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AN IMPROVED WINTNER OSCILLATION CRITERION FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. An iterative technique is used to establish an oscillation theorem for the equation $x''+a(t)x=0$ which relaxes the condition that $a(t)$ satisfy

$$\int^{\infty} \exp\left[-2\int_{t_0}^{t}\int_{s}^{\infty} a(r)\,dr\,ds\right] dt<\infty,$$

without the restriction that

$$\alpha(t)=\int_{t}^{\infty} a(s)\,ds\geq 0.$$

1. **Introduction.** The well-known Wintner [5] oscillation theorem for the second order linear differential equation

$$x''+a(t)x=0 \tag{1}$$

is the following.

THEOREM. *Equation* (1) *is oscillatory if*

$$\int^{\infty} \exp\left[-2\int_{t_0}^{t}\int_{s}^{\infty} a(r)\,dr\,ds\right] dt<\infty. \tag{2}$$

Let $a(t)$ be a (real-valued) continuous function for $t\geq t_0$ in Eq. (1). Eq. (1) is said to be oscillatory or non-oscillatory as one (hence every) solution $x(t)$ of Eq. (1) has or does not have an infinity of zeros for $t\geq t_0$.

In [3], Kameneve has established a sharper theorem by using an iterative technique. Unfortunately, Kamenev's theorem is proved under the condition that $\int_t^\infty a(s)\ ds$ exists and is non-negative while the Wintner theorem simply requires that

$$\alpha(t)=\int_{t}^{\infty} a(s)\,ds \qquad \text{exists.} \tag{3}$$

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The purpose of this paper is to improve two of Wintner's theorems using an iterative technique similar to that of Kamenev. *It will be assumed throughout that* $a(t)$ *is continuous on* (t_0, ∞) *and* (3) *holds.* Before stating our main results we give the following lemmas.

LEMMA 1 (Wintner [5]). *If* (1) *is not oscillatory, then a non-trivial solution* $x(t)$ *of* (1) *satisfies*

$$\int^{\infty} w^2(t)\,dt < \infty, \qquad \lim_{t\to\infty} w(t) = 0 \tag{4}$$

and

$$w(t) = \int_t^{\infty} \{w^2(s) + a(s)\}\,ds, \qquad t \ge t_0. \tag{5}$$

where $w(t) = x'(t)/x(t)$.

LEMMA 2 (Leighton and Morse [4], cf Hartman [2]). *If* (1) *is not oscillatory, then it has a solution* $x_1(t)$ (*a principal solution*) *such that*

$$\int^{\infty} x_1(t)^{-2}\,dt < \infty,$$

and a solution $x_2(t)$ (*a non-principal solution*) *such that*

$$\int^{\infty} x_2(t)^{-2}\,dt = \infty.$$

Using the notation

$$g_+(t) = [g(t)]_+ = \tfrac{1}{2}[g(t) + |g(t)|], \tag{6}$$

we construct the function sequence $\{\alpha_n(t)\}$, $n = 0, 1, \ldots,$ where $\alpha_0(t) = \alpha(t)$,

$$\alpha_1(t) = \int_t^{\infty} [\alpha_0(s)]_+^2\,ds$$

and for $n = 1, 2, \ldots,$

$$\alpha_{n+1}(t) = \int_t^{\infty} [\alpha_0(s) + \alpha_n(s)]_+^2\,ds. \tag{7}$$

2. **Main results**

THEOREM 1. *If there is a positive integer* m *such that the* $\alpha_n(t)$ *are defined for* $n = 0, \ldots, m-1$, *but* $\alpha_m(t)$ *does not exist, then Eq.* (1) *is oscillatory.*

Proof. Suppose that (1) is not oscillatory and $x(t)$ is a non-trivial solution, $x(t) \neq 0$ for $t \ge t_0$. Let $w(t) = x'(t)/x(t)$. We will show that the non-oscillation of

$x(t)$ implies that $\alpha_m(t)<\infty$ for all $m=1,2,\ldots$. By Lemma 1, $w(t)$ must satisfy

$$w(t)=v(t)+\alpha(t),\ v(t)=\int_t^\infty w^2(s)\,ds<\infty. \tag{8}$$

By virtue of (8), we have $w(t)\geq\alpha_0(t)$, hence from (6)

$$w^2(t)\geq[\alpha_0(t)]_+^2. \tag{9}$$

It follows that

$$\alpha_1(t)=\int_t^\infty [\alpha_0(s)]_+^2\,ds\leq\int_t^\infty w^2(s)\,ds=v(t).$$

Inductively, if $\alpha_k(t)\leq v(t)$ for some $k\geq 0$, then from (8),

$$[\alpha_k(t)+\alpha_0(t)]_+^2\leq w^2(t),$$

and from (7) it follows immediately that $\alpha_{k+1}(t)\leq v(t)$, as was to be shown. This completes the proof.

THEOREM 2. *If there is positive number m such that the $\alpha_n(t)$ are defined for $n=0,\ldots,m$ and*

$$\int_t^\infty \exp\left\{-2\int_{t_0}^s [\alpha_0(r)+\alpha_m(r)]\,dr\right\}ds<\infty \tag{10}$$

then Eq. (1) *is oscillatory.*

Proof. Suppose to the contrary that Eq. (1) has a non-oscillatory solution $x(t)>0$ for $t\geq t_0$. Then letting $w(t)=x'(t)/x(t)$, as in Theorem 1, we have $w(t)\geq\alpha_0(t)+\alpha_n(t)$, $n>0$.

Hence,

$$\ln\frac{x(t)}{x(t_0)}\geq\int_{t_0}^t [\alpha_0(s)+\alpha_n(s)]\,ds,\ n>0.$$

Thus

$$x(t)\geq x(t_0)\exp\int_{t_0}^t [\alpha_0(s)+\alpha_m(s)]\,ds, \tag{11}$$

that is, for any non-oscillatory solution,

$$\int_t^\infty [x(s)]^{-2}\,ds\leq[x(t_0)]^{-2}\int_t^\infty \exp\left\{-2\int_{t_0}^s [\alpha_0(r)+\alpha_m(r)]\,dr\right\}ds<\infty.$$

This contradicts the existence of a non-principal solution. Hence, Eq. (1) is oscillatory.

REMARK. Theorem 2 is stronger than Wintner's theorem. In fact, if (2) holds, (10) must hold, but (10) can hold while (2) does not hold. It improves Kamenev's theorem [3], which holds under $\alpha(t)=\int_t^\infty a(s)\,ds\geq 0$.

We illustrate the relationship between Wintner's theorem and Theorem 2 by considering the equation

$$x''+ct^{-2}x=0 \tag{12}$$

where c is a positive constant. It is clear that $\alpha_0(t)=c_0t^{-1}$, and further, that $\alpha_k(t)=c_kt^{-1}$ where we define the sequence c_k by

$$c_0=c$$
$$c_1=c_0^2$$
$$c_{k+1}=(c_0+c_k)^2, \qquad k=1,2,\ldots.$$

Wintner observed that his theorem guarantees oscillation for large t if $c\geq\frac{1}{2}$, whereas, (12) is actually oscillatory for $c>\frac{1}{4}$. To apply Theorem 2, we note that $\alpha_k(t)$ exists for all k. A simple calculation shows that (10) holds if for some k, $c_0+c_k>\frac{1}{2}$. It is readily verified that if $c=c_0\leq\frac{1}{4}$ then $c_0+c_k<\frac{1}{2}$ for all k. If $c_0>\frac{1}{4}$, then we see that

$$c_{k+1}-c_k=c_k^2+(2c_0-1)c_k+c_0^2>c_0-\tfrac{1}{4}$$

so that c_k forms a strictly increasing sequence which must eventually satisfy $c_0+c_k>\frac{1}{2}$. Thus Theorem 2 provides the correct range $c>\frac{1}{4}$.

Another generalization of Wintner's result was given by Hartman [1] and improved on in the versions of [2] published by Hartman (1973) and Birkhauser (1982). These results are sharp when applied to (12), but they do not seem comparable with Theorem 2.

THEOREM 3. *If Eq.* (1) *is non-oscillatory and for some positive number m,*

$$\int_t^\infty \exp\left\{2\int_{t_0}^s [\alpha_0(r)+\alpha_m(r)]\,dr\right\} ds=\infty, \tag{13}$$

then Eq. (1) *does not have an eigensolution of class* L^2, *that is, a solution* ($\neq 0$) *satisfying*

$$\int^\infty x^2(t)\,dt<\infty.$$

Proof. If Eq. (1) is non-oscillatory and $x(t)$ is any solution of Eq. (1), then $x(t)\neq 0$ for $t\geq t_0$. Letting $w(t)=x'(t)/x(t)$, as in Theorem 2, we have that (11) holds. Hence from (12)

$$\int^\infty x^2(t)\,dt\geq\int^\infty \exp\left\{2\int_{t_0}^s [\alpha_0(r)+\alpha_m(r)]\,dr\,ds=\infty\right.$$

for any solution of Eq. (1). This completes the proof.

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