CONVERGENCE OF THE HAUSDORFF MEANS OF DOUBLE FOURIER SERIES*

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In this paper we prove that if $\{s_{m,n}(x, y)\}$ is the sequence of partial sums of the Fourier series of a function $f(x, y)$, which is periodic in each variable and of bounded variation in the sense of Hardy-Krause in the period rectangle, then $\{s_{m,n}(x, y)\}$ converges uniformly to $f(x, y)$ in any closed region D in which this function is continuous at every point. This result is then used to prove that the regular Hausdorff means of the Fourier series of such a function also converge uniformly in such a region.

A simple corollary of these results is that neither the partial sums nor the regular Hausdorff means of the Fourier series of such a function will display the Gibbs phenomenon at any point of continuity of the function.

Let $f(x, y)$ be periodic with period 2π in each variable, and let $s_{m,n}(x, y)$ be the mn-th partial sum of the Fourier series of $f(x, y)$. Then

$$(1) \qquad s_{m,n}(x, y) = \int_{0,0}^{\pi/2,\pi/2} g(s, t) \frac{\sin ps}{s} \frac{\sin qt}{t} \, ds \, dt$$

where

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$$(2)\qquad g(s,t) = \frac{1}{\pi^2}\,\frac{s}{\sin s}\,\frac{t}{\sin t}\,\{f(x+2s,\ y+2t) + f(x+2s,\ y-2t)$$
$$+ f(x-2s,\ y+2t) + f(x-2s,\ y-2t)\}$$

and

$$p = 2m + 1; \quad q = 2n + 1 .$$

If $g(s,t)$ is absolutely bounded, non-negative, and monotonically decreasing in each variable, then a slight modification of the proof given by Hobson [2, p. 705-709] leads to the result

$$(3)\qquad |s_{m,n}(x,y) - f(x,y)| < 2\pi^2\,\phi\,(0,0)$$
$$+ g(0,0)\ \{4\pi/r + 2\,\Sigma_1^{\infty}\ \{\nu(\nu+r)\}^{-1}\} + (2/r)g(\epsilon,\epsilon) + \delta ,$$
$$\text{for } p,q > k' = k'(\delta),$$

where $\delta > 0$ and ϵ are fixed but otherwise arbitrary, $0 < \epsilon < \pi/2$, $r = \min\ \{[p\epsilon/\pi],\ [q\epsilon/\pi]\} - 3$ and $\phi(0,0) = g(0,0) - g(\epsilon,\epsilon)$, taking $(\sin x)/x = 1$ when $x = 0$.

In (3), δ may be chosen arbitrarily small. As for the other terms, all but the first one tend to zero as r tends to infinity, that is, as p and q tend to infinity. Choosing k'' large enough so that for $p,q > k''$, the contribution of these terms is less than δ, it follows that

$$(4)\qquad |s_{m,n}(x,y) - f(x,y)| < 2\pi^2\,\phi\,(0,0) + 2\delta,\quad p,q > \max\{k', k''\} .$$

If, in addition, $f(x,y)$ is continuous at (x,y), then ϵ may be taken small enough so that $\phi\,(0,0) < \delta/2\pi^2$. Then we have that

$$(5)\qquad s_{m,n}(x,y) \to f(x,y),\quad m,\ n \to \infty .$$

Hobson's proof leading to the conclusion expressed in (4) is also valid if only $g(s,t)$ can be expressed as the difference of two non-negative, bounded, monotonically decreasing functions. For then the integral (1) may be expressed as the difference of two such integrals and all the calculations carry through.

An examination of the foregoing indicates that the partial sums of the Fourier series of $f(x,y)$ will converge uniformly to $f(x,y)$ in any region over which $g(s,t)$ may be expressed as the difference of two non-negative, monotonically decreasing functions, provided that these functions can be chosen in such a way that they are uniformly bounded and equicontinuous at the origin. For then the inequality (4) will hold uniformly for all points (x,y) in this region by the uniform boundedness, and ϵ may be chosen small enough so that $\phi(0, 0) < \delta/2\pi^2$ uniformly by the equicontinuity.

THEOREM 1. Let $f(x, y)$ be normalized, periodic with period 2π in each variable, and of bounded variation in the sense of Hardy-Krause in the period cell. If $f(x, y)$ is also continuous at every point of a closed region D, then the partial sums of the Fourier series will converge to it uniformly on D. Consequently, the partial sums of the Fourier series of $f(x, y)$ cannot exhibit the Gibbs phenomenon at any point of continuity of $f(x, y)$.

Proof. The assumptions on $f(x, y)$ are sufficient to allow us to assume that there exists a region D' such that D is contained in the interior of D', both D and D' are rectangular with sides parallel to the coordinate axes, and $f(x, y)$ is continuous at every point of D'. By the foregoing discussion, to prove the theorem, it is sufficient to prove that for each $(x, y) \in D$, $g(s, t)$ can be expressed as the difference of two non-negative, monotonically decreasing functions, $g(s, t) = m(s, t) - p(s, t)$, such that the family $\{m(s, t), p(s, t): (x, y) \in D\}$ is uniformly bounded and equicontinuous at the origin.

Denote the cell determined by the coordinates (a_2, b_2), (a_2, b_1), (a_1, b_2) and (a_1, b_1), $a_1 < a_2$, $b_1 < b_2$, by $[a_2, b_2; a_1, b_1]$. Divide the cell $[2\pi, 2\pi; -2\pi, -2\pi]$ into the nine subcells $[x_1, y_1; -2\pi, -2\pi]$, $[x_2, y_1; x_1, -2\pi]$, $[2\pi, y_1; x_2, -2\pi]$, $[x_1, y_2; -2\pi, y_1]$, $[2\pi, y_2; x_2, y_1]$, $[x_1, 2\pi; -2\pi, y_2]$, $[x_2, 2\pi; x_1, y_2]$, $[2\pi, 2\pi; x_2, y_2]$ and $D' = [x_2, y_2; x_1, y_1]$. There is an overlap along the boundaries in some of these cells. However, this difficulty is easily removed by defining the cells so that they are open along the upper and right hand boundaries whenever there is overlap otherwise.

Restrict $f(x, y)$ to the cell $[x_1, y_1; -2\pi, -2\pi]$ and write

(6) $$e(x, y) = f(x, y) - f(-2\pi, y) - f(x, -2\pi) + f(-2\pi, -2\pi).$$

Then $e(-2\pi, -2\pi) = e(-2\pi, y) = e(x, -2\pi) = 0$. Since $e(x, y)$ is of bounded variation, it can be expressed as the difference of two positively monotonic functions, namely, its positive and negative variation functions. See McShane [3, p.250]. By considering the negatives of these functions, it follows that $e(x, y)$ can be expressed as the difference of two negatively monotonic functions, $e(x, y) = n_1(x, y) - q_1(x, y)$, and choosing these functions so that they are zero along the lines $x = -2\pi$, $y = -2\pi$, it is easy to show that they are monotonically decreasing in each variable.

Now set $e_2(y) = f(-2\pi, y) - f(-2\pi, -2\pi)$ so that $e_2(-2\pi) = 0$.

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Then again $e_2(y)$ may be expressed as the difference of the negatives of its negative and positive variation functions, $e_2(y) = n_2(y) - q_2(y)$, where $n_2(0) = q_2(0) = 0$, and each of these functions is monotonically decreasing. With $e_3(x)$ and its decomposition $n_3(x) - q_3(x)$ similarly defined, we have

$$f(x, y) = \{n_1(x, y) + n_2(y) + n_3(x) + f(-2\pi, -2\pi)\}$$
$$- \{q_1(x, y) + q_2(y) + q_3(x)\}$$
$$= n'(x, y) - q'(x, y),$$

where $n'(x, y)$ and $q'(x, y)$ are each absolutely bounded and monotonically decreasing in each variable.

In a similar manner, $f(x, y)$ may be restricted to each of the remaining eight subcells in turn, and expressed as the difference of two absolutely bounded, monotonically decreasing functions in each cell. The addition of a suitable constant to the decomposition functions in each cell and a combination of the results then yields a decomposition of $f(x, y)$ into two non-negative, monotonically decreasing, bounded functions on the cell $[2\pi, 2\pi; -2\pi, -2\pi]$,

(7) $$f(x, y) = n(x, y) - q(x, y).$$

Note that $n(x, y)$ and $q(x, y)$ are continuous, hence uniformly continuous on D', and that this implies the equicontinuity of the family $\{n(x', y), n(x, y'), q(x, y'), q(x', y): (x', y') \in D'\}$ on D'.

For $(x, y) \in D$, let

(8) $$h(s, t) = g(s, t) - g(s, 0) - g(0, t) + g(0, 0).$$

Then $h(s, 0) = h(0, t) = h(0, 0) = 0$, and again we may express $h(s, t)$ as the difference of the negatives of its negative and positive variation functions, $h(s, t) = m_1(s, t) - p_1(s, t)$, where these functions are monotonically decreasing, and for $0 \leq s, t \leq \pi/2$, they are bounded absolutely and uniformly by $V(f)$, where $V(f)$ denotes the total variation of $f(x, y)$ in the period cell.

Now let

$$M = \max\{n(-2\pi, -2\pi), q(-2\pi, -2\pi)\}.$$

Then $g(s, 0) = m_2(s) - p_2(s)$ and $g(0, t) = m_3(t) - p_3(t)$, where

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(9) $m_2(s) = \frac{2}{\pi^2} \{ \frac{s}{\sin s} \{n(x + 2x, y) - q(x - 2s, y)\} + M(\pi - s)\}$

$p_2(s) = \frac{2}{\pi^2} \{ \frac{s}{\sin s} \{q(x + 2s, y) - n(x - 2s, y)\} + M(\pi - s)\}$,

with $m_3(t)$ and $p_3(t)$ similarly defined, is a decomposition of $g(s, 0)$ and $g(0, t)$. Then

(10) $$g(s, t) = m(s, t) - p(s, t)$$

where

(11) $m(s, t) = m_1(s, t) + V(f) + m_2(s) + m_3(t) + |g(0, 0)|$

$p(s, t) = p_1(s, t) + V(f) + p_2(s) + p_3(t) + g(0, 0) + |g(0, 0)|$

is the required decomposition of $g(s, t)$.

To complete the proof, we examine the family $\{m(s, t), p(s, t) : (x, y) \in D\}$ for the non-negative property, uniform boundedness, monotonicity and equicontinuity at the origin. The functions $m_1(s, t) + V(f) + |g(0, 0)|$ and $p_1(s, t) + V(f) + g(0, 0) + |g(0, 0)|$ are clearly non-negative, monotonically decreasing and uniformly bounded by $2V(f) + 2 \sup |f(x, y)|$. As for the remaining functions on the right side in (11), we consider $m_2(s)$ and remark that the same argument holds for the others. The function $m_2(s)$ is bounded by M in view of (9) and the definition of M. It is non-negative since the functions $\{n(x + 2s, y) - q(x - 2s, y) + \pi M/2\}$ and $M(\pi/2 - s)$ are non-negative. Also, the function $\{n(x + 2s, y) - q(x - 2s, y) + \pi M/2\}$ is monotonically decreasing. To insure the monotonicity after multiplying by the factor $s/\sin s$, the function $M(\pi/2 - s)$ has been added.

We check equicontinuity at the origin. For this purpose, we examine only $m_1(s, t)$ and $m_2(s)$, the argument for $p_1(s, t)$, $m_3(t)$, $p_2(s)$ and $p_3(t)$ being similar, and the constants $V(f)$, $g(0, 0)$ and $|g(0, 0)|$ have no bearing on equicontinuity. Since the variation functions for $f(x, y)$ are continuous, hence uniformly continuous on D', the equicontinuity of the family $\{m_1(s, t) : (x, y) \in D\}$ follows, since $|m_1(0, 0) - m(s, t)|$, $0 \leq s, t \leq \epsilon$, does not exceed the variation of $f(x, y)$ in the cell $[x + \epsilon, y + \epsilon; x - \epsilon, y - \epsilon]$. Also, by the earlier remarks, the family $\{n(x', y), n(x, y'), q(x', y), q(x, y') : (x', y') \in D\}$ is equicontinuous on D'. But this implies the equicontinuity of the family $\{n(x \pm 2s, y), n(x, y \pm 2t), q(x \pm 2s, y), q(x, y \pm 2t) : (x, y) \in D\}$ at the origin. This completes the proof.

THEOREM 2. Let $f(x, y)$ be normalized, periodic with period 2π in each variable, and of bounded variation in the period cell. If $f(x, y)$ is also continuous in a closed region D, then the regular Hausdorff means of the partial sums of the Fourier series of $f(x, y)$ converge to $f(x, y)$ uniformly on D. Consequently, the regular Hausdorff means of the partial sums of the Fourier series of $f(x, y)$ cannot exhibit the Gibbs phenomenon at any point of continuity of $f(x, y)$.

Proof. If $\{s_{m,n}\}$ is any sequence, then the mn-th regular Hausdorff transform of this sequence is given by

$$(12)\quad h_{m,n}(s_{k,\ell}) = \sum_{0,0}^{m,n} \binom{m}{k}\binom{n}{\ell} s_{k,\ell} \int_{0,0}^{1,1} (1-\mu)^{m-k}\mu^{k}(1-v)^{n-\ell}v^{\ell}\, d^2 g(\mu, v),$$

where $g(\mu, v)$ is of bounded variation in the cell $[1, 1; 0, 0]$, $g(1, 1) = 1$ and

$$g(0, 0) = g(\mu, 0) = g(\mu, 0^+) = g(0, v) = g(0^+, v) = 0, \quad 0 \leq \mu, \ v \leq 1.$$

As such, $g(\mu, v)$ may be split up into two positively monotonic functions,

$$g(\mu, v) = p(\mu, v) - n(\mu, v),$$

where $p(\mu, v)$ and $n(\mu, v)$, being the positive and negative variation functions of $g(\mu, v)$ are of bounded variation in the unit cell and $V(g) = V(p) + V(n)$. The expression on the right in (12) may be then split into two parts, one involving an integral with respect to $n(\mu, v)$, and the other an integral with respect to $p(\mu, v)$. Doing this, we have

$$h_{m,n}\{s_{k,\ell}(x, y)\} - f(x, y) = h_{m,n}\{s_{k,\ell}(x, y) - f(x, y)\}$$

$$= \sum_{0,0}^{m,n} \binom{m}{k}\binom{n}{\ell}\{s_{k,\ell}(x, y) - f(x, y)\} \int_{0,0}^{1,1} (1-\mu)^{m-k}\mu^{k}(1-v)^{n-\ell}v^{\ell}\, d^2 p(\mu, v)$$

$$- \sum_{0,0}^{m,n} \binom{m}{k}\binom{n}{\ell}\{s_{k,\ell}(x, y) - f(x, y)\} \int_{0,0}^{1,1} (1-\mu)^{m-k}\mu^{k}(1-v)^{n-\ell}v^{\ell}\, d^2 n(\mu, v).$$

Set $\phi_{k,\ell} = \sup\{|s_{k,\ell}(x, y) - f(x, y)| : (x, y) \in D\}$. Then by Theorem 1, $\{\phi_{k,\ell}\}$ is a null sequence. Also, since $p(\mu, v)$ and $n(\mu, v)$ are of bounded variation in [1, 1; 0, 0], and, being the positive and negative variation function of $g(\mu, v)$, are zero along the coordinate axes, it follows that the Hausdorff transform relative to them is convergence preserving and regular for null sequences. See [1, pp. 17 and 33].

Thus

$$| h_{m,n} \{ s_{k,\ell}(x, y) \} - f(x, y) |$$

$$\leq \sum_{0,0}^{m,n} \binom{m}{k}\binom{n}{\ell} \phi_{k,\ell} \int_{0,0}^{1,1} (1-\mu)^{m-k} \mu^k (1-v)^{n-\ell} v^\ell d^2 p(\mu, v)$$

$$\sum_{0,0}^{m,n} \binom{m}{k}\binom{n}{\ell} \phi_{k,\ell} \int_{0,0}^{1,1} (1-\mu)^{m-k} \mu^k (1-v)^{n-\ell} v^\ell d^2 n(\mu, v)$$

$$\to 0, \quad m, n \to \infty ,$$

uniformly in D. This proves the theorem.

Remark. It is easy to show by example that the results expressed in Theorems 1 and 2 are not the best possible. However, the derivation of the best possible results appears to present some difficulties, so that the widest class of functions for which these theorems hold is an open question at the present time.

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