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A UNIQUENESS THEOREM ON THE DEGENERATE CAUCHY PROBLEM[1]

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In [4] Carroll and the author have treated the following problem

$$u_{tt}+\Lambda^{\alpha}S(t)u_t+\Lambda^{\beta}R(t)u+\Lambda a(t)u+B(t)u = f(t) \tag{1}$$

where Λ is a closed densely defined self-adjoint operator in a separable Hilbert space H with $(\Lambda u, u)\geq c\,\|u\|^2$, $c>0$, $\Lambda^{-1}\in L(H)$ ($L(E, F)$ is the space of continuous linear maps from E to F; in particular, $L(H)=L(H, H)$), $a(t)>0$ for $t>0$, $a(0)=0$ and $S(t)$, $R(t)$, $B(t)\in L(H)$. Various other hypotheses will be given later. It is assumed that all operators commute, and we seek $u\in \mathscr{E}^2(H)$ ($\mathscr{E}^m(H)$ is the space of m-times continuously differentiable functions of t with values in H) satisfying (1) with

$$u(0) = u_t(0) = 0. \tag{2}$$

Existence and uniqueness theorems have been obtained for the problem (1)–(2) in [4]. The purpose of this paper is to prove a somewhat stronger uniqueness theorem for the problem. We also give examples that will be provided for differential equations to which the uniqueness theorem applies but which are not included in any previously known results (e.g. see [1, 4, 6, 8, 10]). A counter-example is also given in order to indicate that the function of degeneracy $a(t)$ can only admit oscillations under a suitable condition.

Notations and symbols in [4] are adopted here except for arbitrary constants. Indeed equations taken from [4] will be renumbered in order to make our paper as self-contained as possible.

In order to apply spectral methods we set that $S(t)=Ss(t)$, $R(t)=Rr(t)$, $B(t)=Bb(t)$, where B, R, S, Λ^{-1} commute and are bounded normal with $b, r, s\in c^0[0, l]$, $a\in c^1[0, l]$. Let $\mathscr{A}$ be the uniformly closed $*$ algebra generated by Λ^{-1}, B, R, S, B^*, R^*, S^* and I. Then we associate with these operators (other than I) the complex spectral variables $z_0, z_1, \ldots, z_6$ (see [3, 4]), $z_0=1/\lambda$ is real. Now the map $m:\Phi_{\mathscr{A}}\to \mathbb{C}^7$ given by $m(\phi)=(\hat{\Lambda}^{-1}(\phi), \hat{B}(\phi), \ldots, \hat{S}^*(\phi))$ is a homeomorphism of the carrier space $\Phi_{\mathscr{A}}$ with the joint spectrum σ of the elements $\Lambda^{-1}, B, \ldots, S^*$ (see [4, 9]). We consider now in connection with (1) the homogeneous equation

$$u_{tt}+\lambda^{\alpha}z_3s(t)u_t+\lambda^{\beta}z_2r(t)u+\lambda a(t)u+z_1b(t)u = 0. \tag{3}$$

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Solutions $Z(t, \tau, z_i, \lambda)$ and $Y(t, \tau, z_i, \lambda)$ of (3) with $Z(\tau, \tau)=1$, $Z_t(\tau, \tau)=0$, $Y(\tau, \tau)=0$, $Y_t(\tau, \tau)=1$ (cf. [2]) will give rise to operators in the von Neumann algebra $\mathscr{A}''$ if Y and Z are continuous in (z_i, λ) for $|z_i|\le c_1$ $(i=1, \ldots, 6)$, $|\lambda|\le R_0$ (R_0 arbitrary), and bounded for $|z_i|\le c_1$, $|z_0|\le 1/c$ (this has been proved in [3]). The constant c_1 is chosen so that $c_1\ge\max\{\|B\|, \|R\|, \|S\|\}$ and then the joint spectrum σ lies within the region $|z_i|\le c_1$ $(i=1, \ldots, 6)$, $|z_0|\le 1/c$ (note that $\lambda\to\infty$ corresponds to $z_0\to 0$).

It is known from classical results (see [5, 6]) that for $0\le\tau\le t\le l<\infty$ there exist unique Z and Y as required, continuous in (t, τ, z_i, λ) in the region $0\le\tau\le t\le l<\infty$, $|z_i|\le c_1$ $(i=1, \ldots, 6)$, $0<z_0\le 1/c$. However, Y and Z may not be analytic single-valued in z_i, λ, because α, β may be fractional. Thus by introducing the Green's operator associated with (3) (see [4, 10] for details) we obtain formally for the solution of (1)–(2):

$$u(t) = \int_\tau^t \mathscr{Y}(t, \xi)f(\xi)\, d\xi \tag{4}$$

where $\mathscr{Y}$ is the operator associating with Y.

For convenience, we now summarize what have been assumed (in [4]): (i) $a\in c^1[0, l]$, (ii) $s, r, b\in c^0[0, l]$, (iii) $P=a'+c_1(r^2+b^2/\lambda)\ge 0$, (iv) $\gamma=\max\{\alpha, \frac{1}{2}\}$, (v) $2\beta\le 1$, (vi) $\text{Re}(z_3 s(t))\ge 0$, (vii) $f/Q=h\in\mathscr{E}^0(D(\Lambda^\gamma))$ $(Q=a^{1/2}q=(a\phi\psi)^{1/2}$, with $\phi(\tau)=\exp(-c_1\int_\tau^l r^2/a\,d\xi)$, $\psi(\tau)=\exp(-(c_1/\lambda)\int_\tau^l b^2/a\,d\xi)$.

According to [4], the solution of (1) with values $u(\tau)$ and $u_t(\tau)$ prescribed, $0<\tau\le t$, can be written as follows:

$$u(t) = \mathscr{Z}(t, \tau)u(\tau)+\mathscr{Y}(t, \tau)u_t(\tau)+\int_\tau^t \mathscr{Z}(t, \xi)f(\xi)\, d\xi, \tag{5}$$

where $\mathscr{Z}$, $\mathscr{Y}$ are operators associating with Z, Y which are solutions of (3).

We recall that two uniqueness theorems were proved in [4]. One of them required strongly $u\in\mathscr{E}^2(H)$, $u/q\in\mathscr{E}^0(D(\Lambda^{\gamma+1/2}))$ and $u_t/Q\in\mathscr{E}^0(D(\Lambda^\gamma))$. On the other hand if $q>0$ a second uniqueness theorem had been given as Theorem 3 in page 254 of [4]. We now state it without proof as follows:

Theorem 3 of [4]. Assume u is a solution of (1)–(2) with $u\in\mathscr{E}^0(D(\Lambda^{\gamma+1/2}))\cap\mathscr{E}^1(D(\Lambda^\gamma))\cap\mathscr{E}^2(H)$ and let $\int r^2/a\,d\xi<\infty$, $\int b^2/a\,d\xi<\infty$ with $(\int_0^t a(\xi)\,d\xi)^\delta/a^{1/2}(t)$ continuous for some $\delta<\frac{1}{2}$. Then u is unique.

It is noted that the second uniqueness theorem of [4] stated above was based upon the following inequality:

$$\|u_t\| \le c_2\left(\int_0^t a(\xi)\, d\xi\right)^{1/2}. \tag{6}$$

One recalls that the degenerate Cauchy Problems of a similar form (e.g. see [7, 8]) were solved with $a(t)$ monotone (monotonicity of $a(t)$ is not necessarily required here or in [1, 4, 10]). However, from (6) with a monotone $a(t)$, it follows immediately that $(t \leq l < \infty)$

$$\|u_t\| \leq c_3 a^{1/2}(t). \tag{7}$$

This means that $\|u_t/a^{1/2}\|$ is bounded for every $t \in [0, l]$. Now $(t, \tau) \to a^{1/2}(\tau)\mathscr{Y}(t, \tau)$ is continuous with values in $L_S(H, D(\Lambda^{1/2}))$ (where the subscript S denotes the strong operator topology), and the term $\mathscr{Y}(t, \tau)u_t(\tau)$ in (5) may be written as

$$\mathscr{Y}(t, \tau)u_t(\tau) = a^{1/2}(\tau)\mathscr{Y}(t, \tau)\frac{u_t(\tau)}{a^{1/2}(\tau)}. \tag{8}$$

But $u_t(\tau)/a^{1/2}(\tau)$ is also continuous. Moreover, since $q>0$ $\mathscr{X}(t, \tau)u(\tau) \to 0$ as $\tau \to 0$. Actually, the above arguments would remain true if $a(t)$ were required to be locally monotone (i.e., $a(t)$ is monotone in a neighborhood of 0). Oscillations in $a(t)$ are then allowed.

In order to facilitate the subsequent arguments, we rewrite (6) as

$$\|u_t/a^{1/2}\| \leq c_2 A^{1/2}(t) \tag{9}$$

where

$$A(t) = \int_0^t a(\xi)\, d\xi / a(t). \tag{10}$$

It is easily seen that the boundedness of $A(t)$ on $[0, l]$ would assure uniqueness for the problem (1)–(2) in view of (8). In fact, if $a(t)$ is assumed to be locally monotone, $A(t)$ is obviously bounded on $[0, l]$. Consequently, we simply replace the condition $(\int_0^t a(\xi)\, d\xi)^\delta / a^{1/2}(t)$ being continuous for some $\delta < \frac{1}{2}$ by $a(t)$ being locally monotone in Theorem 3 of [4]. Thus we have

THEOREM. *Assume* (i), (ii), (iii), (iv), (v), (vi), (vii). *Let u be a solution of* (1)–(2) *with* $u \in \mathscr{E}^0(D(\Lambda^{\gamma+1/2})) \cap \mathscr{E}^1(D(\Lambda^\gamma)) \cap \mathscr{E}^2(H)$, $\int r^2/a d\xi < \infty$, $\int b^2/a d\xi < \infty$. *If* $a(t)$ *is locally monotone, then u is unique.*

It may be worthwhile to point out that in case $a'(0) \neq 0$ or $a(t) \sim kt^n$ (i.e., $\lim_{t\to 0} a(t)/t^n = k > 0$ for $n > 0$), $A(t)$ is evidently bounded. Indeed both cases are special cases of the theorem. For $a(t)$ being locally monotone, some class of non-monotone functions $a(t)$ (such as $a(t) = \exp(-1/t)\ [a_1 + a_2 t^3 \sin^2(1/t)]$, $a(t) = a_1 t^n + a_2 t^p \sin(1/t)$ with $a_1 \geq |a_2|, p > n > 0, p > 2$) would be included.

Finally, it is not hard to see that for a nonmonotone $a(t) = O(t^n)$ with $n > 0$, $A(t)$ may not be bounded; in particular, limsup $A(t) = \infty$. To this end, consider* the function $a_1(t)$ and $a_2(t)$ defined on [0, 1] as follows:

$$a_1(t) = t^3 \sin^2(1/t) + t^5 \cos^2(1/t),$$
$$a_2(t) = t^3 \cos^2(1/t) + t^5 \sin^2(1/t).$$

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* Professor A. Meir constructed these functions for the author when he attended the 1967 Summer Research Institute in Edmonton.

Clearly,

(12) $$a_i(t) \leq t^3, \qquad i = 1, 2$$

and

(13) $$a_1(t)+a_2(t) = t^3+t^5,$$

for all $t \in [0, 1]$. (13) implies that

$$\int_0^t \{a_1(\xi)+a_2(\xi)\}\, d\xi > \int_0^t \xi^3\, d\xi = t^4/4$$

for all $t \in [0, 1]$ and consequently

(14) $$\int_0^t a_1(\xi)\, d\xi > t^4/8$$

or

(15) $$\int_0^t a_2(\xi)\, d\xi > t^4/8$$

or both hold for infinitely many $t \in (0, 1]$. On the other hand it follows from (12) that

$$\int_0^t a_i(\xi)\, d\xi \leq t^4/4, \qquad i = 1, 2$$

for all $t \in [0, 1]$. Now consider two particular sequences $\{t_k\}$, $\{t_k'\}$ of (0, 1) where

$$t_k = 1/2k\pi, \qquad t_k' = 1/(2k-\tfrac{1}{2})\pi, \qquad k = 1, 2, \ldots.$$

(For simplicity, we use the same notations for their subsequences if applicable.)
Evidently,

$$a_1(t_k) = t_k^5, \qquad a_1(t_k') = t_k'^3;$$
$$a_2(t_k) = t_k^3, \qquad a_2(t_k') = t_k'^5.$$

Thus we obtain

(16) $$\frac{\int_0^{t_k'} a_1(\xi)\, d\xi}{a_1(t_k')} \leq \frac{t_k'^4/4}{t_k'^3} = t_k'/4$$

and

(17) $$\frac{\int_0^{t_k} a_2(\xi)\, d\xi}{a_2(t_k)} \leq \frac{t_k^4/4}{t_k^3} = t_k/4.$$

Now if (14) holds for infinitely many $t \in (0, 1]$ and in particular $\{t_k\}$ is considered, then

(18) $$\frac{\int_0^{t_k} a_1(\xi)\, d\xi}{a_1(t_k)} > \frac{t_k^4/8}{t_k^5} = 1/8t_k.$$

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Otherwise, (15) must hold for $\{t_k'\}$ so that we have

$$(19)\quad \frac{\int_0^{t_k} a_2(\xi)\,d\xi}{a_2(t_k')} > \frac{\int_0^{t_k} a_2(\xi)\,d\xi}{t_k'^5} > \frac{t_k^4/8}{t_k'^5} = (t_k/t_k')^4/8t_k' = [(2k-\tfrac{1}{2})/2k]^4/8t_k' \geq (3/4)^4/8t_k'$$

Noting (16) (or (17)) and (18) (or (19)), we conclude that $A(t)$ may be unbounded even if $a(t)=0(t^n)$ for some $n>0$.

The above counter-example shows that $a(t)$ can only admit oscillations conditionally. In the present case $A(t)$ must be bounded on $[0, l]$.

REMARK. The theorem may be regarded as somewhat "best possible" in the sense that the case $a(t)\sim t^n$(for $n>0$) is a special case of the theorem while the case $a(t)=0(t^n)$ (for $n>0$) cannot be generally included in the theorem (see counter-example above).

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