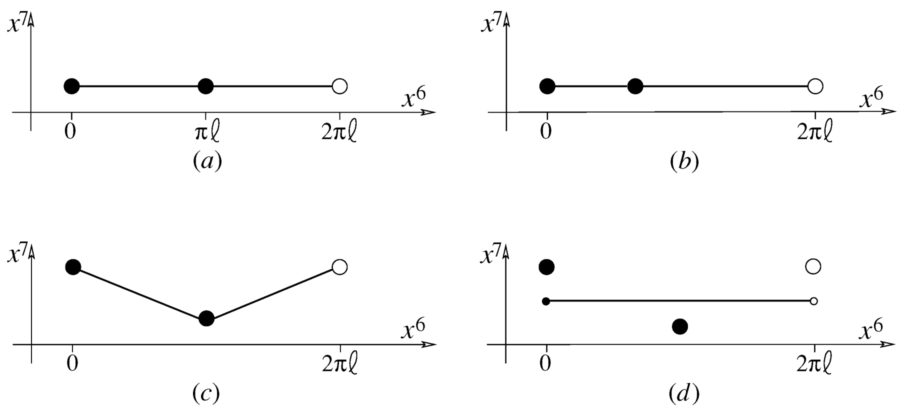

Fig. 13.6. Possible deformations of the brane arrangements, and their gauge theory interpretation. (*a*) The configuration dual to the standard orbifold limit with the traditional 'half unit' of B-flux. (*b*) Varying the distribution of B-flux between segments. Sending it to zero will make the NS5-brane coincide and give an enhanced gauge symmetry. (*c*) Switching on a deformation parameter (an FI term in gauge theory) 'lifts' the Coulomb branch: if there are separated D-brane fragments, supersymmetry cannot be retained. (*d*) First Higgs-ing to make a complete brane allows smooth movement onto the supersymmetric Higgs branch.

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they cannot move in some directions. One use of this is the study of gauge theory on branes with a reduced number of supersymmetries and a reduced number of charged hypermultiplets *à la* Hanany–Witten,[206, 212]. This has many applications[213], some of which we will consider later.

13.4 D-branes as instantons

Consider a D0-brane and N coincident D4-branes. There is a $U(1)$ on the D0 and $U(N)$ on the D4's, which we shall take to be extended in the x^6, x^7, x^8, x^9 directions. The potential terms in the action are

$$\frac{\chi_i^\dagger \chi_i}{(2\pi\alpha')^2}\sum_{a=1}^{5}(X_a - Y_a)^2 + \frac{1}{4g_0^2}\sum_{I=1}^{3}(\chi_i^\dagger \tau^I \chi_i)^2. \tag{13.26}$$

Here a runs over the dimensions transverse to the D4-brane, and X_a and Y_a are respectively the D0-brane and D4-brane positions, and for now we ignore the position of the D0-brane within the D4-branes' world-volume. This is the same action as in the earlier case (11.7), but here the D4-branes have infinite volume and so their D-term drops out relative to that of the D0-brane. We have also written the 0–4 hypermultiplet field χ with a D4-brane index i. (The $SU(2)_\mathrm{R}$ index is suppressed.) The potential (13.26) is exact on grounds of $\mathcal{N}=2$ supersymmetry. The first term is the $\mathcal{N}=2$ coupling between the hypermultiplets χ and the vector multiplet scalars X, Y. The second is the $U(1)$ D-term.

For $N > 1$ there are two branches of moduli space, in direct analogy with the ALE case. The Coulomb branch is $(X \neq Y, \quad \chi = 0)$, which is simply the position of the D0-brane transverse to the D4-branes. There is a mass for χ and so its vev is zero. The Higgs branch $(X = Y, \quad \chi \neq 0)$ represents the physics of the D0-brane being stuck on the world-volume of the D4-branes. The non-zero vev of χ Higgses away the $U(1)$ and some of the $U(N)$.

Let us count the dimension of moduli space. There are $4N$ real degrees of freedom in χ. The vanishing of the $U(1)$ D-term imposes three constraints, and moding by the (broken) $U(1)$ removes another degree of freedom leaving $4N - 4$. There are four moduli for the position of the D0-brane inside the the D4-branes, giving a total of $4N$ moduli. This is in fact the correct dimension of moduli space for an $SU(N)$ instanton when we do not mod out also the $SU(N)$ identifications. For k instantons this dimension becomes $4Nk$.

Another clue that the Higgs branch describes the D0-brane as a D4-brane gauge theory instanton is the fact that the Ramond–Ramond couplings include a term $\mu_4 C_{(1)} \wedge \mathrm{Tr}(F \wedge F)$. As shown in section 9.2, when

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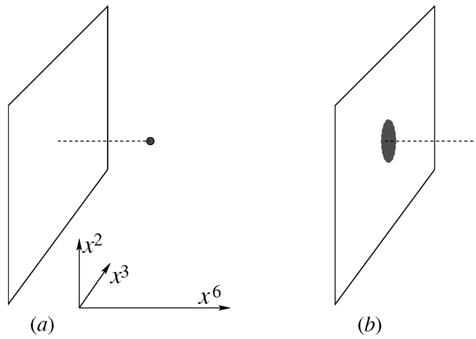

Fig. 13.7. Instantons and the Dp–D$(p+4)$ system. (a) The Coulomb branch of the Dp-brane theory represents a pointlike brane away from the D$(p+4)$-brane. (b) The Higgs branch corresponds to it being stuck inside the D$(p+4)$-brane as a finite sized instanton of the D$(p+4)$-brane's gauge theory.

there is an instanton on the D4-brane it carries D0-brane charge. The position of the instanton is given by the 0–0 fields, while the 0–4 fields should give the size and shape (see figure 13.7).

13.4.1 Seeing the instanton with a probe

Actually, we can really see the resulting instanton gauge fields by using a D-brane as a probe. We will use a D9–D5 system[130] instead of D4–D0, and so we won't have the Coulomb branch, since D9-branes fill all of spacetime, and so the D5-branes cannot move out of them. We will us a D1-brane to probe the D9–D5 system[184]. It breaks half of the supersymmetries left over from the 9–5 system, leaving four supercharges overall. The effective 1+1 dimensional theory is $(0,4)$ supersymmetric and is made of 1–1 fields, which has two classes of hypermulitplets. One represents the motions of the probe transverse to the D5, and the other parallel. The 1–5 and 1–9 fields are also hypermultiplets, while the 9–5 and 5–5 fields are parameters in the model.

In what follows, we shall borrow a lot of the notation of the original papers on the subject[130, 184]. Let us place the D5-branes such that they are pointlike in the x^6, x^7, x^8, x^9 directions. The D1-brane probe will lie along the x^1 direction, as usual.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D1	—	—	•	•	•	•	•	•	•	•
D5	—	—	—	—	—	—	•	•	•	•

This arrangement of branes breaks the Lorentz group up as follows:

$$SO(1,9) \supset SO(1,1)_{01} \times SO(4)_{2345} \times SO(4)_{6789}, \tag{13.27}$$

where the subscripts denote the sub-spacetimes in which the surviving factors act. We may label[207, 184] the world-sheet fields according to how they transform under the covering group:

$$[SU(2)' \times \widetilde{SU(2)}']_{2345} \times [SU(2)_R \times SU(2)_L]_{6789}, \tag{13.28}$$

with doublet indices $(A', \tilde{A}', A, Y)$, respectively.

The analysis that we did for the D1-brane probe in the type I string theory in section 12.2 still applies, but there are some new details. Now ξ_- is further decomposed into ξ_-^1 and ξ_-^2, where superscripts 1 and 2 denote the decomposition into the (2345) sector and the (6789) sector, respectively. So we have that the fermion ξ_-^1 (hereafter called $\psi_-^{A\tilde{A}'}$) is the right-moving superpartner of the four component scalar field $b^{A'\tilde{A}'}$, while ξ_-^2 (called $\psi_-^{A'Y}$) is the right-moving superpartner of b^{AY}. The supersymmetry transformations are:

$$\begin{aligned}
\delta b^{A'\tilde{A}'} &= i\epsilon_{AB}\eta_+^{A'A}\psi_-^{B\tilde{A}'} \\
\delta b^{AY} &= i\epsilon_{A'B'}\eta_+^{AA'}\psi_-^{B'Y}.
\end{aligned} \tag{13.29}$$

In the 1–5 sector, there are four DN coordinates, and four DD coordinates giving the NS sector a zero point energy of zero, with excitations coming from integer modes in the 2345 directions, giving a four component boson. The R sector also has zero point energy of zero, with excitations coming from the 6789 directions, giving a four component fermion χ.

The GSO projections in either sector reduce us to two bosonic states $\phi^{A'}$ and decomposes the spinor χ into left- and right-moving two component spinors, χ_-^A and χ_+^Y, respectively. We see that χ_-^A is the right-moving superpartner of $\phi^{A'}$. Taking into account the fact that there is a D5-brane index for these fields, we can display the components $(\phi^{A'm}, \chi_-^{Am})$ which are related by supersymmetry:

$$\delta\phi^{A'm} = i\epsilon_{AB}\eta_+^{A'A}\chi_-^{Bm}, \tag{13.30}$$

and the $(0,4)$ supersymmetry parameter is denoted by $\eta_+^{A'A}$. Here, m is a D5-brane group theory index. Also, χ_+^Y has components χ_+^{Ym}.

The supersymmetry transformation relating them to the left-moving fields are:

$$\begin{aligned}
\delta\lambda_+^M &= \eta_+^{AA'}C_{AA'}^M \\
\delta\chi_+^{Ym} &= \eta_+^{AA'}C_{AA'}^{Ym},
\end{aligned} \tag{13.31}$$

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where $C^{M}_{AA'}$ and $C^{Ym}_{AA'}$ shall be determined shortly. They will be made of the bosonic 1–1 fields and other background couplings built out of the 5–5 and 5–9 fields.

The 5–5 and 5–9 couplings descend from the fields in the D9–D5 sector. There are some details of those fields which are peculiarities of the fact that we are in type I string theory. First, the gauge symmetry on the D9-branes is $SO(32)$. Also, for k coincident D5-branes, there is a gauge symmetry $USp(2k)$[130], since there is an extra -1 in the action of Ω on D5-brane fields, as explained already[132] in section 8.7. The 5–5 sector hypermultiplet scalars (fluctuations in the transverse $x^{6,7,8,9}$ directions) transform in the antisymmetric of $USp(2k)$, which we call X^{AY}_{mn}, matching the notation in the literature[184]. Meanwhile, the 5–9 sector produces a $(\mathbf{2k}, \mathbf{32})$, denoted h^{Am}_{M}, with m and M as in D5- and D9-brane labels.

Using the form of the transformations (13.31) allows us to write the non-trivial part of the $(0,4)$ supersymmetric 1+1 dimensional Lagrangian containing the 'Yukawa' couplings and the potential of the $(0,4)$ model:

$$\begin{aligned}\mathcal{L}_{\rm tot} = \mathcal{L}_{\rm kinetic} - \frac{i}{4}\int d^2\sigma \Bigg[\lambda^M_+ &\left(\epsilon^{BD}\frac{\partial C^M_{BB'}}{\partial b^{DY}}\psi^{B'Y}_- + \epsilon^{B'D'}\frac{\partial C^M_{BB'}}{\partial \phi^{D'm}}\chi^{Bm}_-\right)\\ &+\chi^{Ym}_+\left(\epsilon^{BD}\frac{\partial C^{Ym}_{BB'}}{\partial b^{DY}}\psi^{B'Y}_- + \epsilon^{B'D'}\frac{\partial C^{Ym}_{BB'}}{\partial \phi^{D'm}}\chi^{Bm}_-\right)\\ &+\frac{1}{2}\epsilon^{AB}\epsilon^{A'B'}\left(C^M_{AA'}C^M_{BB'} + C^{Ym}_{AA'}C^{Ym}_{BB'}\right)\Bigg]. \end{aligned} \qquad (13.32)$$

This is the most general[207] $(0,4)$ supersymmetric Lagrangian with these types of multiplets, providing that the C satisfy the D-flatness condition:

$$C^M_{AA'}C^M_{BB'} + C^{Ym}_{AA'}C^{Ym}_{BB'} + C^M_{BA'}C^M_{AB'} + C^{Ym}_{BA'}C^{Ym}_{AB'} = 0, \qquad (13.33)$$

where $\mathcal{L}_{\rm kinetic}$ contains the usual kinetic terms for all of the fields. Notice that the fields $b^{A'\tilde{A}'}$ and $\psi^{AA'}_-$ are *free*.

Now equation (13.32) might appear somewhat daunting, but is in fact mostly notation. The trick is to note that general considerations can allow us to fix what sort of things can appear in the matrices $C_{AA'}$. The distance between the D1-brane and the D5-branes should set the mass of the 1–5 fields, $\phi^{A'm}$ and its fermionic partners χ^{Am}_-, χ^{Ym}_+. So there should be terms of the form:

$$\phi^m_{A'}\phi^{A'n}(X^{AY}_{mn} - b^{AY}\delta_{mn})^2, \quad \chi^{Am}_-\chi^{Yn}_+(X^{AY}_{mn} - b^{AY}\delta_{mn}), \qquad (13.34)$$

where the term in brackets is the unique translation invariant combination of the appropriate 1–1 and 5–5 fields. There are also 1–5–9 couplings,

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which would be induced by couplings between 1–9, 1–5 and 5–9 fields, in the form $\lambda_+^M \chi_{m-}^A h_{AM}^m$.

In fact, the required Cs which satisfy the requirements (13.33) and give us the coupling which we expect are[184].

$$\begin{aligned} C_{AA'}^M &= h_A^{Mm}\phi_{A'm} \\ C_{AA'}^{Ym} &= \phi_{A'}^n (X_{An}^{Ym} - b_A^Y \delta_n^m). \end{aligned} \tag{13.35}$$

The $(0,4)$ D-flatness conditions (13.33) translate directly into a series of equations for the D5-brane hypermultiplets to act as data specifying an instanton via the 'ADHM description'[208]. The crucial point is[207] that the vacua of the sigma model gives a space of solutions which is isomorphic to those of ADHM.

One can see that one has the right number of parameters as follows: The potential is of the form $V = \phi^2((X-b)^2 + h^2)$. So the term in brackets acts as a mass term for ϕ. The potential vanishes for $\phi = 0$, leaving this space of vacua to be parametrised by X and h, with b giving the position of the D1-brane in the four transverse directions. Let us write $\widehat{X}^{AY}{=}(X^{AY}{-}b^{AY})$ as the centre of mass field.

Notice that for these vacua ($\phi = 0$), the Yukawa couplings are of the form $\sum_a \lambda_+^a B_{Am}^a \chi_-^{Am}$ where $B_{Am}^a = \partial C_{AB'}^a / \partial\phi_{B'm}$, and the index a is the set (M, Y, m). There are $4k$ fermions in χ_- and so this pairs with $4k$ fermions in the set $\lambda_+^a = (\chi_+^{Ym}, \lambda_+^M)$, leaving a subspace of 32 massless modes describing the non-trivial gauge bundle.

The idea is to write the low energy sigma model action for these massless fields. This is done as follows: a basis of massless components is given by v_i^a $(i = 1, \ldots, 32)$ defined by $\sum_a v_v^a B_{Am}^a = 0$, and we choose it to be orthonormal: $\sum_a v_i^a v_j^a = \delta_{ij}$. The basis v_i^a depends on $\widehat{X}$. So substituting $\lambda_+^a = \sum_i v_i^a \lambda_+^i$ into the kinetic energy gives[207]:

$$\lambda_+^a \partial_- \lambda_+^a = \sum_{i,j} \Big\{ \lambda_{+i} \left(\delta_{ij}\partial_- + \partial_- \widehat{X}^\mu A_{\mu,ij} \right) \lambda_{+j} \Big\}, \tag{13.36}$$

where

$$A_{\mu,ij} \equiv A_{BY,ij} = \sum_a v_i^a \frac{\partial v_j^a}{\partial \widehat{X}^{BY}}; \tag{13.37}$$

we have used the x^6, x^7, x^8, x^9 spacetime index μ on our 1–1 field $\widehat{X}^{BY}$ instead of the indices (B, Y), for clarity.

So we see that the second term in (13.36) shows the sigma model couplings of the fermions to a background gauge field A_μ. Since we have generically

$$B^a_{Am}: \quad \left(\widehat{X}^{AY}, h_A^{Mm}\right), \tag{13.38}$$

the orthonormal basis v^a_i is

$$v^a: \quad \left(\frac{h_A^{Mm}}{\sqrt{\widehat{X}^2 + h^2}}, \frac{-\widehat{X}^{AY}}{\sqrt{\widehat{X}^2 + h^2}}\right), \tag{13.39}$$

and from (13.37), it is clear that the background gauge field is indeed of the form of an instanton: the 5–9 field h indeed sets the scale size of the instanton, and the 5–5 field X sets its position.

13.4.2 Small instantons

Notice that this model gives a meaning to the instanton when its size, set by h, drops to zero[130]. This limit of the instanton is simply singular in field theory. Here we see that the size is just the vev of a 9–5 field, for which zero is a perfectly fine value. Generically, in the Dp-D$(p+4)$ description, zero scale size is the place where the Higgs branch joins onto the Coulomb branch representing the Dp-brane becoming pointlike (getting an enhanced gauge symmetry on its world-volume), and moves out of the world-volume of the parent brane. (For $p = 5$ this branch is not present, but the connection is clear via T-duality.)

This supplies a method for making rather different sorts of gauge group for the heterotic or type I string theory, beyond the perturbative $SO(32)$ that we are used to[130]. The inclusion of k type I D5-branes allows for additional $SU(2)^k$ if they are all separated, or if m are coincident, a $USp(2m)\times SU(2)^{k-m}$, as we have seen in section 8.7. In a compact model, the number, k, of D5-branes is restricted by Gauss's law, and we shall see some examples of this in chapter 14. On the heterotic side, we see that this origin of the gauge group is not visible in any perturbative description, and so the description is best done at strong coupling, in terms of type I strings with D5-branes.

Recall from section 12.5 that the chain of dualities involving T-duality between the heterotic strings and type I/heterotic duality leads to the picture of the strongly coupled $E_8 \times E_8$ heterotic string as eleven dimensional supergravity on a line interval. The same reasoning leads to a picture of small instantons in that case too[341]. They are simply M5-branes in a special situation. An ordinary instanton would be embedded in one or other E_8, and this corresponds to the M5-brane being located at one or other end of the interval. In the intermediate picture denoted in figure 12.4, there is a D4-brane which lives inside the worldvolume of the eight D8-branes located at the O8-plane, at one end or the other. The 4–8 strings can take vevs and allow them to fatten up into fully fledged instantons of the $SO(16)$ which will be enhanced with the spinor representation become the E_8 as they become M5-branes when the extra dimension opens up. Setting the 4–8 strings' vev (vacuum expectation value) to zero allows them to give vevs to the 4–4 strings which can move them away from the ends of the interval into the interior. In the fully eleven dimensional picture, this is the M5-brane moving into the interior. The $E_8 \times E_8$ is restored, but there is something extra from the M5-brane, just as in the $SO(32)$ case there was something extra from the D5-branes. In this case, it is not an extra $\mathcal{N} = 2$ six dimensional vector multiplet, $(\mathbf{2}, \mathbf{2}) + 4(\mathbf{1}, \mathbf{1})$ giving extra gauge symmetry, but an extra $\mathcal{N} = 2$ *tensor* multiplet, $(\mathbf{3}, \mathbf{1}) + 5(\mathbf{1}, \mathbf{1})$. Even after the return to the weakly coupled $E_8 \times E_8$ string by shrinking the interval, this structure remains as the result of shrinking an E_8 instanton to zero size.

This is a rather nice result, for many reasons. One is that we see that the number of scalars in the multiplet reflect the fact that the brane (and hence the instanton) indeed has an eleven dimensional origin, representing its strongly coupled roots even after the return to the weakly coupled heterotic string. The other is that the intermediate picture in type IA allows us to use the result that upon dimensional reduction from six to five dimensions (which happens to the $SO(32)$ D5-brane on its way to becoming an M5-brane), a vector multiplet and a tensor multiplet both reduce to the same multiplet (a vector), and so it is possible to make transitions between these multiplets by making a dimension compact and then decompactifying one afterwards[341].

13.5 D-branes as monopoles

Consider the case of a pair of parallel D3-branes, extended in the directions x^1, x^2, x^3, and separated by a distance L in the x^6 direction. Let us now stretch a family of k parallel D1-branes along the x^6 direction, and have them end on the D3-branes. (This is U-dual to the case of D2-branes ending on NS5-branes, as stated earlier in section 13.3.) Let us call the x^6 direction s, and place the D3-branes symmetrically about the origin, choosing our units such that they are at $s = \pm 1$.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D1	—	•	•	•	•	•	$\vert - \vert$	•	•	•
D3	—	—	—	—	•	•	•	•	•	•

This configuration preserves eight supercharges, as can be seen from our previous discussion of fractional branes. Also, a T_6-duality yields a pair of D4-branes (with a Wilson line) in x^1, x^2, x^3, x^6 with k (fractional) D0-branes. (We naively expect that this construction should be related to our previous discussion of instantons, but instead of on $\mathbb{R}^4$, they are on $\mathbb{R}^3 \times S^1$.) We can see it directly from the fact that the presence of the D3- and D1-branes world-volumes place the following constraints on the available supercharges:

$$\epsilon_{\rm L} = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\epsilon_{\rm R}; \quad \epsilon_{\rm L} = \Gamma^0\Gamma^6\epsilon_{\rm R}, \tag{13.40}$$

which taken together give eight supercharges, satisfying the condition

$$\epsilon_{\rm L} = \Gamma^1\Gamma^2\Gamma^3\Gamma^6\epsilon_{\rm L}. \tag{13.41}$$

The 1–1 massless fields are simply the (1+1)-dimensional gauge field $A^\mu(t,s)$ and eight scalars $\Phi^m(t,s)$ in the adjoint of $U(k)$, the latter representing the transverse fluctuations of the branes. There are fluctuations in x^1, x^2, x^3 and others in x^4, x^5, x^7, x^8, x^9. We shall really only be interested in the motions of the D1-brane within the D3-brane's directions x^1, x^2, x^3, which is the Coulomb branch of the D1-brane moduli space. So of the Φ^m, we keep only the three for $m = 1, 2, 3$. There are additionally 1–3 fields transforming in the $(\pm 1, k)$. They form a complex doublet of $SU(2)_{\rm R}$ and are $1 \times k$ matrices. Crucially, these flavour fields are massless only at $s = \pm 1$, the locations where the D1-branes touch the D3-branes. If we were to write a Lagrangian for the massless fields, there will be a delta function $\delta(s \mp 1)$ in front of terms containing those. The structure of the Lagrangian is very similar to the one written for the $p - (p+4)$ system, with the additional features of $U(k)$ non-Abelian structure. Asking that the D-terms vanish, for a supersymmetric vacuum, we get:[209]

$$\frac{d\Phi^i}{ds} - [A_s, \Phi^i] + \frac{1}{2}\epsilon^{ijk}[\Phi^j, \Phi^k] = 0, \tag{13.42}$$

where we have ignored possible terms on the right hand side supported only at $s = \pm 1$. These would arise from the interactions induced by massless 1–3 fields there[210]. We shall derive those effects in another way by carefully considering the boundary conditions in a short while.

If we choose the gauge in which $A_s = 0$, our equation (13.42) can be recognised as the Nahm equations[216], known to construct the moduli space[218] of N $SU(2)$ monopoles, *via* an adaptation of the ADHM construction[208]. The covariant form $A_s \neq 0$, is useful for actually solving for the metric on the moduli space of monopole solutions and for the spacetime monopole fields themselves, as we shall show[211].

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If our k D1-branes were reasonably well separated, we would imagine that the boundary condition at $s=\pm 1$ is clearly $2\pi\alpha'\Phi^i(s=1))=\mathrm{diag}\{x_1^i, x_2^i, \ldots, x_k^i\}$, where x_n^i, $i=1,2,3$ are the three coordinates of the end of the nth D1-brane (similarly for the other end). In other words, the off-diagonal fields corresponding to the 1–1 strings stretching between the individual D1-branes are heavy, and therefore lie outside the description of the massless fields. However, this is not quite right. In fact, it is very badly wrong. To see this, note that the D1-branes have tension, and therefore must be pulling on the D3-brane, deforming its shape somewhat. In fact, the shape must be given, to a good approximation, by the following description. The function $s(\mathbf{x})$ describing the position of the D3-brane along the x^6 direction as a function of the three coordinates x^i should satisfy the equation $\nabla^2 s(x)=0$, where ∇^2 is the three dimensional Laplacian. A solution to this is

$$s = 1 + \frac{c}{|\mathbf{x}-\mathbf{x}_0|}, \tag{13.43}$$

where 1 is the position along the s direction and c and $\mathbf{x}_0$ are constants. So, far away from $\mathbf{x}_0$, we see that the solution is $s=1$, telling us that we have a description of a flat D3-brane. Nearer to $\mathbf{x}_0$, we see that s increases away from 0, and eventually blows up at $\mathbf{x}_0$.

We sketch this shape in figure 13.8(b). It is again our BIon-type solution, described before in section 5.7. The D3-brane smoothly interpolates between a pure D1-brane geometry far away and a spiked shape resembling D1-brane behaviour at the centre. A multi-centred solution is easy to construct as a superposition of harmonic solutions of the above type.

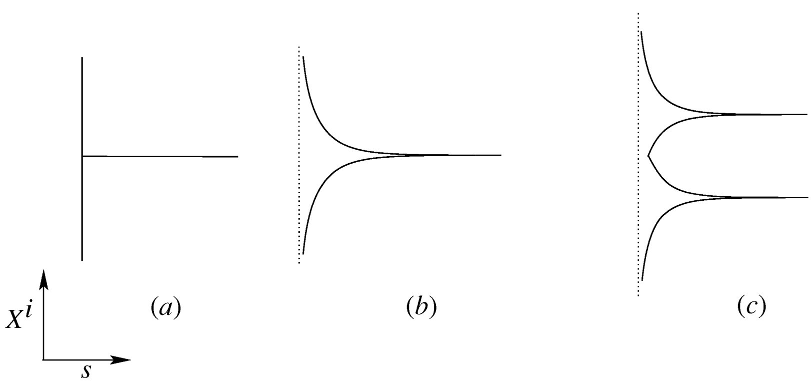

Fig. 13.8. (a) A D3-brane (vertical) with a D1-brane ending on it (horizontal) is actually pulled (b) into a smooth interpolating shape. (c) Finitely separated D1-branes can only be described with non-commutative coordinates (see text)

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Considering two of them, we see that in fact for any finite separation of the D1-branes (as measured far enough along the s-direction), by time we get to $s = 1$, they will be arbitrarily close to each other (see figure 13.8(b)). We therefore cannot forget[213] about the off-diagonal parts of Φ^i corresponding to 1–1 strings stretching between the branes, and in fact we are forced to describe the geometry of the branes' endpoints on the D3-brane using non-Abelian Φ^i. This is another example of the 'natural' occurrence of a non-commutativity arising in what we would have naively interpreted as ordinary spacetime coordinates.

We can see precisely what the boundary conditions must be, since we are simply asking that there be a pole in $\Phi^i(s)$ as $s \to \pm 1$:

$$\Phi^i(s) \to \frac{\Sigma^i}{s \mp 1}, \tag{13.44}$$

and placing this into (13.42), we see that the $k \times k$ residues must satisfy

$$[\Sigma^i, \Sigma^j] = 2i\epsilon_{ijk}\Sigma^k. \tag{13.45}$$

In other words, they must form a k-dimensional representations of $SU(2)$ (in an unusual normalisation). This representation *must* be irreducible, as we have seen. Otherwise it necessarily captures only the physics of m infinitely separated clumps of D1-branes, for the case where the representation is reducible into m smaller irreducible representations.

13.5.1 Adjoint Higgs and monopoles

The problem we have constructed is that of k monopoles[214, 215] of $SU(2)$ spontaneously broken to $U(1)$ *via* an adjoint Higgs field[217] $H = H^a t^a$. Ignoring the centre of mass of the D3-brane pair, this $SU(2)$ is on their world-volume, and the separation is set by the vacuum expectation value (vev), $\sqrt{H^a H^a} = h$ of the Higgs field. In our problem, we have $(2\pi\alpha')h = L/2$, where L is the separation of our D3-branes, where we label the positions of the branes with $L\sigma_3/2$, with a factor of $2\pi\alpha'$ to convert the Higgs field to a length, as we have done before.

Generically, such a model is given by the following effective Lagrangian:

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2}\mathrm{Tr}(D_\mu H D^\mu H) - V(H), \tag{13.46}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu], \qquad D_\mu H = \partial_\mu H + e[A_\mu, H],$$

where H is valued in the adjoint of the gauge group, and there is the usual gauge invariance ($g(\mathbf{x}) \in SU(2)$):

$$A_\mu \to g^{-1}A_\mu g + g^{-1}\partial_\mu g; \quad H \to g^{-1}Hg. \tag{13.47}$$

N.B. Note that we have put explicitly a coupling e into the model so that we can see how the physics depends on it. Our usual conventions do not have that coupling there, but this is for convenience. We will remove it at a later juncture.

The potential V is taken to be positive (but see later), and a typical choice is $V \sim \lambda(\mathrm{Tr}(H \cdot H)/2 - h^2)$. We can imagine a non-zero vev for H which breaks the $SU(2)$ to $U(1)$, such as $H = h\sigma_3$, or, in an $SO(3)$ language, $\vec{H} = (0, 0, h)$. The $U(1)$ left over is the σ_3 generator, or just a rotation about the x^3 axis. Notice that the family of values of H which minimise V form an S^2 of radius h, each point on the S^2 being equivalent, as they can be reached by an $SU(2)$ rotation from any other point. This S^2 is the coset space $SU(2)/U(1)$.

More interestingly, we shall seek static solutions with configurations of H and A which have non–trivial dependence on the spatial coordinate $\mathbf{x} = (x_1, x_2, x_3)$. Let us work in a gauge in which $A_0 = 0$, and seek static configurations of finite energy. The Lagrangian reduces to a potential energy density:

$$\mathcal{L} = -\mathcal{E} = -\frac{1}{4}F^a_{ij}F^{aij} - \frac{1}{2}D_iH^aD^iH^a - V(H^aH^a). \tag{13.48}$$

Each term, being positive define, must give a finite value as the result for integrating it over all space. In particular, V requires us to have that H approaches a constant value, h at infinity. We can think of the choice of $H(x)$ at infinity as a map from the sphere at infinity to the vacuum manifold. This map can in fact wind the S^2 of H-vacua around the S^2 at infinity k times, where k is an integer. (The fancy way of saying this is that $\pi_2(SU(2)/U(1)) = \mathbb{Z}$.) This will give a stable solution whose magnetic charge will turn out to be a fixed number times k.

A standard choice for $k = 1$ is that of 't Hooft and Polyakov[214]

$$\text{as} \quad r \longrightarrow \infty, \quad H_i \longrightarrow h\frac{x_i}{r^2} \quad \text{and} \quad A_{ai} \longrightarrow \epsilon_{aij}\frac{x^j}{r^2}, \tag{13.49}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$. To seek lowest energy configurations, a spherical ansatz

$$H = \hat{x}_i\sigma_i hF(r); \qquad A_{ai} = \epsilon_{aij}\frac{\hat{x}_j}{r}G(r), \tag{13.50}$$

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where finite energy requires that $F(r)$ and $G(r)$ tend to unity as $r \to \infty$ and $F'(r) \to 0$ in the limit.

The gauge field strength can then be seen to go at infinity as

$$F_{aij} = \frac{\epsilon_{ijk}x_k x_a}{r^4}.$$

If we pick a gauge where the unbroken $U(1)$ is in the three-direction, we see that, since the magnetic field vector is

$$B_i = \frac{1}{2}\epsilon_{ijk}F^{jk},$$

the magnetic charge is given by

$$g = \frac{1}{4\pi}\int_{S^2} B_i dS_i = \frac{1}{e}, \tag{13.51}$$

which fixes the relation between the magnetic charge and the winding solution. It is possible to show that higher winding, which is a topological invariant, will simply give integers times $1/e$, but we will not do that here, and refer the reader instead to the literature.

A special class of solutions to this model are the *Bogomol'nyi–Prasad–Summerfeld* solutions[61, 62], which have the smallest energy that such a solution can posses, the lower bound being set by the magnetic charge and the potential V. In supersymmetric cases, V actually vanishes, and we recover the familiar situation which we have been seeing all through this book, which is a supersymmetric solution whose mass is essentially equal to its charge (in appropriate units). See insert 13.1 for a discussion.

13.5.2 BPS monopole solution from Nahm data

In fact, we can construct the Higgs field and gauge field of BPS monopole solutions of the 3+1 dimensional gauge theory directly from the Nahm data as follows. Given $k\times k$ Nahm data $(\Phi^1, \Phi^2, \Phi^3) = 2\pi\alpha'(T_1, T_2, T_3)$ solving equation (13.42), there is an associated differential equation for a $2k$ component vector $\mathbf{v}(s)$:

$$\left\{\mathbf{1}_{2N}\frac{d}{ds} + \left(\frac{x^a}{2}\mathbf{1}_k + iT_a\right)\otimes\sigma^a\right\}\mathbf{v} = 0.$$

There is a unique solution normalisable with respect to the inner product

$$\langle \mathbf{v}_1, \mathbf{v}_2\rangle = \int_{-1}^{1} \mathbf{v}_1^\dagger \mathbf{v}_2 ds.$$

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Insert 13.1. The prototype BPS object

Let us see why BPS monopoles are BPS in the sense that we have been using in many places in this book. This is in fact the *original* BPS solution. The energy density that we presented in equation (13.48) can be written in a suggestive way:

$$\mathcal{E} = \frac{1}{4}\left(F_{aij} \pm \epsilon_{ijk}D_kH_a\right)^2 \pm \epsilon_{ijk}F_{aij}D_kH_a + V(H),$$

as can be checked by direct reexpansion (we've just completed the square). The second term in this form can be written as

$$\pm\epsilon_{ijk}D_k(F_{aij}H_a) = \pm\frac{1}{2}\epsilon_{ijk}\partial_k(F_{aij}H_a),$$

where we have used the Bianchi identity for the electromagnetic field strength. Since we are interested in the total energy, observe that if we integrate the second term, we get:

$$\pm\frac{1}{2}\epsilon_{ijk}\int d^3x\partial_k(F_{aij}H_a) = \pm\frac{1}{2}\epsilon_{ijk}\int_{S^2} F_{aij}H_a dS_k,$$

but this just integrates the magnetic field at infinity, in the $U(1)$ picked out by H^a, (which we can choose, as before, to be in the three-generator), and so the integral gives $\pm 4\pi gh$. Since all of the other terms are manifestly positive, we have the bound on the total energy

$$E \geq 4\pi|g|h.$$

Well, it is easy to see how to saturate this bound. We can make the first term vanish with the *Bogomol'nyi condition*:

$$F_{aij} = \mp\epsilon_{ijk}D_kH_a,$$

and then choose to make V as small as we can, which means that $\lambda \ll e$. In fact, we know that in supersymmetric cases, we have that V vanishes for supersymmetry preserving vacua, and so we can saturate the bound precisely in such a situation, giving an energy (mass) for the monopole which is equal to the charge, in appropriate units. Actually, putting the condition above directly into the ansatz (13.50) gives a soluble first order differential equation, with solution written in equations (13.54) and (13.55). We will obtain this solution directly from Nahm data in the next section.

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In fact, the space of normalisable solutions to the equation is four dimensional, or complex dimension two. Picking an orthonormal basis $\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2$, we construct the Higgs and gauge potential as:

$$\begin{aligned}\mathbf{H} &= i\begin{bmatrix}\langle s\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_1\rangle & \langle s\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2\rangle \\ \langle s\widehat{\mathbf{v}}_2, \widehat{\mathbf{v}}_1\rangle & \langle s\widehat{\mathbf{v}}_2, \widehat{\mathbf{v}}_2\rangle\end{bmatrix}, \\ A_i &= \begin{bmatrix}\langle \widehat{\mathbf{v}}_1, \partial_i\widehat{\mathbf{v}}_1\rangle & \langle \widehat{\mathbf{v}}_1, \partial_i\widehat{\mathbf{v}}_2\rangle \\ \langle \widehat{\mathbf{v}}_2, \partial_i\widehat{\mathbf{v}}_1\rangle & \langle \widehat{\mathbf{v}}_2, \partial_i\widehat{\mathbf{v}}_2\rangle\end{bmatrix}.\end{aligned} \tag{13.52}$$

The reader may notice a similarity between this means of extracting the gauge and Higgs fields, and the extraction (13.36)(13.37) of the instanton gauge fields in the previous section. This is not an accident. The Nahm construction is in fact a hyper-Kähler quotient which modifies the ADHM procedure. The fact that this arrangement of branes is T-dual to that of the p-$(p+4)$ system is the physical realisation of this fact, showing that the basic families of hypermultiplet fields upon which the construction is based (in the brane context) are present here too.

It is worth studying the case $k = 1$, for orientation, and since we can get an exact solution for this value. In this case, the solutions T_i are simply real constants $(2\pi\alpha')\Phi_i = -ia_i/2$, having the meaning of the position of the monopole at $\mathbf{x} = (a_1, a_2, a_3)$. Let us place it at the origin. Furthermore, as this situation is spherically symmetric, we can write $\mathbf{x} = (0, 0, r)$. Writing components $\mathbf{v} = (w_1, w_2)$, we get a pair of simple differential equations with solution

$$w_1 = c_1 e^{-rs/2}, \quad w_2 = c_2 e^{rs/2}. \tag{13.53}$$

An orthonormal basis is given by

$$\widehat{\mathbf{v}}_1 \ : \left(c_1 = 0, c_2 = \sqrt{\frac{r}{e^{2r}-1}}\right); \quad \widehat{\mathbf{v}}_2 \ : \left(c_2 = 0, c_1 = \sqrt{\frac{r}{1-e^{-2r}}}\right)$$

and the Higgs field is simply:

$$\begin{aligned}\mathbf{H}(r) &= \hat{x}_i\sigma_i\frac{\varphi(r)}{r}, \quad \text{with} \\ \varphi(r) &= \frac{r}{(e^{2r}-1)}\int_{-1}^{1} s e^{rs} ds = r\coth r - 1,\end{aligned} \tag{13.54}$$

while the gauge field is:

$$A_i(r) = \epsilon_{ijk}\sigma_j\hat{x}_k\frac{\sinh r - r}{r^2\sinh r}. \tag{13.55}$$

This is the standard one-monopole solution of Bogomol'nyi, Prasad and Sommerfield, the prototypical 'BPS monopole'[61, 62]. (See insert 13.1.)

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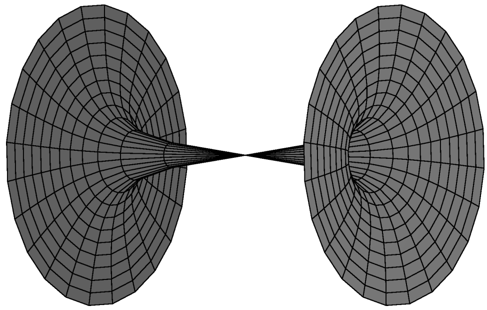

Fig. 13.9. A slice through part of two (horizontal) D3-branes with a (vertical) D1-brane acting as a single BPS monopole. This is made by plotting the Higgs field of the exact BPS solution.

We can insert the required dimensionful quantities:

$$\varphi(r) \to \varphi\left(Lr/4\pi\alpha'\right), \tag{13.56}$$

to get the Higgs field:

$$\mathbf{H} = \frac{\sigma_3}{r}\varphi\left(\frac{Lr}{4\pi\alpha'}\right) \longrightarrow \frac{L}{4\pi\alpha'}\sigma_3, \quad \text{as} \quad r \to \infty, \tag{13.57}$$

showing the asymptotic positions of the D3-branes to be $\pm L/2$, after multiplying by $2\pi\alpha'$ to convert the Higgs field (which has dimensions of a gauge field) to a distance in x^6. A picture of the resulting shape[59, 220] of the D3-brane is shown in figure 13.9.

There is also a simple generalisation of the purely magnetic solution which makes a 'dyon', a monopole with an additional n units of electric charge[221]. It interpolates between the magnetic monopole behaviour we see here and the spike electric solution we found in section 5.7. It is amusing to note[47] that an evaluation of the mass of the solution gives the correct formula for the bound state mass of a D1-string bound to n fundamental strings, as it should, since an electric point source is in fact the fundamental string.

13.6 The D-brane dielectric effect

13.6.1 Non-Abelian world-volume interactions

Consider the familiar non-Abelian term in the D-brane's world-volume action corresponding to the familiar scalar potential of the Yang–Mills theory. This of course appears in the Yang–Mills theory in the usual way, and can be thought of as resulting from the reduction of the ten dimensional Yang–Mills theory. It also arises as the leading part of the expansion

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of the $\det(Q^i_j)$ term in the non-Abelian Born–Infeld action, in the case when the brane is embedded in the trivial flat background $G_{\mu\nu} = \eta_{\mu\nu}$, as discussed in section 5.6:

$$V = \tau_p \,\mathrm{Tr}\sqrt{\det(Q^i{}_j)} = N\tau_p + \frac{\tau_p(2\pi\alpha')^2}{4}\mathrm{Tr}([\Phi^i, \Phi^j]\,[\Phi^j, \Phi^i]) + \cdots, \quad (13.58)$$

where $i = p+1, \ldots, 9$. As we have discussed in a number of cases before, the simplest solution extremising V is that the Φ^i all commute, in which case we can write them as diagonal matrices $\Phi^i = (2\pi\alpha')^{-1}X^i$, where $X^i = \mathrm{diag}(x^i_1, x^i_2, \ldots, x^i_N)$. The interpretation is that x^i_n is the coordinate of the nth Dp-brane in the X^i direction; we have N parallel flat Dp-branes, identically oriented, at arbitrary positions in a flat background, $\mathbb{R}^{9-p}$. The centre of mass of the Dp-branes is at $x^i_0 = \mathrm{Tr}(X^i)/N$. The potential is $N\tau_p$, which is simply the sum of all of the rest energies of the branes. We shall discard it in much of what follows.

When we look for situations with non-zero commutators, things become more complicated in interesting ways, giving us the possibility of new interesting extrema of the potential in the presence of non-trivial backgrounds. This is because the commutators appear in many parts of the worldvolume action, and in particular appear in couplings to the R–R fields, as we have seen in section 9.7. Furthermore, the background fields themselves depend upon the transverse coordinates X^i even in the Abelian case, and so will expected to depend upon their non-Abelian generalisation.

In general, this is all rather complicated, but we shall focus on one of the simpler cases as an illustration of the rich set of physical phenomena waiting to be uncovered[51]. Imagine that we have N Dp-branes in a constant background R–R $(p+4)$-form field strength $G_{(p+4)} = dC_{(p+3)}$, with non-trivial components:

$$G_{01\ldots pijk} \equiv G_{tijk} = -2f\varepsilon_{ijk} \quad i, j, k \in \{1, 2, 3\}\,. \quad (13.59)$$

(We have suppressed the indices $1, \ldots, p$, as there is no structure there, and will continue to do so in what follows.) Let the Dp-brane be pointlike in the directions x^i $(i = 1, 2, 3)$, and extended in p other directions. None of these Dp-branes in isolation is an electric source of this R–R field strength. Recall, however, that there is a coupling of the Dp-branes to the R–R $(p+3)$-form potential in the non-Abelian case, as shown in equation (9.34). We will assume a static configuration, choosing static gauge

$$\zeta^0 = t, \quad \zeta^\mu = X^\mu, \quad \text{for } \mu = 1, \ldots, p, \quad (13.60)$$

and get (see equation (9.34)):

$$
\begin{aligned}
&(2\pi\alpha')\mu_p \int \mathrm{Tr}\, P\,[i_\Phi i_\Phi C] \\
&\quad = (2\pi\alpha')\mu_p \int dt \mathrm{Tr}\left[\Phi^j\Phi^i\left(C_{ijt}(\Phi,t) + (2\pi\alpha')C_{ijk}(\Phi,t)\, D_{\mathrm{t}}\Phi^k\right)\right].
\end{aligned}
$$

We can now do a 'non-Abelian Taylor expansion'[48, 253] of the background field about Φ^i. Generally, this is defined as:

$$
F(\Phi^i) = \sum_{n=0}^{\infty} \frac{(2\pi\alpha')^n}{n!}\Phi^{i_1}\dots\Phi^{i_n}\partial_{x^{i_1}}\dots\partial_{x^{i_n}}F(x^i)|_{x=0}, \tag{13.61}
$$

and so:

$$
\begin{aligned}
C_{ijk}(\Phi,t) &= C_{ijk}(t) + (2\pi\alpha')\Phi^k\partial_k C_{ijk}(t) \\
&\quad + \frac{(2\pi\alpha')^2}{2}\Phi^l\Phi^k\partial_l\partial_k C_{ijk}(t) + \cdots.
\end{aligned} \tag{13.62}
$$

Now since $C_{ijt}(t)$ does not depend on Φ^i, the quadratic term containing it vanishes, since it is antisymmetric in (ij) and we are taking the trace. This leaves the cubic parts:

$$
\begin{aligned}
&(2\pi\alpha')^2\mu_p \int dt\, \mathrm{Tr}\left(\Phi^j\Phi^i\left[\Phi^k\partial_k C_{ijt}(t) + C_{ijk}(t)\, D_{\mathrm{t}}\Phi^k\right]\right) \\
&\quad = \frac{1}{3}(2\pi\alpha')^2\mu_p \int dt\, \mathrm{Tr}\left(\Phi^i\Phi^j\Phi^k\right) G_{tijk}(t),
\end{aligned} \tag{13.63}
$$

after an integration by parts. Note that the final expression only depends on the gauge invariant field strength, $G_{(p+4)}$. Since we have chosen it to be constant, this interaction (13.63) is the only term that need be considered, since of the higher order terms implicit in equation (13.62) will give rise to terms depending on derivatives of G.

13.6.2 Stable fuzzy spherical D-branes

Combining equation (13.63) with the part arising in the Dirac–Born-Infeld potential (13.58) yields our effective Lagrangian. This is a static configuration, so there are no kinetic terms and so $\mathcal{L} = -V(\Phi)$, with

$$
V(\Phi) = -\frac{(2\pi\alpha')^2\tau_p}{4}\mathrm{Tr}([\Phi^i,\Phi^j]^2) - \frac{1}{3}(2\pi\alpha')^2\mu_p\mathrm{Tr}\left(\Phi^i\Phi^j\Phi^k\right) G_{tijk}(t). \tag{13.64}
$$

Let us substitute our choice of background field (13.59). The Euler–Lagrange equations $\delta V(\Phi)/\delta\Phi^i = 0$ yield

$$
[[\Phi^i,\Phi^j],\Phi^j] + f\varepsilon_{ijk}[\Phi^j,\Phi^k] = 0. \tag{13.65}
$$

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Now of course, the situation of N parallel static branes, $[\Phi^i, \Phi^j] = 0$ is still a solution, but there is a far more interesting one[51]. In fact, the non-zero commutator:

$$[\Phi^i, \Phi^j] = f\,\varepsilon_{ijk}\Phi^k, \tag{13.66}$$

is a solution. In other words we can choose

$$\Phi^i = -i\,\frac{f}{2}\,\Sigma^i \tag{13.67}$$

where Σ^i are any $N \times N$ matrix representation of the $SU(2)$ algebra (we have chosen a non-standard normalisation for convenience)

$$[\Sigma^i, \Sigma^j] = 2i\,\epsilon_{ijk}\,\Sigma^k. \tag{13.68}$$

The $N \times N$ irreducible representation of $SU(2)$ has

$$(\Sigma^i)^2 = \frac{1}{3}(N^2-1)\mathbf{I}_{N\times N} \quad \text{for } i = 1, 2, 3, \tag{13.69}$$

where $\mathbf{I}_{N\times N}$ is the identity. Now the value of the potential (13.64) for this solution is

$$V_N = -\frac{\tau_p(2\pi\alpha')^2 f^2}{6}\sum_{i=1}^{3}\mathrm{Tr}[(\Phi^i)^2] = -\frac{(2\pi)^{-p+2}\alpha'^{\frac{3-p}{2}} f^4}{12g}N(N^2-1). \tag{13.70}$$

So our non-commutative solution solution has *lower energy* than the commuting solution, which has $V = 0$ (since we threw away the constant rest energy). This means that the configuration of separated Dp-branes is unstable to collapse to the new configuration.

What is the geometry of this new configuration? Well, the Φs are the transverse coordinates, and so we should try to understand their geometry, despite the fact that they do not commute. In fact, the choice (13.67) with the algebra (13.68) is that corresponding to the non-commutative or 'fuzzy' two-sphere[252] . The radius of this sphere is given by

$$R_N^2 = (2\pi\alpha')^2\frac{1}{N}\sum_{i=1}^{3}\mathrm{Tr}[(\Phi^i)^2] = \pi^2\alpha'^2 f^2(N^2-1), \tag{13.71}$$

and so at large N: $R_N \simeq \pi\alpha' f N$. The fuzzy sphere construction may be unfamiliar, and we refer the reader to the references for the details[252]. It suffices to say that as N gets large, the approximation to a smooth sphere improves.

Note that the *irreducible* $N \times N$ representation is not the only solution. A reducible $N \times N$ representation can be made by direct product

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of k smaller irreducible representations. Such a representation gives a $\mathrm{Tr}[(\Sigma^i)^2]$ which is less than that for the irreducible representation (13.66), and therefore yields higher values for their corresponding potential. Therefore, these smaller representations representations, corresponding geometrically to smaller spheres, are unstable extrema of the potential which again would collapse into the single large sphere of radius R_N. It is amusing to note that we can adjust the solution representing an sphere of size n by

$$\Phi^i = -i\frac{f}{2}\,\Sigma^i_n + x^i\,\mathbf{I}_{n\times n}. \tag{13.72}$$

This has the interpretation of shifting the position of its centre of mass by x^i.

What we have constructed is a D$(p+2)$-brane with topology $\mathbb{R}^p \times S^2$. The $\mathbb{R}^p$ part is where the N Dp-branes are extended and the S^2 is the fuzzy sphere. There is no net D$(p+2)$-brane charge, as each infinitesimal element of the spherical brane which would act as a source of $C_{(p+3)}$ potential has an identical oppositely oriented (and hence oppositely charged) partner. There is therefore a 'dipole' coupling due to the separation of these oppositely oriented surface elements. This type of construction is useful in matrix theory (described in chapter 16), where one can construct for example, spherical D2-brane backgrounds in terms of N D0-branes variables[253, 254, 255].

13.6.3 Stable smooth spherical D-branes

One way[59, 51] to confirm that we have made a spherical brane at large N, is to *start* with a spherical D$(p+2)$-brane, (topology $\mathbb{R}^p\times S^2$) and bind N Dp-branes to it, aligned along an $\mathbb{R}^p$. We can then place it in the background R–R field we first thought of and see if the system will find a static configuration keeping the topology $\mathbb{R}^p \times S^2$, with radius R_N. Failure to find a non-zero radius as a solution of this probe problem would be a sign that we have not interpreted our physics correctly.

Let us write the ten dimensional flat space metric with spherical polar coordinates on the part where the sphere is to be located (x^1, x^2, x^3):

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \sum_{i=4}^{9}(dx^i)^2. \tag{13.73}$$

Our constant background fields in these coordinates is (again, suppressing the $1,\dots,p$ indices):

$$G_{tr\theta\phi} = -2fr^2\sin\theta \quad \text{and so} \quad C_{t\theta\phi} = \frac{2}{3}fr^3\sin\theta. \tag{13.74}$$

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As we have seen many times before, N bound Dp-branes in the D$(p+2)$-brane's world-volume correspond to a flux due to the coupling:

$$(2\pi\alpha')\mu_{p+2}\int_{\mathcal{M}^3} C_{(p+1)}\wedge F = \frac{\mu_p}{2\pi}\int dt\, C_{(p+1)}\wedge F, \tag{13.75}$$

where $C_{(p+1)}$ is the R–R potential to which the Dp-branes couple, and is not to be confused with the $C_{(p+3)}$ we are using in our background, in (13.74). We need exactly N Dp-branes, so let us determine what F-flux we need to achieve this. If we work again in static gauge, with the D$(p+2)$-brane's world-volume coordinates in the interesting directions being:

$$\zeta^0 = t, \quad \zeta^1 = \theta, \quad \zeta^2 = \phi, \tag{13.76}$$

then

$$F_{\theta\phi} = \frac{N}{2}\sin\theta, \tag{13.77}$$

is correctly normalised magnetic field to give our desired flux.

We now have our background, and our N bound Dp-branes, so let us seek a static solution of the form

$$r = R \quad \text{and } x^i = 0, \text{ for } i = 4,\ldots,9. \tag{13.78}$$

The world volume action for our D$(p+2)$-brane is:

$$S = -\tau_{p+2}\int dt\, d\theta\, d\phi\, e^{-\Phi}\det{}^{\frac{1}{2}}(-G_{ab}+2\pi\alpha' F_{ab}) + \mu_{p+2}\int C_{(p+3)}. \tag{13.79}$$

Assuming that we have the static trial solution (13.78), inserting the fields (13.74), a trivial dilaton, and the metric from (13.73), the potential energy is:

$$\begin{aligned} V(R) &= -\int d\theta\, d\phi\, \mathcal{L} \\ &= 4\pi\tau_{p+2}\left(\left[R^4 + \frac{(2\pi\alpha')^2N^2}{4}\right]^{\frac{1}{2}} - \frac{2f}{3}R^3\right) \\ &= N\tau_p + \frac{2\tau_p}{(2\pi\alpha')^2N}R^4 - \frac{4\tau_p}{3(2\pi\alpha')}fR^3 + \cdots. \end{aligned} \tag{13.80}$$

In the above we expanded the square root assuming that $2R^2/(2\pi\alpha' N) \leq 1$, and kept the first two terms in the expansion. As usual we have substituted $\tau_p = 4\pi^2\alpha'\tau_{p+2}$.

The constant term in the potential energy corresponds to the rest energy of N Dp-branes, and we discard that as before in order to make our comparison. The case $V = 0$ corresponds to $R = 0$, the solution representing flat Dp-branes. Happily, there is another extremum:

$$R = R_N = \pi\alpha' f N \quad \text{with } V = -\frac{(2\pi)^{-p+2}\alpha'^{\frac{3-p}{2}} f^4}{12 g_s} N^3.$$

To leading order in $1/N$, we see that we have recovered the radius (and potential energy) of the non-commutative sphere configuration which we found in equations (13.71) and (13.70). It is appropriate that it only matches at large N, since the fuzzy geometry only approximates a smooth one in this limit.

As noted before, this spherical D$(p+2)$-brane configuration carries no net D$(p+2)$-brane charge, since each surface element of it has an antipodal part of opposite orientation and hence opposite charge. However, as the sphere is at a finite radius, there is a finite dipole coupling.

There is one major limitation of this whole discussion which is worth remarking upon. There is no such solution as a constant flux in flat space. So the analysis above need to be taken with a pinch of salt. Actually, this sort of brane expansion mechanism was anticipated in an earlier supergravity study before the identification of the precise world-volume mechanisms behind it[59] and so it is worthwhile revisiting the supergravity technology. A flux would create a gravitational back-reaction and so the flat metric that we've been using should really be replaced by some other metric. The prototype such solution of four dimensional Einstein–Maxwell gravity is the Melvin solution[224], which has an infinite magnetic flux threading a four dimensional universe.

This is the sort of solution which we need, with the flux identified with the R–R sector. There is a Kaluza–Klein version of the Melvin solution [225] and this fact has been used[226] to make a $C^{(1)}$ R–R flux solution of type IIA using a reduction from eleven dimensions, and other related solutions. Doing this with a twist allows one to include M5-branes, which upon reduction give a solution representing D4-branes expanding dielectrically into D6-branes via the dielectric mechanism in the magnetic $C^{(1)}$ flux.

All of this that we have described here is the D-brane analogue[51] of the dielectric effect in electromagnetism. If we place Dp-branes in a background R–R field under which the Dp-branes would normally be regarded as neutral, the external field 'polarises' the Dp-branes, making them puff out into a (higher dimensional) non-commutative world-volume geometry. Just as in electromagnetism, where an external field may induce a separation of charges in neutral materials, the D-branes respond through the production of electric dipole and possibly higher multipole couplings via the non-zero commutators of the world-volume scalars.

There is clearly a rich set of physical phenomena to be uncovered by considering non-commuting Φs. Already there have been applications of this mechanism to the understanding of a number of systems, such as large N gauge theory via the AdS/CFT correspondence and other gauge/gravity duals.[256]

The phenomena of branes being able to deform their shape and change their dimensionality, turning into other branes, etc., is a very important direction to explore further. This represents a quite mature physical arena, but currently we are limited to only a few solutions, and quite indirect description. It is possible that we need a whole new language to efficiently describe this physics, which may well be formulated directly in terms of non-commutative variables at the outset.

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