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ON SPROTT'S EXAMPLE OF AN ANCILLARY STATISTIC

BY
PETER TAN(1)

1. For statistical inference on the unknown parameter θ of a probability distribution with a known form of its density function $f(x \mid \theta)$, the late Sir R. A. Fisher suggests that inference should be based on the observed value of a sufficient statistic when it exists. In the absence of a sufficient statistic, fiducial probability statements about θ can still be made, according to Fisher, if we can find an ancillary statistic so that "the most likely estimate could be made exhaustive by means of the ancillary values" [1, p. 138].

Putting aside the much debated question if the fiducial argument is acceptable ("most observers have regarded the fiducial argument as essentially dead" writes Dempster [2, p. 368]), we still face the problem of how to discover an ancillary statistic for a given family of probability distributions.

Fisher illustrates the two types of inference problems mentioned in the first paragraph of this paper by using as examples in [1] the joint probability distribution of two independent univariate normal variables with known variance but unknown means θ and ϕ. He shows by geometrical arguments that if the point (θ, ϕ) in a plane lies in a straight line, a sufficient statistic exists while if the point (θ, ϕ) lies on a circle, an ancillary statistic can be found.

Sprott [3, 1961] considers the joint distribution of two independent sufficient statistics for normal and gamma distributions respectively and derives an ancillary statistic and the corresponding fiducial distribution for their common parameter. He further shows that the resulting fiducial distribution is the same distribution a posteriori obtained by Bayes' theorem.

Both Fisher's and Sprott's examples belong to a wide class of probability distributions with a natural group structure which Fraser [4] calls structural models. For these models classical probability distributions for the parameter, called structural probability distributions by Fraser, can be obtained without explicit use of sufficient or ancillary statistics. Structural probability distributions are obtained conditional on the orbit under the group of transformations, which is in fact the unique maximal invariant statistic with respect to the group G [5, p. 148].

In what follows we shall analyze Sprott's example by the structural method in [4]. It will be seen that the orbital statistic is exactly Sprott's ancillary statistic

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while the structural probability distribution of θ is identical with Sprott's distribution a posteriori obtained by Bayes' theorem.

2. Let T_1 and T_2 be two independent statistics with joint probability element

$$f(T_1, T_2 \mid \theta, \phi)\, dT_1\, dT_2 = L_1(T_1 \mid \theta) L_2(T_2 \mid \phi)\, dT_1\, dT_2$$

where

$$L_1(T_1 \mid \theta) = (2\pi n)^{-1/2} \exp[-(T_1 - n\theta)^2/2n], \quad -\infty < T_1 < \infty,$$
$$L_2(T_2 \mid \phi) = [(\phi T_2)^{m-1}/\Gamma(m)]\phi \exp[-\phi T_2], \quad T_2 > 0,$$
$$\phi = c \exp[k\theta], \quad -\infty < \theta < \infty.$$

Sprott uses the transformations

$$T_1 = nU_2, \qquad T_2 = \exp[-k(U_1 + U_2)]$$

and shows that $U_1 = -T_1/n - (\ln T_2)/k$ is an ancillary statistic.

To apply the structural method, we make the following changes of variables. Let $T_3 = \ln T_2$, $-\tau = \ln \phi = \ln c + k\theta$. Then the probability element of T_3 is

$$L(T_3 \mid \tau)\, dT_3 = [1/\Gamma(m)] \exp[m(T_3 - \tau) - e^{T_3 - \tau}]\, dT_3.$$

Let $\gamma = n\theta$. Then $\tau = a\gamma + b$, where $a = -k/n$, $b = -\ln c$.

The probability distribution of $T = (T_1, T_3)$ with probability element

$$f(T)\, dT = L_1(T_1 \mid \gamma/n) L_3(T_3 \mid \tau)\, dT_1\, dT_3$$

is easily seen to belong to a structural model described by an error variable $E = (E_1, E_3)$ with the known probability element

$$\begin{aligned} f(E)\, dE &= L_1(E_1 \mid \gamma = 0) L_3(E_3 \mid \tau = b)\, dE_1\, dE_3, \\ &= (2\pi n)^{-1/2} \exp[-E_1^2/2n]\, dE_1 \times [1/\Gamma(m)] \exp[m(E_3 - b) - e^{E_3 - b}]\, dE_3, \end{aligned} \tag{1}$$

and by a structural equation $T = g_\gamma E$, where g_γ is an element of a unitary group G of transformations of R^2 onto R^2:

$$G = \{g_t : -\infty < t < \infty\},$$
$$g_t T = g_t(T_1, T_3) = (T_1 + t, T_3 + at).$$

The orbit of T under G is defined to be the set

$$GT = \{g_t T : -\infty < t < \infty\}.$$

All points $E = (E_1, E_3)$ on the orbit GT have the common characteristic

$$E_3 - aE_1 = T_3 - aT_1 = D_3, \text{ say.}$$

$D = (D_1, D_3) = (0, D_3)$ represents the orbit GT.

The position of each point E in GT can be described by the position variable

$$[E] = E_1$$

since $[g_t E] = E_1 + t = [E] + t$ for all t.

The conditional distribution of $[E]=E_1$ given the orbit indexed by D or D_3 is easily found to have the probability element (see [4, p. 59]) directly obtained from (1):

$$\begin{aligned} g([E]\colon D)\, d[E] &= k(D) f(E)\, d[E] \\ &= K(D_3) \exp\left[-E_1^2/2n + m(D_3 + aE_1 - b) - e^{D_3 + aE_1 - b}\right] dE_1 \end{aligned}$$

where $k(D)$ and $K(D_3)$ are constants depending only on D or D_3.

From the structural equation $T = g_\gamma E$ we obtain

$$[T] = [E] + \gamma \quad \text{or} \quad T_1 = E_1 + \gamma = E_1 + n\theta,$$

which leads to the structural probability element for the parameter θ:

$$\begin{aligned} g(\theta\colon D_3) d\theta &= K(D_3) \exp\left[-(T_1 - n\theta)^2/2n + m(D_3 + aT_1 - an\theta - b)\right. \\ &\qquad \left. - e^{D_3 + aT_1 - an\theta - b}\right] n\, d\theta \\ &= K(T_1, T_2) \exp\left[-(T_1 - n\theta)^2/2n + mk\theta - c\, e^{k\theta} T_2\right] d\theta \end{aligned} \tag{2}$$

where $K(T_1, T_2)$ is the normalizing constant dependent on T_1 and T_2 only.

The final expression of $g(\theta\colon D_3)$ in (2) is identical with $b(\theta \mid T_1, T_2)$ obtained by Sprott as the posterior distribution of θ derived by Bayes' theorem. We note that $g(\theta\colon D_3)$ depends on the orbital statistic $D_3 = T_3 - aT_1 = \ln T_2 + (kT_1)/n$, and $-D_3/k = U_1$ is the ancillary statistic derived by Sprott.

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Carleton University,
Ottawa, Ontario