PRIMITIVE IDEMPOTENT MEASURES ON COMPACT SEMITOPOLOGICAL SEMIGROUPS

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1. Introduction

For a semigroup S let $I(S)$ be the set of idempotents in S. A natural partial order of $I(S)$ is defined by $e \leqq f$ if $ef = fe = e$. An element e in $I(S)$ is called a *primitive idempotent* if e is a minimal non-zero element of the partially ordered set $(I(S), \leqq)$. It is easy to see that an idempotent e in S is primitive if and only if, for any idempotent f in S, $f = ef = fe$ implies $f = e$ or f is the zero element of S. One may also easily verify that an idempotent e is primitive if and only if the only idempotents in eSe are e and the zero element. We let $\Pi(S)$ denote the set of primitive idempotent in S.

In what follows S is a compact semitopological semigroup (i.e. the multiplication is separately continuous). Let $P(S)$ denote the set of probability measures on S. Then $P(S)$ forms a compact semitopological semigroup under convolution and the weak* topology (see, for example, [1] and [6]).

Primitive idempotent probability measures on compact semigroups have been studied by several authors, for example, in [2], [3] and [8]. Some intrinsic characterizations of primitive idempotent measures on various classes of compact semigroups may be found in [5]. In this paper, we shall give some characterizations of primitive idempotent measures in $P(S)$ and indicate how some results in compact semigroups continue to hold in the semitopological case.

It is known that every compact semitopological semigroup S has a minimal (two-sided) ideal $K(S)$. For $\mu \in P(S)$ we write supp μ for the support of μ. Then for μ, ν in $P(S)$, we have

$$\text{supp}\, \mu\nu = \overline{(\text{supp}\, \mu)(\text{supp}\, \nu)};$$

where the bar denotes the closure see, for example, [6] and [10]. For μ in $I(P(S))$ we denote the minimal/ideal of the compact semitopological semigroup supp μ by K_μ.

2. The minimal ideal of $P(S)$

In this section we shall describe the measures in $K(P(S))$. It turns out that there is a close relation between $\Pi(P(S))$ and $K(P(S))$ (see § 3). As the methods

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we use are familiar, we shall only sketch the proofs.

Let μ be in $I(P(S))$ and let e be an idempotent in $K_\mu \cap K(S)$. Then $E_\mu = I(K_\mu e)$, $F_\mu = I(eK_\mu)$, $E = I(K(S)e)$, $F = I(eK(S))$ are compact subsemigroups and $G_\mu = eK_\mu e$, $G = eK(S)e$ are compact groups. Pym [10] decompose μ as

$$\mu = \mu_E \mu_G \mu_F,$$

where μ_E has support E_μ, μ_G is the (normalized) Haar measure of G_μ, and μ_F has support F_μ.

LEMMA 1. *Let* $\mu \in I(P(S))$ *and let* $\operatorname{supp} \mu \cap K(S) \neq \emptyset$. *Then*
(1) *there is an idempotent in* $K_\mu \cap K(S)$.
(2) $\operatorname{supp} \mu \subset \bar{K}(S)$.

PROOF. (1) Suppose $\operatorname{supp} \mu \cap K(S) \neq \emptyset$. Then $\operatorname{supp} \mu \cap K(S)$ is an ideal in $\operatorname{supp} \mu$. Hence $\operatorname{supp} \mu \cap K(S)$ contains K_μ and so contains an idempotent e (say) in K_μ (see [1] II 3.4). Therefore $e \in K_\mu \cap K(S)$.

(2) Since $K(S) \supset \operatorname{supp} \mu \cap K(S) \supset K_\mu$ and $\bar{K}_\mu = \operatorname{supp} \mu$ (see, for example, [10] Lemma 2), we see

$$\bar{K}(S) \supset \bar{K}_\mu = \operatorname{supp} \mu.$$

Let

$$H = \{\mu : \mu \in I(P(S)),\ \operatorname{supp} \mu \cap K(S) \neq \emptyset \text{ and } \mu_G \text{ is the Haar measure of } G\};$$

where we decompose μ with respect to an idempotent $e \in K_\mu \cap K(S)$.

LEMMA 2. *Let* $\mu \in H$, $\nu \in P(S)$. *Then* $\mu\nu\mu = \mu$.

PROOF. We note first that $\operatorname{supp} \mu\nu \subset \bar{K}(S)$ and that $eK(S)e = e\bar{K}(S)e$. Now

$$\mu\nu\mu = \mu(\mu\nu)\mu = \mu_E \mu_G \mu_F (\mu\nu) \mu_E \mu_G \mu_F.$$

One may easily verify that $\operatorname{supp}(\mu_F \mu\nu\mu_E)$ has support in G and so $\mu_F \mu\nu\mu_E$ is annihilated by μ_G. Therefore $\mu\nu\mu = \mu$.

LEMMA 3. *H is an ideal in* $P(S)$.

PROOF. Let μ be in H and let ν be in $P(S)$. Then $\mu\nu, \nu\mu$ are idempotent measures. We prove that $\mu\nu$ is in H, the proof for the other is similar. Let $\tau = \mu\nu$. Then $\operatorname{supp} \tau \cap K(S) \neq \emptyset$. Let e be an idempotent in $K_\tau \cap K(S)$ and decompose τ as $\tau_E \tau_G \tau_F$ with respect to e. It is easy to see, by Lemma 2, that $\tau\lambda\tau = \tau$ for each λ in $P(S)$. Let m_G be the Haar measure of G and let δ_e be the unit point mass at e. Then,

$$\tau = \tau(\delta_e m_G \delta_e)\tau = \tau_E \tau_G \tau_F \delta_e m_G \delta_e \tau_E \tau_G \tau_F.$$

But now $\tau_F \delta_e$ and $\delta_e \tau_E$ have supports in G and so is annihilated by m_G. Therefore

$$\tau = \tau_E m_G \tau_F,$$

completing the proof.

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We can now describe the measures in the minimal ideal $K(P(S))$ of $P(S)$.

THEOREM 1. *Let μ be in $P(S)$. Then the following conditions are equivalent.*

(1) $\mu \in K(P(S))$.

(2) $\mu \in H$.

(3) $\mu P(S)\mu = \mu$

(4) $\mu^2 = \mu$ *and* $(\operatorname{supp} \mu)\, x\, (\operatorname{supp} \mu) = \operatorname{supp} \mu$ *for each x in S.*

We shall only prove (4) implies (2). It follows from the assumption $(\operatorname{supp} \mu)\, x\, (\operatorname{supp} \mu) = \operatorname{supp} \mu$ for each x in S that $\operatorname{supp} \mu \cap K(S) \neq \emptyset$. Let $\mu = \mu_E \mu_G \mu_F$ be the Pym's decomposition of μ with respect to an idempotent $e \in K_\mu \cap K(S)$. Let $\nu = \delta_e m_G \delta_e$; where m_G is the Haar measure of G. Now, by the assumption again, we see $\operatorname{supp} \mu\nu\mu \subset \operatorname{supp} \mu$. Hence, by ([10], Lemma 3),

$$\begin{aligned} \mu &= \mu(\mu\nu\mu)\mu = \mu\nu\mu \\ &= \mu_E \mu_G \mu_F \delta_e m_G \delta_e \mu_E \mu_G \mu_F = \mu_E m_G \mu_F. \end{aligned}$$

That is $\mu \in H$, completing the proof.

3. Central idempotents and primitive idempotents in $P(S)$

THEOREM 2. *Let μ be a central idempotent in $P(S)$. Then* $\operatorname{supp} \mu$ *is a compact group normal in S.*

PROOF. It follows from the centrality of μ and the separate continuity of multiplication that $x(\operatorname{supp} \mu) = (\operatorname{supp} \mu)x$ for each x in $\operatorname{supp} \mu$. Hence $x(\operatorname{supp} \mu)$ is an ideal of $\operatorname{supp} \mu$ and so $x(\operatorname{supp} \mu) \supset K_\mu$. Therefore $x(\operatorname{supp} \mu) \supset \overline{K}_\mu = \operatorname{supp} \mu$. On the other hand, $\operatorname{supp} \mu = (\operatorname{supp} \mu)(\operatorname{supp} \mu) \supset x(\operatorname{supp} \mu)$. We conclude that

$$\operatorname{supp} \mu = x(\operatorname{supp} \mu) = (\operatorname{supp} \mu)x$$

for each x in $\operatorname{supp} \mu$. That is, $\operatorname{supp} \mu$ is an algebraic group (this is implicit in the proof of ([7] Theorem 9.16) and so is a compact group (see [1] II 2.1).

COROLLARY. *$P(S)$ has a zero element if and only if $K(S)$ is a group.*

We omit the proof of the corollary, all we need is to point out that $K(S)$ is a group if and only if $K(S)$ is a compact group (see [1] II 4.16) and that the Haar measure m of $K(S)$ is the zero element of $P(S)$.

We now come to our description of the primitive idempotent measures. In the case when the minimal ideal of S is not a group, one may easily verify, by Theorem 1 and the above corollary, that $\Pi(P(S)) = K(P(S))$. It thus remains for us to discuss the case in which the minimal ideal of S is a group.

THEOREM 4. *Let $K(S)$ be a group and let μ be a non-zero idempotent in $P(S)$. Then the following conditions are equivalent.*

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(1) $\mu \in \Pi(P(S))$.

(2) *For each closed subsemigroup S' containing* $\operatorname{supp}\mu$, *the minimal ideal $K(S')$ of S' satisfies either*

  (i) $K(S') = K(S)$ *or*

  (ii) $\operatorname{supp}\mu \cap K(S') \neq \emptyset$ *and μ_G is the Haar measure of G, where we regard μ as an idempotent measure on S'.*

(3) *For each closed subsemigroup S' containing* $\operatorname{supp}\mu$, *the minimal ideal $K(S')$ of S' satisfies either*

  (i) $K(S') = K(S)$ *or*

  (ii) $\operatorname{supp}\mu \cap K(S') \neq \emptyset$ *and* $\overline{(\operatorname{supp}\mu)\,x\,(\operatorname{supp}\mu)} = \operatorname{supp}\mu$ *for each x in $K(S')$.*

(4) *Let S' be a closed subsemigroup such that $S' = \overline{K}(S')$ and that $S' = \overline{S'(\operatorname{supp}\mu)} = \overline{(\operatorname{supp}\mu)S'}$. Then $S' = K(S)$ or $S' = \operatorname{supp}\mu$.*

The first part of the proof follows closely the proof of this result for the compact semigroup case ([2] Theorem 4.6).

PROOF. (1) implies (2). Let $\mu \in \Pi(P(S))$ and let S' be a closed subsemigroup containing $\operatorname{supp}\mu$. Suppose first that $K(S')$ is a group. We denote by m' the Haar measure of the compact group $K(S')$ and regard it as a measure on S. Now by the assumption that $\operatorname{supp}\mu \subset S'$, we see $m' = m'\mu = \mu m'$. Therefore $m' = m$ (i.e. the zero of $P(S)$) or $m' = \mu$. Hence $K(S') = K(S)$ or $K(S') = \operatorname{supp}\mu$. If $K(S') = \operatorname{supp}\mu$, then since $K(S')$ is a group, we see that μ is the Haar measure of $K(S')$ and $\mu = \mu_G$.

Suppose next that $K(S')$ is not a group. Then the relation $\mu P(S')\mu \subset \mu P(S)\mu$ and the assumption that $K(S')$ is not a group combine to yield that the measure μ is primitive in $P(S')$. Hence, by the remarks before this theorem, μ is in $K(P(S'))$ and so, by Theorem 1, μ is in H. Thus (2) holds.

(2) implies (3). This follows from Theorem 1 immediately.

(3) implies (4). Suppose S' is a closed subsemigroup such that $S' = \overline{K}(S')$ and that $S' = \overline{S'(\operatorname{supp}\mu)} = \overline{(\operatorname{supp}\mu)S'}$. Let $S_0 = S' \cup \operatorname{supp}\mu$. Then S_0 is a closed subsemigroup containing $\operatorname{supp}\mu$. Since S' is an ideal of S_0, we see $S' \supset K(S_0)$. But now $K(S') \cap K(S_0) \supset K(S')K(S_0) \neq \emptyset$ implies that $K(S') \cap K(S_0)$ is an ideal of $K(S')$ and so $K(S') = K(S') \cap K(S_0) \subset K(S_0)$. It follows that $S' = \overline{K}(S') = \overline{K}(S_0)$. By assumption (3), $S' = \overline{K}(S_0) = \overline{K}(S) = K(S)$ or $\operatorname{supp}\mu \cap K(S_0) \neq \emptyset$ and $\overline{(\operatorname{supp}\mu)\,x\,(\operatorname{supp}\mu)} = \operatorname{supp}\mu$ for each x in $K(S_0)$. Suppose $\operatorname{supp}\mu \cap K(S_0) \neq \emptyset$ and $\overline{(\operatorname{supp}\mu)\,x\,(\operatorname{supp}\mu)} = \operatorname{supp}\mu$ for each x in $K(S_0)$. Then $\operatorname{supp}\mu \subset S'$. Now, by a simple result of ([1] II 3.1), we see

$$\operatorname{supp}\mu = \overline{(\operatorname{supp}\mu)K(S_0)(\operatorname{supp}\mu)}$$
$$\supset \overline{S'(\operatorname{supp}\mu)} = S'.$$

We conclude that $S' = \operatorname{supp}\mu$ and (3) implies (4).

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(4) implies (1). Suppose ν is an idempotent in $P(S)$ such that $\nu = \mu\nu = \nu\mu$. Then

$$\text{supp}\,\nu = \overline{(\text{supp}\,\mu)(\text{supp}\,\nu)} = \overline{(\text{supp}\,\nu)(\text{supp}\,\mu)}.$$

But now since supp $\nu = \bar{K}_\nu$, we see supp $\nu = K(S)$ or supp $\nu =$ supp μ. If supp $\nu = K(S)$, then $\nu = m$. If supp $\nu =$ supp μ then, by ([10] Lemma 3), $\mu = \mu\nu\mu = \nu\mu = \nu$, completing the proof.

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