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Locally Compact Pro- C^* -Algebras

Massoud Amini

Abstract. Let X be a locally compact non-compact Hausdorff topological space. Consider the algebras $C(X)$, $C_b(X)$, $C_0(X)$, and $C_{00}(X)$ of respectively arbitrary, bounded, vanishing at infinity, and compactly supported continuous functions on X . Of these, the second and third are C^* -algebras, the fourth is a normed algebra, whereas the first is only a topological algebra (it is indeed a pro- C^* -algebra). The interesting fact about these algebras is that if one of them is given, the others can be obtained using functional analysis tools. For instance, given the C^* -algebra $C_0(X)$, one can get the other three algebras by $C_{00}(X) = K(C_0(X))$, $C_b(X) = M(C_0(X))$, $C(X) = \Gamma(K(C_0(X)))$, where the right hand sides are the Pedersen ideal, the multiplier algebra, and the unbounded multiplier algebra of the Pedersen ideal of $C_0(X)$, respectively. In this article we consider the possibility of these transitions for general C^* -algebras. The difficult part is to start with a pro- C^* -algebra A and to construct a C^* -algebra A_0 such that $A = \Gamma(K(A_0))$. The pro- C^* -algebras for which this is possible are called *locally compact* and we have characterized them using a concept similar to that of an approximate identity.

1 Multipliers of Pedersen's Ideal

The C^* -algebra of continuous functions vanishing at infinity has always been a source of motivations for many abstract aspects of the theory of C^* -algebras. Developing a measure theory for C^* -algebras, Gert K. Pedersen found a minimal dense ideal inside each C^* -algebra which plays the role of functions of compact support in the commutative case [Pd66]. We use this ideal and objects related to it frequently in the next section. Therefore we would like to give more details about this ideal here. G. K. Pedersen in his Ph.D. thesis in the mid 60's investigated a non-commutative measure theory for C^* -algebras [Pd64], [Pd66, I–IV]. Because most of the interesting measures are not finite, he had to choose one of the following options: either to consider (infinite) measures on the pure state space of the C^* -algebra or to regard measures as extended valued linear functionals (now called *weights*) on the C^* -algebra. The second approach seems more effective, but then he had to make sure that these weights would be finite on a fairly large subalgebra. The Pedersen ideal is one candidate. He proved that any C^* -algebra, A , has a dense two-sided ideal $K(A)$ which is minimal (indeed minimum) among all dense hereditary ideals of A , where hereditary means that any positive element of A majorized by an element of $K(A)$ actually belongs to $K(A)$ (Later it was proved that it is indeed minimal (minimum) among all dense ideals). Two classical examples are $A = C_0(X)$, the algebra of continuous functions vanishing at infinity, and $B = K(H)$, the algebra of compact operators, for which the Pedersen ideal is $K(A) = C_{00}(X)$, the continuous functions of compact support,

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and $K(B) = F(H)$, the algebra of finite rank operators. In general, one can explicitly construct the Pedersen ideal as follows. For a C^* -algebra A , let

$$(1) \quad K_0(A) = \{x \in A^+ : xy = x, \text{ some } y \in A^+\},$$

$$(2) \quad K_+(A) = \left\{ x \in A^+ : \exists n \geq 1, \exists x_1, \dots, x_n \in K_0(A) \ x \leq \sum_{k=1}^n x_k \right\}.$$

Then $K(A) = \text{span } K_+(A)$ is a minimal (the minimum) dense ideal of A . It is well known that there always exist an approximate identity of A inside any given dense ideal [Mur]. In particular there is an approximate identity of A consisting of elements of $K(A)$. Indeed in the *separable* case we can even choose the approximate identity inside $K_0(A)$. More precisely, any separable C^* -algebra A has a countable approximate identity (e_n) which is *canonical* in the sense that $e_n \geq 0$ and $e_n e_{n+1} = e_n$, for each n [CF]. In particular $e_n \in K_0(A)$ by definition.

Let J be a topological (complex) algebra with approximate identity. By a (*double*) *multiplier* (or *double centralizer*) of J we mean a pair (S, T) of functions from J to J such that

$$xS(y) = T(x)y,$$

for all $x, y \in J$. One can then show that S and T have to be linear and, respectively, *left* and *right multiplier* [LT, 2.10], *i.e.*,

$$S(xy) = S(x)y, \quad T(xy) = xT(y) \quad (x, y \in J).$$

We denote the set of all (left, right) multipliers of J by $\Gamma(J)$ ($\Gamma_\ell(J)$, $\Gamma_r(J)$, respectively). Then $\Gamma(J)$ is a vector space under the natural operations, and an algebra under the multiplication $(S, T)(U, V) = (SU, VT)$. If J is a $*$ -algebra, then so is $\Gamma(J)$, under the involution $(S, T)^* = (S^*, T^*)$, where $S^*(x) = S(x^*)^*$, for each $x \in J$. It is obvious that $\Gamma(J) \subseteq \Gamma_\ell(J) \times \Gamma_r(J)$. If J is complete, all three of these spaces could be identified with subspaces of J^{**} , and $\Gamma(J) = M(J) = \Gamma_\ell(J) \cap \Gamma_r(J)$. If J is a normed algebra with a contractive approximate identity, and $(S, T) \in \Gamma(J)$, then S is bounded if and only if T is bounded and $\|S\| = \|T\|$. If this is the case, we say that (S, T) is *bounded* and put $\|(S, T)\| = \|S\|$. We denote the set of all bounded (left, right) multipliers of J by $M(J)$ ($M_\ell(J)$, $M_r(J)$, respectively). If J is complete, then $\Gamma(J) = M(J)$.

Let A_0 be a C^* -algebra and $A_{00} = K(A_0)$ be its Pedersen ideal. We are particularly interested in $\Gamma(A_{00})$. One reason is that this algebra contains some unbounded elements which are important in functional analytic applications. $\Gamma(A_{00})$ has been extensively studied by A. J. Lazar and D. C. Taylor [LT] and N. C. Phillips [Ph88b]. We sketch some of the main results from these references.

Every bounded multiplier on A_{00} extends uniquely to a multiplier on A_0 . A_0 could be naturally embedded in $\Gamma(A_{00})$ by identifying $a \in A_0$ with $(S_a, T_a) \in \Gamma(A_{00})$ defined by

$$S_a(b) = ab, \quad T_a(b) = ba.$$

It was first pointed out by N. C. Phillips that $\Gamma(A_{00})$ could be represented as a projective limit of unital C^* -algebras. Indeed

$$\Gamma(A_{00}) = \varprojlim_I M(I),$$

where I runs over all closed two sided ideals of A_0 singly generated by elements of (an approximate identity contained in) A_{00}^+ [Ph88b].

There are different topologies on $\Gamma(A_{00})$. Lazar and Taylor introduced the κ -topology on $\Gamma(A_{00})$. This is the topology induced by the seminorms

$$(S, T) \mapsto \|S(a)\| \quad \text{and} \quad (S, T) \mapsto \|T(a)\|,$$

where a runs over A_{00} . A sequence $(x_i) \subseteq \Gamma(A_{00})$ converges to $x \in \Gamma(A_{00})$ if and only if $\|x_i a - x a\| \rightarrow 0$ and $\|a x_i - a x\| \rightarrow 0$, as $i \rightarrow \infty$, for each $a \in A_{00}$. Also by Phillips' result we have the *projective topology* induced by the norm topologies of C^* -algebras $M(I)$ on $\Gamma(A_{00})$. It follows from [Ph88b, Theorem 4] that the projective topology is stronger than the κ -topology.

It was already noticed by Lazar and Taylor that $\Gamma(A_{00})$ could be represented as an algebra of densely defined linear operators on a Hilbert space. We describe this in some more details. Let A_0 be a C^* -algebra which is non degenerately represented on a Hilbert space H (i.e., the closed linear span of $A_0 H = \{a\zeta : a \in A_0, \zeta \in H\}$ is equal to H). Let $H' = \text{span}(A_{00}H)$. This is clearly a dense subspace of H . We introduce an algebra of operators on H' as follows

$$B(A_{00}, H') = \{T \in L(H') : xT \text{ and } Tx \in B(H') \cap A_{00} \ (x \in A_{00})\}$$

where we have identified elements of $B(H')$ with their unique extension to H (This is from [LT], but they don't use this notation). This is obviously an algebra under the usual operations. Take $T \in L(H')$ and let T^* be the (possibly unbounded) adjoint of this densely defined operator restricted to H' . Let $x \in A_{00}$, $\zeta \in H$, and $\eta \in H'$, then $y = x^*T \in A_{00}$ and $\langle T\eta, x\zeta \rangle = \langle \eta, y^*\zeta \rangle$, so T^* is defined on the whole H' and $T^*(x\zeta) = (x^*T)\zeta$. Under this involution $B(A_{00}, H')$ becomes a $*$ -algebra. Then one can show that $\Gamma(A_{00}) \cong B(A_{00}, H')$ as $*$ -algebras [LT]. Indeed to each $u \in \Gamma(A_{00})$ there corresponds $\hat{u} \in L(H')$ defined by $\hat{u}(x\zeta) = (ux)\zeta$. Since there is an approximate identity for A_0 in A_{00} , \hat{u} is well defined. Now for each $x \in A_{00}$, $x\hat{u} + \hat{u}x$ is the restriction of $xu + ux$ to H' , and so $\hat{u} \in B(A_{00}, H')$. On the other hand, for each $T \in B(A_{00}, H')$, define $S(x)$ and $T(x)$ for $x \in A_{00}$ to be extensions of Tx and xT to H and observe that $\hat{u} = T$, for $u = (S, T) \in \Gamma(A_{00})$. It is now easy to check that $u \mapsto \hat{u}$ is a $*$ -isomorphism.

Let's make an immediate observation: If $\text{span}(A_{00}H) = H$, then $\Gamma(A_{00}) = M(A_0)$. Indeed, in this case, for each $T \in \Gamma(A_{00})$, T^* is defined everywhere, and so bounded. Therefore T has to be bounded also.

2 Locally Compact σ - C^* -Algebras

We were aiming at the problem of transitions between the categories of C^* -algebras and pro- C^* -algebras (For more details on the theory of C^* -algebras and pro- C^* -

algebras, we refer the reader to [Mur] and [Ph88c] and references therein, respectively). Before dealing with this problem, however, we should overcome a technical difficulty. Starting with a C^* -algebra A_0 , one can simply get the other three algebras by putting $A_{00} = K(A_0)$, $A_b = M(A_0)$, and $A = \Gamma(A_{00})$, but starting with a pro- C^* -algebra A it is not clear how we can appropriately associate a C^* -algebra A_0 to it (specially if we want them to be related by the relation $A = \Gamma(K(A_0))$). Indeed, even in the commutative case, this is not possible in general. The commutative unital σ - C^* -algebras are exactly the algebras $C(X)$ of all continuous functions on a countably compactly generated Hausdorff topological space X , with the compact-open topology [Ph88a, 5.7]. But X need not be locally compact and so $C_0(X)$ is not necessarily a C^* -algebra. However, we show that it is possible to distinguish a subcategory of σ - C^* -algebras for which one can make the desired transition. These are naturally called *locally compact σ - C^* -algebras*. This subcategory would contain all unital C^* -algebras.

Let's first consider the commutative case. For a (locally compact) topological space X , the relation between compactness type conditions on X and existence of special types of approximate identities in $C_0(X)$ has been extensively studied. Here we quote some of these results to motivate our approach. A countable approximate identity (e_n) in a Banach $*$ -algebra is called *well-behaved* if, for each n and each strictly increasing subsequence of indices (n_i) , there is $N \geq 0$ such that $e_n e_{n_k} = e_n e_{n_l}$ ($k, l \geq N$) [Ty72], [CD]. The compactness type conditions on a locally compact topological space X are related to the existence of certain approximate identities on $C_0(X)$. H. S. Collins and J. R. Dorroh showed that X is σ -compact if and only if $C_0(X)$ has a countable canonical (in the sense of Definition 2.5) approximate identity [CD]. Collins and Fontenot showed that X is paracompact if and only if $C_0(X)$ has a strictly totally bounded (canonical) approximate identity [CF], and conjectured that this is also equivalent to the condition that $C_0(X)$ has a well-behaved approximate identity (this was proved by R. F. Wheeler [Whe]). They also showed that if X is pseudo-compact and $C_0(X)$ has a well-behaved approximate identity, then X is compact. Later, Fontenot and Wheeler showed that X is paracompact if and only if $C_0(X)$ has a weakly compact approximate identity [FW].

Now let $A = \varprojlim_n A_n$ be a unital (with unit 1) σ - C^* -algebra and $\pi_n: A \rightarrow A_n$ be the corresponding homomorphisms and A_b denote the (unital) C^* -algebra of bounded elements of A , that is

$$A_b = \{a \in A : \sup_n \|\pi_n(a)\|_n < \infty\}.$$

Definition 2.1 An element $x \in A$ is called *strongly bounded* if $Ax \subseteq A_b$. We denote the set of all such elements by A_{sb} .

Then $A_{sb} \subseteq A_b$ is clearly a two-sided ideal of A .

Definition 2.2 An element $x \in A^+$ is called *multiplicatively bounded* (m.b.) if $Ax \subseteq A_b$. A sequence in A^+ is called *multiplicatively bounded* if all of its elements are multiplicatively bounded.

Note that in the above definition, since x is positive (and so self adjoint), then the above condition would imply $xA \subseteq A_b$. In particular $x^2 \in A_{sb}$.

Definition 2.3 Let A be a σ - C^* -algebra. A *support algebra* of A is a two-sided ideal A_{00} of A of the form

$$A_{00} = \bigcup_n Ae_n^2A,$$

where e_n 's are some given multiplicatively bounded elements of A^+ (compare with [Lin]).

Each support algebra of A is clearly a two sided ideal of A . Also it is always contained in A_{sb} . This ideal then induces a topology on A as follows.

Definition 2.4 The A_{00} -topology on A is the weakest topology such that the maps from A to A_b of the form

$$a \mapsto xa \quad \text{and} \quad a \mapsto xa,$$

are continuous for each $x \in A_{00}$, where A_b has its norm topology. A sequence $(a_i) \subseteq A$ converges to $a \in A$ in the A_{00} -topology if and only if $\|xa_i - xa\|_\infty \rightarrow 0$ and $\|a_ix - ax\|_\infty \rightarrow 0$, as $i \rightarrow \infty$, for each $x \in A_{00}$.

Proposition 2.1 Let (e_n) be a m.b. sequence in A and let A_{00} be the corresponding support algebra of A . Consider the following conditions:

- (1) (e_n) is an approximate identity of A in the A_{00} -topology,
- (2) $e_n \rightarrow 1$ in the A_{00} -topology,
- (3) (e_n) is an approximate identity of A_{00} in the norm topology.

Then (1) \Rightarrow (2) \Leftrightarrow (3).

Moreover if

$$\forall n \forall a \in A \quad \|\pi_n(a)\|_n \geq \|e_n a\|_\infty,$$

then every approximate identity in the projective topology is an approximate identity in the A_{00} -topology.

Proof The fact that (1) implies (2) is immediate (as $1 \in A$). The equivalence of (2) and (3) is just the definition of the A_{00} -topology. The last statement follows from the fact that condition mentioned above says that the projective topology is stronger than the A_{00} -topology. Indeed, if (x_i) be a sequence in A and $x_i \rightarrow x \in A$ in the projective topology, then for each n and $a, b \in A$ we have

$$\|ae_n^2b(x_i - x)\|_\infty \leq \|ae_n\|_\infty \|\pi_n(b)\|_n \|\pi_n(x_i - x)\|_n \rightarrow 0,$$

as $i \rightarrow \infty$. Now each element of A_{00} is a finite sum of elements of the form ae_nb , and we conclude that $x_i \rightarrow x$ in the A_{00} -topology. ■

Definition 2.5 Let $(e_n) \subseteq A^+$ be a m.b. sequence in A and let A_{00} be the corresponding support algebra. Then (e_n) is called *canonical* if $e_m e_n = e_n$, for each $n < m$.

Definition 2.6 A unital σ - C^* -algebra A is called *locally compact* if it has a presentation $A = \varprojlim_n A_n$ as an inverse limit of a countable family of C^* -algebras such that the corresponding morphisms $\pi_n: A \rightarrow A_n$ are surjective, and there is a multiplicatively bounded, canonical sequence $(e_n)_{n \in \mathbb{N}}$ with $0 \leq e_n \leq 1$ ($n \in \mathbb{N}$), such that if $A_{00} = \bigcup_n Ae_n^2A$ is the corresponding support algebra of A , then $e_n \rightarrow 1$ in the A_{00} -topology, A is complete in the A_{00} -topology, and it satisfies the following compatibility condition:

$$\forall n \forall a \in A \quad \sup\{\|e_nba\|_\infty : b \in A, \|e_nb\|_\infty \leq 1\} \geq \|\pi_n(a)\|_n \geq \|e_na\|_\infty.$$

Note that the above condition implies its right hand side version (i.e., with e_n multiplied from right). This follows from the fact that π_n is involutive and A is a $*$ -algebra.

In order to get a better idea about these definitions, let's look at some classical examples.

Example 2.1 Let $A = C(\mathbb{R})$ be the σ - C^* -algebra of all continuous functions on the real line. For each $n \geq 1$, let $A_n = C[-n, n]$ be the unital C^* -algebra of all continuous functions on the compact interval $[-n, n]$. For $m \geq n$ we have the morphism $\pi_{nm}: A_m \rightarrow A_n$ which sends a continuous function on $[-m, m]$ to its restriction on $[-n, n]$. Then (A_n, π_{nm}) is an inverse system of C^* -algebras with

$$C(\mathbb{R}) = \varprojlim_n C[-n, n],$$

where the morphism $\pi_n: C(\mathbb{R}) \rightarrow C[-n, n]$ is also defined by restriction. Then, for $f \in C(\mathbb{R})$, $\|\pi_n(f)\| = \|f|_{[-n, n]}\|$, so $\|f\|_\infty$ is nothing but the usual sup-norm. In particular, the set of bounded elements of A is exactly the (unital) C^* -algebra $A_b = C_b(\mathbb{R})$ of bounded continuous functions on \mathbb{R} . Also it is easy to see that the strongly bounded elements are exactly those of compact support, namely $A_{sb} = C_{00}(\mathbb{R})$. Also a sequence $(f_n) \subseteq C(\mathbb{R})$ converges to $f \in C(\mathbb{R})$ in the projective topology if and only if it uniformly converges to f on each interval $[-n, n]$. Since each compact subset of \mathbb{R} is already contained in one of these intervals, the projective topology is nothing but the topology of uniform convergence on compact sets (or the so called *compact-open* topology).

Now consider the sequence (e_n) in $C(\mathbb{R})$, where $0 \leq e_n \leq 1$ is the *bump function* which is 1 on $[1 - n, n - 1]$ and zero off $[-n, n]$. Let's observe that this is an approximate identity of $C(\mathbb{R})$ in the projective topology. Given $m \geq 1$ and $f \in C(\mathbb{R})$, we need to show that $e_n f$ converges to f uniformly on $[-m, m]$. But this becomes obvious when we note that $e_n = 1$ on $[-m, m]$ for $n \geq m + 1$. Now as e_n 's have compact support, this sequence is multiplicatively bounded. For each n , Ae_n consists of those elements of A whose support is inside the support of e_n , so the corresponding support algebra A_{00} of A is nothing but $C_{00}(\mathbb{R})$. Hence the A_{00} -topology is also the compact-open topology and the two topologies coincide. Also for $n > m$, e_n is 1 on the support of e_m , so $e_n e_m = e_m$, i.e., we have a canonical approximate identity. Moreover, given $f \in C(\mathbb{R})$ and $n \geq 1$, e_n is dominated by the characteristic function

of the interval $[-n, n]$, so in particular $\|fe_n\|_\infty \leq \|f|_{[-n,n]}\|_\infty$. The other inequality in the compatibility condition holds for e_n replaced by e_{n+1} (see the remark after Theorem 3.1).

Example 2.2 Let $A_n = \ell_n^\infty$ be the direct sum of n copies of \mathbb{C} and for $m \geq n$, let $\pi_{nm}: A_m \rightarrow A_n$ be the projection onto the first n components. This is an inverse system of C^* -algebras with the inverse limit being the algebra $A = c$ of all sequences. Then $\pi_n: A \rightarrow A_n$ is simply the projection onto the first n components. Also $A_b = \ell^\infty$ and A_{sb} is the set of all sequences with finitely many non zero components. The projective topology on A is just the topology of pointwise convergence; a sequence of sequences converges to a sequence in projective topology if and only if for each n the sequence of the n -th components converges to the n -th component in \mathbb{C} . Here $e_n = (1, \dots, 1, 0, 0, \dots)$ with the first n components 1 and the rest 0, forms a multiplicatively bounded canonical approximate identity of A in projective topology which satisfies the compatibility condition. The corresponding support algebra is $A_{00} = A_{sb}$ and the A_{00} -topology is the same as the projective topology.

Example 2.3 Let $A = B(\ell^2)$ be the C^* -algebra of all bounded operators on the separable Hilbert space ℓ^2 . This is a unital σ - C^* -algebra in a trivial way (it is the inverse limit of the inverse system consisting of only one unital C^* -algebra, namely $B(\ell^2)$ itself with the connecting homomorphism being identity!). In this case every element is (strongly) bounded, so $A_b = A_{sb} = A$. The projective topology coincides with the norm topology, so an approximate identity in projective topology is just a sequence which converges to I in norm. Since all elements are bounded in this case, every approximate identity is multiplicatively bounded. The trivial approximate identity consisting only of the identity operator I is canonical and satisfies the compatibility condition. This choice of approximate identity gives $A_{00} = A$ and the two topologies coincide.

There is, however, a more interesting choice of approximate identity. Take the presentation $B(\ell^2) = \varprojlim_n A_n$, where $A_n = B(\ell^2)$ and $\pi_{mn}: A_m \rightarrow A_n$ is the identity, for each $m, n \geq 1$. Then again $A_b = A_{sb} = A$ and the projective topology is the norm topology. For each n , put $e_n = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \in B(\ell^2)$. Then $0 \leq e_n \leq I$ and (e_n) is a canonical multiplicatively bounded sequence in $B(\ell^2)$. The corresponding support algebra $A_{00} = \bigcup_n B(\ell^2)e_nB(\ell^2) = F(\ell^2)$ is the algebra of finite rank operators. Hence the A_{00} topology is the strong*-topology on $B(\ell^2)$. Now note that (e_n) is an approximate identity of $B(\ell^2)$ only in this weaker topology. To see that the compatibility condition holds in this case, one only needs to observe that $\pi_n = \text{id}$ and, given an infinite matrix $a \in B(\ell^2)$, $e_n a$ is obtained from a by fixing the first n rows and making the rest zero.

Example 2.4 Let $M_n = M(n, \mathbb{C})$ be the unital C^* -algebra of all $n \times n$ complex matrices, and

$$A_n = M_{1,\dots,n} = \sum_{k=1}^n \oplus M_k,$$

be the C^* -algebra of all block matrices with increasing blocks of size from 1 to n .

Then for each $m \geq n$, the projection $\pi_{nm}: A_m \rightarrow A_n$ on the first n blocks of the top left hand corner is a morphism of C^* -algebras, and (A_n, π_{nm}) is an inverse system of C^* -algebras. The corresponding σ - C^* -algebra

$$A = M_{\mathbb{N}} = \varprojlim_n \sum_{k=1}^n \oplus \mathbb{M}_k = \prod_n \mathbb{M}_n,$$

is the algebra of all infinite block matrices with an increasing sequence of blocks of size $1, 2, \dots$. Also $\pi_n: A \rightarrow A_n$ is simply the projection onto the first n blocks. A typical element of A is of the form $M = (M_1, \dots, M_n, \dots)$, where $M_n \in \mathbb{M}_n$. Then $\pi_n(M) = (M_1, \dots, M_n)$ has norm $\sup_{1 \leq i \leq n} \|M_i\|$. Therefore A_b consists of those block matrices M for which $\sup_i \|M_i\| < \infty$. Also A_{sb} consists of the block matrices with only finitely many non zero blocks. A sequence of infinite block matrices converges in the projective topology to an infinite block matrix if and only if for each n , the n -th block of the elements of the sequence converges to the n -th block of that element in norm. If I_n denote the $n \times n$ identity matrix, then the sequence (e_n) , where $e_n = (I_1, I_2, \dots, I_n, 0, 0, \dots)$, forms a multiplicatively bounded, canonical approximate identity of A in the projective topology which satisfies the compatibility condition. Here $A_{00} = A_{sb}$ and the two topologies coincide.

One expects that commutative locally compact unital σ - C^* -algebras should be of the form $C(X)$ for a σ -compact, locally compact, (completely) Hausdorff topological space X . We show that this is true. Let's first see what is the general form of a commutative unital (not necessarily locally compact) σ - C^* -algebras. We know that each commutative unital σ - C^* -algebras is of the form $C(X)$ for some Hausdorff κ_ω -space X (recall that X is called a κ_ω -space, or *countably compactly generated*, if it is the union of an increasing sequence of compact subsets which determine the topology). Indeed the contravariant functor $X \mapsto C(X)$ is an equivalence of categories. Moreover a Hausdorff κ_ω -space is automatically completely regular (and so completely Hausdorff) [Ph88a, Proposition 5.7]. Also each κ_ω -space is σ -compact (by definition), but σ -compact κ -spaces are not κ_ω -spaces necessarily [Ph88a, 5.8].

Lemma 2.1 *If X is a σ -compact Hausdorff topological space and the σ - C^* -algebra $C(X)$ has an approximate identity in the pointwise convergence topology consisting of functions of compact support, then X is locally compact.*

Proof Let $(f_n) \subseteq C_{00}(X)$ be an approximate identity of $C(X)$ in the pointwise convergence topology. Let F_n be the support of f_n and let $V_n = \text{int}(F_n)$ be the interior of F_n . Since each V_n is an open set with compact closure, it is enough to show that $\bigcup_n V_n = X$. Let $x \in X$ and let $f \in C(X)$ be the constant function 1. Then $f_n(x) = (f_n f)(x) \rightarrow f(x) = 1$ as $n \rightarrow \infty$. Hence, there is n with $f_n(x) \geq \frac{1}{2}$, i.e., $x \in V_n$. This proves the claim and finishes the proof. ■

Lemma 2.2 *Let X be a completely regular topological space and let K be a compact subset of X . Let $f \in C(X)$ have the property that fg is bounded for each $g \in C(X)$ and $\|fg\|_\infty \leq \|g\|_\infty$. Then f has compact support.*

Proof Let $F = \text{cl}(V)$ be the support of f , where $V = \{x \in X : f(x) \neq 0\}$. Since X is Hausdorff, every closed subset of a compact set is compact, therefore we only need to show that $V \subseteq K$. Assume that, on the contrary, the open set $V \setminus K$ is nonempty. Choose $x \in V \setminus K$, then there is a function $g \in C(X)$ such that $g(x) = 1$ and $g = 0$ off $V \setminus K$ (X is completely regular). In particular, $g = 0$ on K and so $\|fg\|_\infty \leq \|g|_K\|_\infty = 0$. Hence $fg = 0$ everywhere. But $fg(x) = f(x)g(x) = f(x) \neq 0$, as $x \in V$, which is a contradiction. ■

Theorem 2.1 *Let X be a Hausdorff κ_ω -space. Then the commutative σ - C^* -algebra $C(X)$ is locally compact if and only if X is locally compact.*

Proof Let X be locally compact. We already know that $C(X) = \varprojlim_n C(K_n)$ for an increasing (ordered by inclusion) sequence (K_n) of compact subsets of X satisfying $X = \bigcup_n K_n$ with $\pi_n(f) = f|_{K_n}$ ($f \in C(X)$) [Ph88a]. Since X is locally compact we may assume that there are open sets V_n with compact closure such that $K_n \subseteq V_{n+1} \subseteq K_{n+1}$, for each n . (Take an open covering of K_n by open sets with compact closure and choose a finite subcover. Let K'_n be the closure of the union of the elements of this subcover, then K'_n 's are compact neighborhoods whose union is X . Next to make this sequence increasing, one can replace K'_n with $K''_n = \bigcup_1^n K'_i$). By the Urysohn lemma, for each $n \in \mathbb{N}$, there is an element $e_n \in C_{00}(X)$ such that $0 \leq e_n \leq 1$ and $e_n = 1$ on K_n and $e_n = 0$ off V_{n+1} . This is an approximate identity for $C_0(X)$ which satisfies all of the conditions in Definition 2.6, so $C(X)$ is a locally compact σ - C^* -algebra.

Conversely let $C(X) = \varprojlim_n C(K_n)$ be locally compact as a σ - C^* -algebra, where $\{K_n\}$ is an increasing sequence of compact subsets of X which determine the topology. Then $C(X)$ has an approximate identity in compact-open topology (and so in pointwise convergence topology) which satisfies the compatibility condition of Definition 2.6. Now X is a Hausdorff κ_ω -space and so it is also completely regular and σ -compact. Now by Lemma 2.2, each member of this approximate identity has compact support, and so X is locally compact by Lemma 2.1. ■

Now let A be a locally compact σ - C^* -algebra, let (e_n) be an approximate identity of A satisfying the conditions of Definition 2.6, and let A_{00} be the corresponding support algebra of A . Let A_0 be the norm closure of A_{00} in A_b . Then A_0 is a (possibly non unital) C^* -subalgebra of A_b . We want to show that indeed $A_{00} = K(A_0)$. First we need some lemmas.

Lemma 2.3 *With above notations, for each n*

$$Ae_n^2A = A_0e_n^2A_0 = A_{00}e_n^2A_{00}.$$

Proof For each n we have

$$Ae_n^2A = Ae_{n+1}e_n^2e_{n+1}A \subseteq AA_{00}e_n^2A_{00}A \subseteq A_{00}e_n^2A_{00} \subseteq A_0e_n^2A_0 \subseteq Ae_n^2A,$$

and so all three algebras are equal. ■

Lemma 2.4 *Let A, A_{00}, A_0 and (e_n) be as above. Then (e_n) is an approximate identity of A_0 in the norm topology.*

Proof We know that $e_n \rightarrow 1$ in the A_{00} -topology. Hence, by Proposition 2.1 it forms an approximate identity for A_{00} . Now recall that $e_n \in A_b$ and $0 \leq e_n \leq 1$. By this and the fact that $A_{00} \subseteq A_0$ is norm dense, the result follows immediately. Indeed, if $a \in A_0$ and $\epsilon > 0$ are given, then there is $b \in A_{00}$ such that $\|a - b\|_\infty < \epsilon$. Now $b \in A_{00}$ and so by above lemma,

$$\exists n_0 \forall n \geq n_0 \quad \|be_n - b\|_\infty < \epsilon.$$

Also $\|be_n - ae_n\|_\infty \leq \|\pi_n(b - a)\|_n \leq \|a - b\|_\infty < \epsilon$. Combining these inequalities we get $\|ae_n - a\|_\infty < 3\epsilon$, for each $n \geq n_0$. Hence (e_n) is a right approximate identity for A_0 . Similarly one can show that it is also a left approximate identity. ■

Next we show that A_{00} is indeed the Pedersen ideal of A_0 .

Lemma 2.5 *With above notations, $K(A_0) = A_{00}$.*

Proof Since A_{00} is a dense ideal of A_0 , we have $K(A_0) \subseteq A_{00}$. Now given n , by the fact that (e_n) is canonical, $e_n e_{n+1} = e_n$ and hence $e_n \in K_0(A_0) \subseteq K(A_0)$. But then, as $K(A_0) \subseteq A_0$ is a two sided ideal, $A_0 e_n \cup e_n A_0 \subseteq K(A_0)$. Hence, by Lemma 2.3, $A_{00} = \bigcup_n A_0 e_n^2 A_0 = \bigcup_n A_0 e_n^2 A_0 \subseteq K(A_0)$. ■

Let us check the above examples to see what is A_0 in each example.

Example 2.5 Let $A = C(\mathbb{R})$, then $A_b = C_b(\mathbb{R})$ and for $0 \leq e_n \leq 1$ being the bump function which is 1 on $[1 - n, n - 1]$ and zero off $[-n, n]$, we get $A_{00} = C_{00}(\mathbb{R})$. Hence $A_0 = C_0(\mathbb{R})$.

Example 2.6 Consider the σ - C^* -algebra $A = c$ of all sequences of complex numbers, then $A_b = \ell^\infty$, and for $e_n = (1, \dots, 1, 0, 0, \dots)$ we get A_{00} as the ideal of all sequences with finitely many non zero components. Hence $A_0 = c_0$.

Example 2.7 Let $A = B(\ell^2)$ with norm topology. Then $A_b = A$. If the approximate identity consists of the identity I only, then $A_{00} = A_0 = A$. In the case that we choose the approximate identity consisting of the elements

$$e_n = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \in B(\ell^2),$$

we get $A_{00} = F(\ell^2)$ and so $A_0 = K(\ell^2)$.

Example 2.8 For the σ - C^* -algebra $A = \prod_n \mathbb{M}_n$, we have $A_b = \sum_n^{\ell^\infty} \oplus \mathbb{M}_n$ and, if we choose the approximate identity $e_n = (I_1, I_2, \dots, I_n, 0, 0, \dots)$, then

$$A_{00} = \bigcup_n \left(\sum_{i=1}^n \oplus \mathbb{M}_i \right)$$

consists of the block matrices with only finitely many non zero blocks. Hence $A_0 = \sum_n^{c_0} \oplus M_n$.

Note that what is common between these examples is that $A = \Gamma(K(A_0))$. Indeed as a typical example of locally compact σ - C^* -algebras we show that multiplier algebra of the Pedersen ideal of any C^* -algebra is a locally compact σ - C^* -algebra. Before that, we need a trivial lemma which we state without proof.

Lemma 2.6 *Let X be a Banach space and let $X_0 \subseteq X$ be a dense subspace. Let $T_0: X_0 \rightarrow X_0$ be a bounded linear map. Then T_0 extends uniquely to a bounded linear map $T \in B(X)$ and*

$$\|T\| = \sup\{\|T_0(x)\| : x \in X_0, \|x\| \leq 1\}.$$

Let A_0 be a C^* -algebra and $A_{00} = K(A_0)$. For each $a \in A_0$, we put $L_a = \text{cl}(A_0a)$, $R_a = \text{cl}(aA_0)$, and $I_a = \text{cl}(L_aR_{a^*})$, where the closures are taken in the norm topology of A_0 . Consider all linear maps $S: L_a \rightarrow L_a$ and $T: R_{a^*} \rightarrow R_{a^*}$ and put

$$M_a = \{(S, T) : yS(x) = T(y)x \text{ for each } x \in L_a, y \in R_{a^*}\}.$$

Then S, T are automatically bounded and M_a is a C^* -algebra. Indeed $M_a \cong M(I_a)$, for each $a \in A_0^+$.

Theorem 2.2 *For each σ -unital C^* -algebra A_0 , the σ - C^* -algebra $A = \Gamma(K(A_0))$ is locally compact.*

Proof Recall that for each (countable contractive) canonical approximate identity (e_n) of A_0 contained in $K(A_0)^+$ (we can always choose (e_n) to be canonical [CF]) we have $\Gamma(K(A_0)) = \varprojlim_n M_{e_n}$ such that the corresponding morphism $\pi_n: A \rightarrow A_n = M_{e_n}$ sends $(S, T) \in \Gamma(K(A_0))$ to $(S|_{L_{e_n}}, T|_{R_{e_n}})$. Also $A_b = M(A_0)$ [Ph88b]. Let A_{00} be the corresponding support algebra of $A = \Gamma(K(A_0))$; that is $A_{00} = \bigcup_n Ae_n^2A$. Then we claim that $A_{00} = K(A_0)$.

Indeed, $K(A_0)$ is an ideal in A [LT], so $A_{00} = \bigcup_n Ae_n^2A \subseteq AK(A_0)A \subseteq K(A_0)$. On the other hand, A_{00} contains the approximate identity (e_n^2) of A_0 , so it is dense in A_0 (in the norm topology). Being a dense two sided ideal, then $A_{00} \supseteq K(A_0)$, and the claim is proved. In particular the A_{00} -topology on A is just the κ -topology. The fact that (e_n) is an approximate identity of A_0 then implies that $e_n \rightarrow 1 \in A$ in the κ -topology. We know that A is complete in the κ -topology [LT, 3.8]. We can always choose (e_n) to be canonical [CF]. The fact that (e_n) is multiplicatively bounded follows immediately, as for each $n, e_nA \subseteq K(A_0)A \subseteq K(A_0) \subseteq A_b$.

Finally for the compatibility conditions observe that, for each n and each $x = (S, T) \in \Gamma(K(A_0))$, we have

$$\|\pi_n((S, T))\|_n = \|(S|_{L_{e_n}}, T|_{R_{e_n}})\|_n = \|S|_{L_{e_n}}\|.$$

Now recall that Ae_n is a dense linear subspace of L_{e_n} and so by the above lemma, we have

$$(3) \quad \|S|_{L_{e_n}}\| = \sup\{\|S(be_n)\|_\infty : b \in A, \|be_n\|_\infty \leq 1\}$$

$$(4) \quad = \sup\{\|xbe_n\|_\infty : b \in A, \|be_n\|_\infty \leq 1\}.$$

But $e_{n+1}e_n = e_n$ and $\|e_{n+1}e_n\|_\infty \leq 1$, and so the last supremum is clearly not less than $\|xe_{n+1}e_n\|_\infty = \|xe_n\|_\infty$, which proves one inequality of the compatibility condition. For the other, the above calculation shows that indeed equality holds. ■

In the next section we will show that the converse is also true.

3 An Equivalence of Two Categories

In this section we show that there is an equivalence of categories between the category of σ -unital C^* -algebras and the category of locally compact σ - C^* -algebras. We start with a non unital σ -unital C^* -algebra A_0 . Put $A_{00} = K(A_0)$. Recall that for each $a \in A$, $L_a = \text{cl}(A_0a)$, $R_a = \text{cl}(aA_0)$, and $I_a = \text{cl}(L_aR_{a^*})$, where closures are taken in the norm topology of A_0 . Also recall that for linear maps $S: L_a \rightarrow L_a$ and $T: R_{a^*} \rightarrow R_{a^*}$,

$$M_a = \{(S, T) : yS(x) = T(y)x \text{ for each } x \in L_a, y \in R_{a^*}\}$$

is a C^* -algebra and $M_a \cong M(I_a)$, for each $a \in A_0^+$. If $0 \leq a \leq b$ are in A_0 then $L_a \subseteq L_b$, $R_a \subseteq R_b$, $I_a \subseteq I_b$ and the restriction map defines a $*$ -homomorphism from M_b to M_a . Also for $K_+ = K_+(A_0)$,

$$\bigcup_{a \in K_+} I_a \supseteq \bigcup_{a \in K_+} L_a = \bigcup_{a \in K_+} R_a = K(A_0),$$

[Ph88b].

Now $K(A_0)$ is a minimal dense ideal, so for each approximate identity (e_n) of A_0 we have $K(A_0) \subseteq \bigcup_n I_{e_n}$, and so

$$A_0 \cong \varinjlim_n I_{e_n},$$

as C^* -algebras. Now let $A = \Gamma(K(A_0))$. Then A is a σ - C^* -algebra, and

$$A \cong \varprojlim_n M(I_{e_n}),$$

as σ - C^* -algebras. Also $A_b = b(A) = M(A_0)$ [Ph88b].

Now, using Dauns-Hofmann's theorem and its generalization to σ - C^* -algebras, we have that if $X = \text{Prim}(A_0)$, then

$$Z(A_b) \cong C_b(X), \quad Z(A) \cong C(X).$$

Of course, $Z(A_0) = C_0(X)$ and $Z(A_{00}) = C_{00}(X)$ does not hold in general.

Next let's consider the reverse situation. We want to start with a σ - C^* -algebra $A = \varprojlim_n A_n$, with corresponding morphisms $\pi_n: A \rightarrow A_n$, and get a C^* -algebra A_0 such that $A \cong \Gamma(K(A_0))$. We do this when A is locally compact. In this case there is a support algebra A_{00} in A such that if A_0 be the norm closure of A_{00} inside $A_b = b(A)$, then $K(A_0) = A_{00}$. Moreover, for the corresponding m.b. canonical net (e_n) in A , we show that (e_n) forms an approximate identity of A_0 in norm topology and so by [Ph88b, Theorem 4] we have

$$\Gamma(A_{00}) \cong \varprojlim_n M(I_{e_n}),$$

as σ - C^* -algebras. Finally we show that indeed

$$\Gamma(A_{00}) \cong A,$$

as σ - C^* -algebras.

Let's us begin with two well known results about inverse limits. These are true for uncountable systems (in more general categories) also, but we state them only for the countable case. The proofs are quite standard. The first lemma is an immediate consequence of the definition of inverse limit. We sketch the proof of the second lemma.

Lemma 3.1 *If $A = \varprojlim_n (A_n, \pi_n)$ is a σ - C^* -algebra and $\pi_n(A)$ is C^* -subalgebra of A_n , then $A \cong \varprojlim_n (\pi_n(A_n), \pi_n)$, as σ - C^* -algebras.*

Lemma 3.2 *If $A = \varprojlim_n (A_n, \phi_n)$ and $B = \varprojlim_n (B_n, \psi_n)$ are σ - C^* -algebras with connecting morphisms $\phi_{n,n+1}: A_{n+1} \rightarrow A_n$ and $\psi_{n,n+1}: B_{n+1} \rightarrow B_n$, and for each n , there are morphisms $\alpha_n: A_n \rightarrow B_n$ and $\beta_n: B_{n+1} \rightarrow A_n$ such that*

$$\alpha_n \phi_{n,n+1} = \psi_{n,n+1} \alpha_{n+1},$$

then $A \cong B$, as topological $$ -algebras.*

Proof By the universal property of inverse limits, there are maps $j: A \rightarrow B$ and $i: B \rightarrow A$ such that for each n ,

$$\psi_n j = \alpha_n \phi_n,$$

and

$$\phi_n i = \beta_n \psi_{n+1},$$

and

$$\alpha_n \beta_n = \psi_{n,n+1}.$$

Hence

$$\psi_n j i = \alpha_n \phi_n i = \alpha_n \beta_n \psi_{n+1} = \psi_{n,n+1} \psi_{n+1} = \psi_n,$$

for each n , and so $ji = \text{id}_B$. Similarly we get $ij = \text{id}_A$ and the proof is complete. ■

Since A_{00} is an ideal of A , each element of A is a multiplier of A_{00} . This way A maps into $\Gamma(A_{00})$. The morphism $j: A \rightarrow \Gamma(A_{00})$ maps a to the multiplier (S_a, T_a) . Now the compatibility condition tells us that j is a continuous embedding.

Lemma 3.3 *If A is locally compact, then j is one-one.*

Proof If $x \in A$ and $ax = 0$ for each $a \in A_{00}$, then in particular $e_n bx = 0$, for each n and $b \in A$. Hence by the compatibility condition, $\pi_n(x) = 0$. Having this for each n , we get $x = 0$. ■

Now we can identify A with a subalgebra of $\Gamma(A_{00})$. Let the mappings $\pi_n: A \rightarrow A_n$ and $\pi'_n: \Gamma(A_{00}) \rightarrow M_{e_n}$ be the corresponding morphisms, where the second is as in the Theorem 2.2 (denoted by π_n there).

Proposition 3.1 *With the above notations, if A is locally compact then*

$$\|\pi_n(x)\|_n \leq \|\pi'_n(x)\|_n \leq \|\pi_{n+1}(x)\|_{n+1} \quad (x \in A).$$

In particular, the embedding $j: A \rightarrow \Gamma(A_{00})$ is continuous and has continuous left inverse with respect to the corresponding projective topologies.

Proof Fix n and $x \in A$. Recall from the proof of Theorem 2.2 that

$$\|\pi'_n(x)\|_n = \sup\{\|xae_n\|_\infty : a \in \Gamma(A_{00}), \|ae_n\|_\infty \leq 1\} \geq \|\pi_n(x)\|_n,$$

by the compatibility condition. Let $a \in \Gamma(A_{00})$ be such that $\|ae_n\|_\infty \leq 1$. Then by the compatibility condition,

$$\begin{aligned} \|xae_n\|_\infty &= \|xae_n e_{n+1}\|_\infty \\ &\leq \|\pi_{n+1}(xae_n)\|_{n+1} \\ &= \|\pi_{n+1}(x)\pi_{n+1}(ae_n)\|_{n+1} \\ &\leq \|\pi_{n+1}(x)\|_{n+1} \|\pi_{n+1}(ae_n)\|_{n+1} \\ &\leq \|\pi_{n+1}(x)\|_{n+1} \|ae_n\|_\infty \\ &\leq \|\pi_{n+1}(x)\|_{n+1}, \end{aligned}$$

which implies that $\|\pi'_n(x)\|_n \leq \|\pi_{n+1}(x)\|_{n+1}$. ■

It is not easy to show that j is onto. Instead we construct another morphism from $\Gamma(A_{00})$ to A indirectly. For this purpose we need a slightly different version of Phillips' result. First some preliminary results.

Recall that

$$\Gamma(K(A_0)) \simeq \varprojlim_{n \in \mathbb{N}} (M(I_{e_n}), \pi'_n) \simeq \varprojlim_{n \in \mathbb{N}} (M_{e_n}, \phi_n),$$

where mappings π'_n are defined by $\pi'_n = \psi_n^{-1} \circ \phi_n$, where

$$\phi_n: \Gamma(K(A_0)) \rightarrow M_{e_n} \quad \text{and} \quad \psi_n: M(I_{e_n}) \rightarrow M_{e_n},$$

are both defined by restriction, and the second is an isomorphism [Ph88b]. Now let $B_n = \phi_n(A_0)$. Then B_n is a C^* -subalgebra of M_{e_n} and $\phi_n: A_0 \rightarrow B_n$ is a surjective $*$ -homomorphism of C^* -algebras, in particular $\phi_n(K(A_0)) = K(B_n)$ [Pd66, II].

The C^* -algebras B_n also provide us another inverse system. Indeed the restriction of ϕ_{nm} to B_m (still denoted by ϕ_{nm}) gives us a surjective morphism $\phi_{nm}: B_m \rightarrow B_n$ which extends uniquely to a surjective morphism $\phi_{nm}: M(B_m) \rightarrow M(B_n)$. Also for each n , $\phi_n(\Gamma(K(A_0))) = M(B_n)$ [LT,5.4], hence by Lemma 3.1 we have

Proposition 3.2 *With the above notations,*

$$\Gamma(K(A_0)) \simeq \varprojlim_n (M(B_n), \phi_n),$$

as σ - C^* -algebras, such that the corresponding morphisms $\phi_n: \Gamma(K(A_0)) \rightarrow M(B_n)$ are all surjective.

We emphasize that the advantage of this presentation is in the fact that the morphisms ϕ_n are all surjective. Now recall that $\pi'_n: \Gamma(K(A_0)) \rightarrow M(I_{e_n})$ is defined by $\pi'_n = \psi_n^{-1} \circ \phi_n$, where $\psi_n: M(I_{e_n}) \rightarrow M_{e_n}$, defined by restriction, is a C^* -algebra isomorphism (and so an isometry). Therefore by Proposition 3.1 we have

Proposition 3.3 *With the above notations, for each n we have*

$$\|\pi'_n(x)\|_n = \|\phi_n(x)\|_n \quad (x \in \Gamma(K(A_0))),$$

and

$$\|\pi_n(a)\|_n \leq \|\phi_n(a)\|_n \leq \|\pi_{n+1}(a)\|_{n+1} \quad (a \in A),$$

where in the last equation we have identified A with its image under $j: A \rightarrow \Gamma(K(A_0))$.

In particular the $\|\cdot\|_\infty$ of $\Gamma(A_{00})$ restricted to A coincides with the $\|\cdot\|_\infty$ of A , namely

$$\sup_n \|\pi_n(a)\|_n = \sup_n \|\phi_n(a)\|_n \quad (a \in A).$$

The following lemma is the only place we use the fact that a locally compact σ - C^* -algebra A is complete with respect to the corresponding A_{00} -topology. Recall that the elements the multiplier algebra of a Banach algebra B are automatically bounded and we used the notation $M(B)$ for the multiplier algebra. This is not the case when B is merely a normed algebra. Therefore we use the notations $\Delta(B)$ and $\Gamma(B)$ to distinguish between the bounded multiplier algebra and the multiplier algebra.

Lemma 3.4 *With the above notations, if A is locally compact and A_{00} is the corresponding support algebra of A , then $\Delta(A_{00}) \subseteq A \subseteq \Gamma(A_{00})$.*

Proof Let $x \in \Delta(A_{00})$ be self adjoint and let (e_n) be as in Definition 2.6. Then we claim that the sequence $(e_n x) \subseteq A$ is A_{00} -Cauchy. Note that, since x is a multiplier of A_{00} and $e_n \in A_{00}$, we have $e_n x \in A_{00} \subseteq A$. Now, for each $a \in A_{00}$, we have

$$\|a(e_n x - e_m x)\|_\infty \leq \|a(e_n - e_m)\|_\infty \|x\|_\infty \rightarrow 0,$$

and

$$\|(e_n x - e_m x)a\|_\infty = \|e_n(xa) - e_m(xa)\|_\infty \rightarrow 0,$$

as $m, n \rightarrow \infty$. All of this calculation is done inside $\Gamma(A_{00})$. In particular $\|\cdot\|_\infty = \sup_n \|\phi_n(\cdot)\|_n$. But from the above lemma this calculation is valid inside A also (because $\sup_n \|\pi_n(\cdot)\|_n = \sup_n \|\phi_n(\cdot)\|_n$). This completes the proof of the claim.

Now, as A is A_{00} -complete, there is a self adjoint element $y \in A$ such that $e_n x \rightarrow y$ in A with respect to the A_{00} -topology. In particular, for each $a \in A_{00}$, we have $\|e_n xa - ya\|_\infty \rightarrow 0$ in A . On the other hand, $\|e_n xa - xa\|_\infty \rightarrow 0$ in $\Gamma(A_{00})$. Again all these nets are inside A and the $\|\cdot\|_\infty$ of $\Gamma(A_{00})$ restricted to A coincides with $\|\cdot\|_\infty$ of A . Therefore $ya = xa \in A_{00}$. But x and y are self adjoint and A_{00} is an $*$ -algebra, hence $ay = ax \in A_{00}$. Therefore $x = y$ as elements of $\Gamma(A_{00})$ and the proof is finished. ■

The first part of the following lemma is a modification of [LT, 5.4].

Lemma 3.5 *With the above notations, for each n*

$$\phi_n(\Delta(K(A_0))) = \phi_n(\Gamma(K(A_0))) = \Gamma(K(B_n)) = \Delta(K(B_n)) \subseteq M_{e_n}.$$

Moreover if A is locally compact, then we have $j(A) \subseteq \Gamma(K(A_0))$ and $\phi_n(j(A)) = \Delta(K(B_n)) = M(B_n)$.

Proof Recall that A_0 (and so its quotient B_n) is a σ -unital C^* -algebra. Now the second equality and the inclusion holds for any C^* -algebra A_0 by [LT, 5.4]. The first equality is proved in [LT, 5.4] under the assumption that A_0 is separable. However, they use this assumption only to make sure that the surjection $\phi_n: A_0 \rightarrow B_n$ extends to a surjective morphism between the multiplier algebras (see [APT, 4.2] wrongly quoted in [LT, 5.4] as Theorem 9.2). But this holds also for σ -unital C^* -algebras (this is a special case of [Ph88a, 5.11]).

Now let A be locally compact and A_{00} be the support algebra of A (which we have chosen and fixed), then we know that $j(A) \subseteq \Gamma(A_{00})$. Hence $\phi_n(j(A)) \subseteq \Delta(K(B_n))$ by the first part. Now by the above paragraph, $\phi_n(M(A_0)) = M(B_n)$, hence by Lemma 3.4, we have $\phi_n(j(A)) \supseteq \phi_n(\Delta(K(A_0))) = \Delta(K(B_n))$. Also $\phi_n(j(A)) \subseteq \phi_n(\Gamma(K(A_0))) = \Delta(K(B_n))$, and we are done. ■

Now recall that if A is locally compact then the morphisms $\pi_n: A \rightarrow A_n$ are surjective (cf. Definition 2.6). Also in this case Lemma 3.5 tells us that $\phi_n \circ j: A \rightarrow M(B_n)$ is surjective for each n . Now by the same lemma the morphisms

$$\tilde{\pi}_n: A_{n+1} \rightarrow M(B_n); \quad \pi_{n+1}(a) \mapsto \phi_n j(a)$$

and

$$\hat{\pi}_n: M(B_n) \rightarrow A_n; \quad \phi_n j(a) \mapsto \pi_n(a)$$

are well defined, norm continuous, and surjective. Now we are ready to prove the main result of this section.

Theorem 3.1 Let $A = \varprojlim_{n \in \mathbb{N}} A_n$ be a locally compact σ - C^* -algebra. Let $(e_n) \subseteq A_{sb}$ and $A_{00} = \bigcup A e_n^2 A$ denote the corresponding approximate identity and support algebra. Let A_0 denote the C^* -subalgebra of A_b which is the norm closure of A_{00} in A_b . Then $A_{00} = K(A_0)$ and $A \cong \Gamma(K(A_0))$, as σ - C^* -algebras, and $A_b \cong M(A_0)$, as C^* -algebras. Moreover the C^* -algebra A_0 is unique if we require that the isomorphism from A onto $\Gamma(K(A_0))$ be A_{00} - κ -bi-continuous.

Proof The fact that $A_{00} = K(A_0)$ is Lemma 2.5. Now from the last two isomorphisms, the second follows immediately from the first and the fact that $\Gamma(K(A_0))_b = M(A_0)$ [Ph88b]. For the first statement, by Lemma 3.2, we only need to observe that, for each n ,

$$\pi_{n,n+1} \tilde{\pi}_{n-1} = \phi_{n-1,n} \hat{\pi}_n.$$

But this follows directly from the definition of the maps $\tilde{\pi}_n$ and $\hat{\pi}_n$, and the fact that each map $\phi_n j$ is onto (by Lemma 3.5). For the last statement, take two C^* -algebras A_0 and B_0 such that $\Gamma(K(A_0))$ and $\Gamma(K(B_0))$ are both isomorphic to A with the isomorphisms being A_{00} - κ and B_{00} - κ bi-continuous, respectively. Then the composition of these would be a κ - κ -bi-continuous isomorphism between $\Gamma(K(A_0))$ and $\Gamma(K(B_0))$. Hence A_0 and B_0 are isomorphic by [LT, Theorem 7.10]. ■

Remark 3.1 It is clear from the above proof that the *compatibility condition* in the definition of locally compact σ - C^* -algebras could be replaced with the following

$$\|e_{n_{k-1}} a\|_\infty \leq \|\pi_k(a)\|_k \leq \sup\{\|e_{n_k} b a\|_\infty : b \in A, \|e_{n_k} b\|_\infty \leq 1\} \quad (a \in A, k \in \mathbb{N}),$$

for some (infinite) subsequence $\{n_k\}$ of \mathbb{N} . This, in particular, justifies Example 2.1.

The above theorem with the previously mentioned result of N. C. Phillips gives a correspondence between the *objects* of the given categories. Namely

Corollary 3.1 A σ - C^* -algebra A is locally compact if and only if there is a C^* -algebra A_0 such that $A \cong \Gamma(K(A_0))$, as σ - C^* -algebras.

Proof If A is locally compact and A_0 is as in the above theorem, then $A \cong \Gamma(K(A_0))$. The other direction is Theorem 2.2. ■

If $A = C(X)$ is a locally compact commutative σ - C^* -algebra, where X is a Hausdorff κ_ω -space, then the above result tells us that there exists a C^* -subalgebra A_0 of A_b such that $A \cong \Gamma(K(A_0))$, as σ - C^* -algebras. But $A_0 = C(Y)$, for a locally compact σ -compact Hausdorff space Y and so $C(X)$ and $C(Y)$ are isomorphic. In general this does *not* imply that X and Y are homeomorphic. But here both X and Y are σ -compact and so they are real compact (see [GJ, 8.2] for the definition and proof). In particular, X and Y are homeomorphic [GJ, 10.6]. Hence X is locally compact. This gives an alternative proof of (the difficult direction of) Theorem 2.1. Also it shows that the uniqueness part of the Theorem 3.1 holds in the commutative case.

Corollary 3.2 *If $A = \varprojlim_n A_n$ is a locally compact σ - C^* -algebra and A_{00} is support algebra of A , then the projective topology is stronger than the A_{00} -topology.*

Next let us consider the issue of morphisms. Let A_0, B_0 be C^* -algebras and $\phi_0: A_0 \rightarrow B_0$ be a $*$ -homomorphism, then ϕ_0 does not lift to a $*$ -homomorphism of the corresponding multiplier algebras unless it is surjective. To turn around this difficulty people usually consider

$$\text{Mor}(A_0, B_0) = \{\phi_0: A_0 \rightarrow M(B_0) : \phi_0(A_0)B_0 \subseteq B_0 \text{ is dense}\}$$

as the family of *morphisms* from A_0 to B_0 . Such morphisms are called *non degenerate*. These morphisms have a unique extension to (strictly continuous) unital $*$ -homomorphisms between multiplier algebras.

Definition 3.1 Let A_0 and B_0 be as above, a morphism $\phi_0: A_0 \rightarrow M(B_0)$ is called *strictly nondegenerate* if $\phi_0(A_{00})B_0 \supseteq B_{00}$, where A_{00} and B_{00} are the Pedersen ideals of A_0 and B_0 , respectively.

Now let A, B be the corresponding unital σ - C^* -algebras, *i.e.*, $A = \Gamma(A_{00})$ and $B = \Gamma(B_{00})$, where A_{00} and B_{00} are the Pedersen ideals of A and B , respectively. A *morphism* from A to B is a unital $*$ -homomorphism such that

$$\forall m \exists n \|\phi(a)\|_m \leq \|a\|_n, \quad (a \in A).$$

This condition plays a crucial role. It implies that each representation of a σ - C^* -algebra $A = \varprojlim_\alpha A_\alpha$ factors through some A_n . It also ensures that the GNS-construction yields a representation of A .

Definition 3.2 We say that ϕ is *nondegenerate* if $\phi(A_{00})B \subseteq B_0$ is dense. We say that ϕ is *strictly nondegenerate* if $\phi(A_{00})B \supseteq B_{00}$.

Now let us start with a morphism $\phi_0 \in \text{Mor}(A_0, B_0)$. We wish to extend ϕ_0 to a morphism $\phi: A \rightarrow B$.

Lemma 3.6 *With the above notation, if ϕ_0 is (strictly) nondegenerate then we have $B_0\phi_0(A_{00})B_0 = B_{00}$ ($B_0\phi_0(A_{00}) = \phi_0(A_{00})B_0 = B_{00}$, respectively).*

Proof Since ϕ_0 preserves the spectral theory, $\phi_0(A_{00}) \subseteq B_{00}$. On the other hand, $\phi_0(A_{00})$ is clearly a dense ideal of $\phi_0(A_0)$. Therefore $B_0\phi_0(A_{00})B_0$ is an ideal of B_0 which is dense in $B_0\phi_0(A_0)B_0$. Now if ϕ_0 is non degenerate then $\phi_0(A_0)B_0$ is dense in B_0 , so $B_0\phi_0(A_0)B_0$ is dense in $B_0B_0 = B_0$, *i.e.*, $B_0\phi_0(A_{00})B_0$ is a dense ideal of B_0 and so contains B_{00} . But $B_0\phi_0(A_{00})B_0 \subseteq B_0B_{00}B_0 \subseteq B_{00}$, hence the equality holds. If ϕ_0 is strictly nondegenerate then $\phi_0(A_{00})B_0 \supseteq B_{00}$. The converse inequality follows from the fact that $\phi_0(A_{00}) \subseteq B_{00}$. Hence $\phi_0(A_{00})B_0 = B_{00}$. Now the right hand side is self adjoint and the adjoint of the left hand side is $B_0\phi_0(A_{00})$, hence $B_0\phi_0(A_{00}) = B_{00}$. The other equality is proved similarly. ■

Remark 3.2 When ϕ_0 is surjective, one gets the better result $\phi_0(A_{00}) = B_{00}$, first proved by G. K. Pedersen.

Now let ϕ_0 be strictly nondegenerate. Then it extends to a map $\phi: A = \Gamma(A_{00}) \rightarrow B = \Gamma(B_{00})$ given by

$$\phi(x)\phi_0(a)b = \phi_0(xa)b, \quad b\phi_0(a)\phi(x) = b\phi_0(ax) \quad (x \in A, a \in A_{00}, b \in B_0).$$

Again, as ϕ preserves the spectral theory, we have $\phi(A_b) \subseteq B_b$, and so ϕ induces a $*$ -homomorphism $\phi_b: A_b \rightarrow B_b$. By uniqueness, ϕ_b is the same as the extension of ϕ_0 to $A_b = M(A_0)$. Next observe that ϕ is strictly nondegenerate. Indeed $\phi(A_{00})B \supseteq \phi_0(A_{00})B_0 \supseteq B_{00}$.

Now we want to examine the other direction. This time a strictly non degenerate morphism $\phi: A \rightarrow B$ is given, and we are aiming to show that there exist a strictly nondegenerate morphism $\phi_0: A_0 \rightarrow B_0$ which coincides with the restriction of ϕ to A_0 . Indeed the fact that ϕ is a morphism in particular implies that $\phi(A_b) \subseteq B_b$, so if we take ϕ_0 to be the restriction of ϕ to A_0 , then ϕ_0 is norm continuous (indeed of norm ≤ 1) and $\phi_0(A_0) \subseteq \phi(A_b) \subseteq B_b$.

Now ϕ preserves the spectral theory, hence $\phi(A_{00}) \subseteq B_{00}$, and so $\phi(A_{00})B \subseteq B_{00}B \subseteq B_{00}$. Hence $\phi(A_{00})B = \phi(A_{00})B_0 = B_{00}$. We need to show that $\phi_0(A_{00})B_0 \supseteq B_{00}$. But since ϕ_0 is the restriction of ϕ to A_0 , we have $\phi_0(A_{00})B_0 = \phi(A_{00})B_0 = B_{00}$. Therefore we have shown that, given a strictly nondegenerate morphism $\phi: A \rightarrow B$, of σ - C^* -algebras the restriction ϕ_0 of ϕ to A_0 is a strictly nondegenerate morphism of C^* -algebras. Conversely, each strictly nondegenerate element $\phi_0 \in \text{Mor}(A_0, B_0)$ uniquely extends to a strictly nondegenerate morphism $\phi: A \rightarrow B$.

Let's summarize these observations as follows.

Theorem 3.2 *There is an equivalence of categories between the category of σ -unital C^* -algebras and strictly nondegenerate $*$ -homomorphisms and the category of locally compact σ - C^* -algebras and strictly nondegenerate $*$ -homomorphisms. This functor assigns to each σ -unital C^* -algebra the σ - C^* -algebra of multipliers of its Pedersen ideal. In particular, for each σ -compact Hausdorff topological space X , it sends the commutative C^* -algebra $C_0(X)$ to the commutative σ - C^* -algebra $C(X)$.*

Proof We have established a one to one correspondence between the objects and morphisms of the category of locally compact σ - C^* -algebras and the category of σ -unital C^* -algebras in Theorems 2.2 and 3.1. The fact that this is indeed an equivalence of categories then follows from the trivial observation that $\iota_B\phi_0 = \phi\iota_A$. The other statements are trivial. ■

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Department of Mathematics
Tarbiat Modarres University
P.O. Box 14115-175
Tehran
Iran
e-mail: mamini@modares.ac.ir

Department of Mathematics
and Statistics
University of Saskatchewan
Saskatoon, Saskatchewan
S7N 5E6
e-mail: mamini@math.usask.ca