FREE PRODUCTS OF HOPFIAN LATTICES

Dedicated to the memory of Hanna Neumann

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1. Introduction

In this paper we are going to prove the following results:

THEOREM 1. *There exist two bounded hopfian lattices such that their* $\{0,1\}$-*free product is not hopfian.*

THEOREM 2. *There exist two hopfian lattices such that their free product is not hopfian.*

In Theorem 2 free product (coproduct, sum) has its usual meaning (see, for instance, [4]); in Theorem 1 we use the usual definition but all lattices are assumed to be bounded (that is, having a least element 0 and largest element 1) and all homomorphisms are assumed to be $\{0,1\}$-homomorphisms (that is, homomorphisms preserving 0 and 1).

Recall, that a lattice L (group, ring, and so on) is called *hopfian* if L is not isomorphic to any proper quotient L/Θ (proper means that $\Theta \neq \omega$); or equivalently, if any onto endomorphism of L is an automorphism. A remarkable result of Evans [3] states that every finitely presentable lattice is hopfian. A finitely generated non-hopfian lattice has recently been found by Wille (unpublished).

This paper grew out of a colloquium lecture of H. Neumann in which she reported on hopfian groups, in particular on the result of [2], namely that the free product of two finitely generated hopfian groups is hopfian. The problem whether this is true in general was left open. Theorem 2 of this paper gives a negative answer to this question for lattices. Later it was shown by the second author [13] that a modification of the set theoretic scheme of this paper (see section 3) could be used to settle the original group theoretic problem.

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The lattices we construct for Theorem 2 are all $> \aleph_0$ generated. It is an open problem whether this result could be improved. Could the two lattices be constructed to be $\aleph_0$-generated; one finitely generated, one $\aleph_0$-generated; one finitely presented, one $\aleph_0$-generated; both finitely generated; or the ultimate: one finitely generated and the other finitely presented? The last two possibilities are ruled out if the following conjecture is true: the free product of two bounded hopfian lattices is hopfian again.

The basic idea of the construction is to use a set theoretic scheme which provides two "hopfian objects" whose union is not hopfian. The technical work in section 3 converts this scheme into two hopfian graphs whose disjoint union is not hopfian.

Graphs can be turned into lattices using our earlier paper [7]; this conversion preserves the properties of being hopfian and being non-hopfian, while the disjoint union of graphs is transformed into the $\{0,1\}$-free product of the corresponding lattices. Thus the results of section 3 yield Theorem 1. To obtain Theorem 2 it is necessary to do some more work about free products of lattices and this is accomplished in section 4. All the lattice theoretic lemmas are proved in section 2.

2. Lattice theoretic lemmas

By a *graph* $\langle X;R\rangle$ we mean a nonvoid set X with a symmetric binary relation R such that $\{a,a\}\in R$ for no $a\in X$ (that is, an undirected graph with no loops). A *triangle* of $\langle X;R\rangle$ is a three element subset $\{a_0,a_1,a_2\}$ of X such that $\{a_0,a_1\},\{a_1,a_2\},\{a_2,a_0\}\in R$. All graphs in this paper will be assumed to have the property that every vertex is contained in at least one triangle.

Let $\langle X;R\rangle$ be a graph, let $F_{0,1}(X)$ be the bounded lattice freely generated by X, and let Θ be the smallest congruence relation on $F_{0,1}(X)$ such that $x\wedge y\equiv 0\,(\Theta)$ and $x\vee y\equiv 1\,(\Theta)$ for all $\{x,y\}\in R$. The lattice $M(X,R)=F_{0,1}(X)/\Theta$ constructed in [7] has the following properties:

(i) $X\subseteq M(X,R)$ and no two distinct elements of X are comparable in $M(X,R)$;

(ii) X generates $M(X,R)$;

(iii) $\{a,b\}$ is a complemented pair in $M(X,R)$ if and only if either $\{a,b\}=\{0,1\}$ or $\{a,b\}\in R$ (this follows from [1] or [5]).

Since $M(X,R)$ is the "most free" lattice satisfying (i)–(iii), every compatible mapping $\phi:\langle X;R\rangle\to\langle Y;S\rangle$ extends uniquely to a $\{0,1\}$-homomorphism $M(\phi):M(X,R)\to M(Y,S)$; in [7] we proved that every $\{0,1\}$-homomorphism $f:M(X,R)\to M(Y,S)$ is equal to $M(\phi)$ for some compatible mapping $\phi:\langle X;R\rangle\to\langle Y;S\rangle$.

The lattice $M(X,R)$ can also be described as a subset of $F_{0,1}(X)$ for which

(iv) if $a,b\in M(X,R)$ then $a\leqq b$ in $M(X,R)$ if and only if $a\leqq b$ in $F_{0,1}(X)$;

(v) the 0 and 1 of $M(X,R)$ is the 0 and 1 of $F_{0,1}(X)$, respectively;

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(vi) if $a, b \in M(X, R) - \{0, 1\}$, then $a \equiv b(\Theta)$ if and only if $a = b$ (see [1] or [5]).

(vii) if $a, b, c \in M(X, R) - \{0, 1\}$ and if c is a lower bound of $\{a, b\}$, then $a \wedge b$ formed in $F_{0,1}(X)$ lies in $M(X, R)$ and is equal to the meet of a, b in $M(X, R)$; dually for the join.

Now we start proving our lemmas.

Lemma 1. *X is the set of all join and meet-irreducible elements of $M(X, R)$.*

Proof. It is known (Whitman [14]) that any $x \in X$ is irreducible in $F_{0,1}(X)$. If $x = y \vee z$ in $M(X, R)$, $y \neq x$, $z \neq x$, then $\{y, z\}$ is disjoint with $\{0, 1\}$ and so $x = y \vee z$ in $F_{0,1}(X)$ by (vii), a contradiction. It follows from (ii) that no other element is irreducible.

Let M_5 denote the five element modular nondistributive lattice, see Figure 1. If $\{a_0, a_1, a_2\}$ is a triangle in $\langle X; R \rangle$, then $\{0, a_0, a_1, a_2, 1\}$ is a sublattice isomorphic to M_5; we call it the *sublattice associated with the triangle* $\{a_0, a_1, a_2\}$.

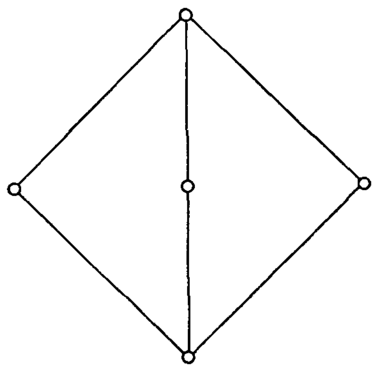

Figure 1

The following result is implicit in [12]:

Lemma 2. *A sublattice of $M(X, R)$ is isomorphic with M_5 if and only if it is associated with a triangle of $\langle X; R \rangle$.*

Proof. "If" being trivial we prove "only if". Let $\{o, a, b, c, i\}$ be a sublattice of $M(X, R)$ isomorphic to M_5 with the bounds o and i. If $o = 0$ and $i = 1$, then by (iii), $\{a, b, c\}$ is a triangle of $\langle X; R \rangle$ as required. If $o \neq 0$ and $i \neq 1$, then by (vii), $\{o, a, b, c, i\}$ is also a sublattice of $F_{0,1}(X)$ which is known ([10] and [14]) not to have such sublattices. Finally, let $o \neq 0$ and $i = 1$ (or dually). Then $o = a \wedge b = a \wedge c$ in $F_{0,1}(X)$ by (vii). Therefore, by a result of Jónsson [10], $o = a \wedge (b \vee c)$ in $F_{0,1}(X)$. Since $b \vee c \equiv 1(\Theta)$ we conclude that $a \equiv o(\Theta)$, contradicting (vi).

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COROLLARY 3. *$M(X,R)$ has no sublattice isomorphic to the lattice of Figure 2 or to $M_5 \times 2$.*

PROOF. Indeed, if $\{o, a_1, a_2, b, c, i\}$ is a sublattice isomorphic to the lattice of Figure 2, then by Lemma 2, $\{a_i, b, c\}$ is a triangle for $i = 1,2$ and so $a_1, a_2 \in X$. Since $a_1 < a_2$ this contradicts (i). Also, if a sublattice is isomorphic to $M_5 \times 2$, then we find a sublattice isomorphic to M_5 not including $\{0,1\}$, contradicting Lemma 2.

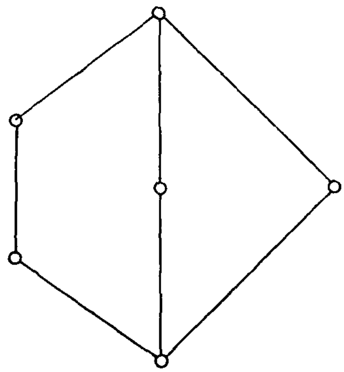

Figure 2

Call a graph $\langle X; R\rangle$ *hopfian* if every compatible map of $\langle X; R\rangle$ onto itself is an automorphism.

LEMMA 4. *$\langle X; R\rangle$ is hopfian if and only if $M(X,R)$ is hopfian.*

PROOF. Let $\langle X; R\rangle$ be hopfian and let f be an onto endomorphism of $M(X,R)$. Then there exists a compatible map $\phi : X \to X$ such that $f = M(\phi)$. If ϕ is not onto, then $x \notin \phi(X)$ for some $x \in X$. But $f(M(X,R))$ is generated by $\phi(X)$ by (ii), x is irreducible by Lemma 1, hence $x \notin f(M(X,R))$, contradicting that f is onto. Therefore ϕ is onto. Since $\langle X; R\rangle$ is hopfian, ϕ is an automorphism and so there is another automorphism ϕ' such that $\phi\phi'$ and $\phi'\phi$ are the identity map on X. Thus, both $M(\phi\phi') = M(\phi)M(\phi')$ and $M(\phi'\phi) = M(\phi')M(\phi)$ are equal to the identity map of $M(X,R)$ and so $f = M(\phi)$ is an automorphism, which was to be proved. Conversely, if $\langle X; R\rangle$ is not hopfian, then there is a compatible onto map $\phi : X \to X$ that is not an automorphism. Now if $M(X,R)$ is hopfian, then $M(\phi)$ is an automorphism, hence it has an inverse $M(\phi')$. Just as before, ϕ' is an inverse of ϕ, a contradiction. This completes the proof of the lemma.

Lemma 4 is the Reduction Theorem. Combined with the next result it completely reduces Theorem 1 to a statement on graphs.

LEMMA 5. *Let $\langle X_i; R_i\rangle$ be graphs $i = 1,2$, and let $\langle X; R\rangle$ be their disjoint union. Then $M(X,R)$ is the $\{0,1\}$-free product of $M(X_1,R_1)$ and $M(X_2,R_2)$.*

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PROOF. Let $\phi_i : X_i \to X$ be the natural embedding of X_i into X for $i = 1, 2$. Then $M(\phi_i)$ is a $\{0, 1\}$-homomorphism of $M(X_i, R_i)$ into $M(X, R)$. Now let L be a bounded lattice and let f_i be a $\{0, 1\}$-homomorphism of $M(X_i, R_i)$ into L for $i = 1$, 2. Let ψ_i be the restriction of f_i to X_i $(i = 1, 2)$ and define $\psi : X \to L$ by $\psi(x) = \psi_i(x)$ for $x \in X_i$ $(i = 1$ or $2)$. If $\{x, y\} \in R$, then $\psi(x)$ and $\psi(y)$ are complementary in L (since $\{x, y\} \in R$ for *no* $x \in X_1$ and $y \in X_2$) and so there is a unique homomorphism $g : M(X, R) \to L$ extending ψ. Since $gM(\phi_i)$ and f_i agree on X_i we obtain $gM(\phi_i) = f_i$ for $i = 1, 2$. We have verified that $M(X, R)$ is the $\{0, 1\}$-free product of $M(X_1, R_1)$ and $M(X_2, R_2)$.

Finally, we shall need in section 4 some results on free products. We start with a result of Lakser [11]:

LEMMA 6. *Let L_i, $i \in I$ be lattices and let L be a free product of the L_i, $i \in I$. Let A be a sublattice of L isomorphic to M_5. Then either $A \subseteq L_i$ for some $i \in I$ or some L_i has a sublattice isomorphic to the lattice of Figure* 2 *or to* $M_5 \times 2$.

Combining Lemma 6 with Corollary 3 and Lemma 2 we obtain

COROLLARY 7. *Let L be a free product of the $M(X_i, R_i)$, $i \in I$. If M is a sublattice of L and M is isomorphic to M_5, then $M \subseteq M(X_i, R_i)$ for some $i \in I$ and M is associated with a triangle of $\langle X_i; R_i \rangle$.*

We close this section with a generalization of Lemma 1:

LEMMA 8. *Let L_i, $i \in I$ be lattices and let L be a free product of the L_i, $i \in I$. If $a \in L_i$ is join-irreducible in L_i, then it is also join-irreducible in L and dually.*

PROOF. For the proof of this lemma we have to assume that the reader is familiar with the notation and results of [6], in particular with pp. 233–235. Let a be join-reducible in L, $a = b \vee c$. Let p and q be polynomials with $\langle p \rangle = b$, $\langle q \rangle = c$. Then $a \subseteq p \vee q$ and $p \subseteq a$, $q \subseteq a$. Of the rules 3.(1)–3.(6) only two may be applied to $a \subseteq p \vee q$, namely 3.(5) and 3.(2). If 3.(5) applies, then $a \subseteq p$ or $a \subseteq q$, and so $b = \langle p \rangle = a$ or $c = \langle q \rangle = a$. If $a \subseteq p \vee q$ because of 3.(2), then $a^{(j)} \leqq (p \vee q)_{(j)} = p_{(j)} \vee q_{(j)}$ for some $j \in I$. However, $a^{(j)}$ exists only for $j = i$, hence $a^{(i)} = a \leqq p_{(i)} \vee q_{(i)}$. Furthermore, $p_{(i)} \leqq a$ and $q_{(i)} \leqq a$, so $p_{(i)} \vee q_{(i)} = a$. In view of the join-irreducibility of a in L_i we get $p_{(i)} = a$ or $q_{(i)} = a$, say $p_{(i)} = a$. So $p_{(i)} \subseteq p \subseteq a$ and therefore $a = p_{(i)} \subseteq p \subseteq p^{(i)} \subseteq a^{(i)} = a$, yielding $b = \langle p \rangle = a$, as required.

COROLLARY 9. *Let L be a free product of the $M(X_i, R_i)$, $i \in I$. Then $\bigcup (X_i \mid i \in I)$ can be characterized as the set of all irreducible elements of L.*

3. The scheme and its graph representations

Let N be the set of all positive integers and let Z be the set of integers. Set

$$C_n = \{n\} \times Z.$$

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If there is no danger of confusion, we write $a \in C_n$ for $\langle n, a\rangle$. Let us consider the sets C_n as arranged in Figure 3. There are two arrows pointing at C_n for each $n \in N$. The arrow $C_{2n} \to C_n$ represents the map $x \mapsto 2x$, the arrow $C_{2n+1} \dashrightarrow C_n$ represents the map $x \mapsto 2x+1$. For $m, n \in N$ let us define $f_{mn} : C_m \to C_n$ as follows:

(i) if $m = n$, then f_{mm} is the identity map on C_m; (ii) if there is a sequence of arrows from C_m to C_n let f_{mn} be the composition of the maps represented by the arrows; (iii) otherwise, f_{mn} is not defined.

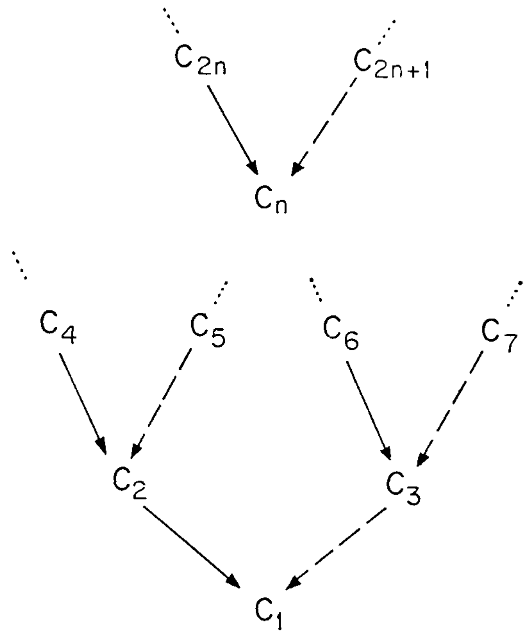

Figure 3

Observe, that between C_m and C_n there is at most one sequence of arrows, so if f_{mn} is defined it is defined uniquely. Now set

$$A_0 = \bigcup(C_{2k} \mid k \geqq 1),\ A_1 = \bigcup(C_{2k+1} \mid k \geqq 0),$$
$$A_2 = A_0 \cup A_1 = N \times Z.$$

Call a map $g : A_j \to A_j (j = 0, 1, \text{ or } 2)$ *admissible* if for any $C_m \subseteq A_j$, g restricted to C_m agrees with some f_{mn}. Call A_j *hopfian* if every admissible onto map has an inverse which is also admissible, or equivalently, if the only admissible onto map is the identity map.

LEMMA 10. *A_0 and A_1 are hopfian but A_2 is not hopfian.*

PROOF. Define g on A_2 as follows: g on C_1 is the identity map; g is $f_{2n.n}$ on C_{2n} for $n \geqq 1$ and g is $f_{2n+1.n}$ on C_{2n+1} for $n \geqq 1$. g is onto since

$$C_n = f_{2n.n}(C_{2n}) \cup f_{2n+1.n}(C_{2n+1}),$$

for every $n \in N$. But g is not one-to-one as every element of C_1 is the image of two distinct elements. Hence A_2 is not hopfian. It is somewhat more complicated to show that A_0 and A_1 are hopfian. We will show it for A_0. We start with an observation:

If $f_{nk}(x) = -1 \in C_k$ and $n = 2k$ or $n = 2k+1$, then $x = -1$ and $n = 2k+1$. Consequently,

$$\text{if } f_{ik}(x) = -1 \in C_k, \text{ then } i \text{ is odd.}$$

Now let $g : A_0 \to A_0$ be an admissible onto map. We claim that g is the identity map on A_0 (so we prove more than hopfian). Indeed, if g is not the identity map on A_0, then there is a k such that g restricted to C_k is f_{km} for some $m \neq k$. We claim that $-1 \in C_k$ is not in $g(A_0)$. Indeed, if $-1 = g(x)$, then $-1 = f_{nk}(x)$ for some $n \in N$ and $C_n \subseteq A_0$. But we have shown above that this would imply that n is odd, contradicting $C_n \subseteq A_0$.

Observe, that -1 is the only element of C_k not necessarily in $g(A_0)$; the proof of the fact that A_1 is hopfian requires 0 instead of -1 and is analogous to the proof above.

Now we start constructing the graphs. First, some definitions. Let $\langle X; R\rangle$ be a graph. The vertices x, y are said to be *triangle-connected* if there is a sequence $T_0, T_1, \cdots, T_n$ of triangles such that $x \in T_0$, $y \in T_n$ and $T_i \cap T_{i+1} \neq \phi$ for $i = 0, 1, \cdots, n-1$. *Triangle-connected components* of a graph and *triangle-connected graphs* are defined in the obvious fashion. Finally, a graph is *rigid* if the identity map is the only compatible map of the graph into itself. The following result is a weak version of a theorem of Hell [9]:

LEMMA 11. *For every infinite cardinal* $\mathfrak{m}$ *there exists a triangle-connected rigid graph of cardinality* $\mathfrak{m}$.

The idea of the transformation of the scheme into graphs is to replace each element in C_n by a copy of an infinite triangle-connected rigid graph $\langle V; E\rangle$ and to add some more edges so that the only compatible maps will be of the form

$$(\text{identity on } V) \times f_{mn}.$$

So let $\langle V; E\rangle$ be an infinite triangle-connected rigid graph and assume that $N \subseteq V$. For every $n \in N$ we define the graph

$$H_n = \langle V \times Z; S_n\rangle$$

as follows (see Figure 4 which indicates the edges listed under (b) as segments of the horizontal lines between two consecutive dots):

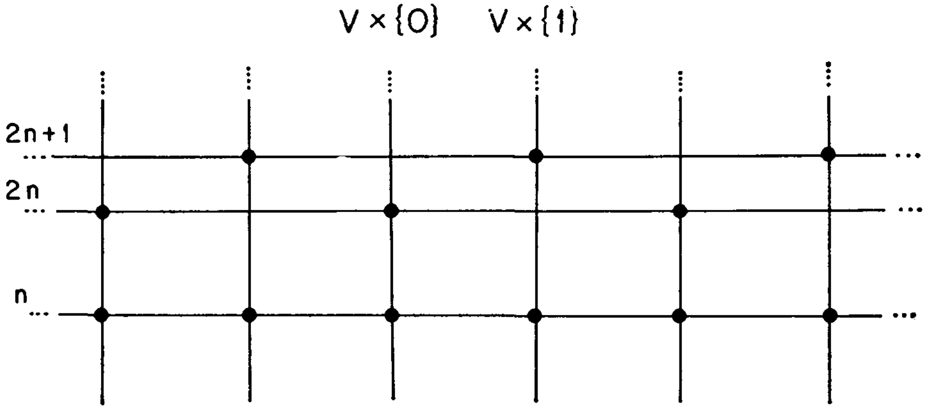

Figure 4

(a) for $i \in Z$ and $\{v, v'\} \in E$ we set

$$\{\langle v, i\rangle, \langle v', i\rangle\} \in S_n;$$

(b) for all $m \in N$ such that f_{mn} is defined and for all $i \in Z$

$$\{\langle m, f_{mn}(i)\rangle, \langle m, f_{mn}(i+1)\} \in S_n;$$

(c) no other pairs are in S_n.

(a) shows that $V \times \{i\}$ with S_n restricted to $V \times \{i\}$ is isomorphic to $\langle V; E\rangle$. The edges under (b) will be called *mixed*. For each $m \in N$, the set $\{m\} \times Z$ is called the *level m of* H_n. Observe that a mixed edge of H_n is always on a level, in fact on a level m with $m \geqq n$.

H_n is *connected* since $\langle V; E\rangle$ is; in fact, any two copies of $\langle V; E\rangle$ are connected on the level n.

Now define the map g_{mn} for all $\langle m, n\rangle$ for which f_{mn} is defined:

$$g_{mn} = (\text{identity map on } V) \times f_{mn},$$

that is,

$$g_{mn}\langle v, z\rangle = \langle v, f_{mn}(z)\rangle.$$

g_{mn} is obviously a compatible map of H_m into H_n. The converse of this statement is the crucial step in this section.

LEMMA 12. *Let h be a compatible map of H_m into H_n. Then $h = g_{mn}$.*

PROOF. Since all mixed edges are on a level it follows easily that all the triangles of H_m are contained in the $V \times \{i\}, i \in Z$. Thus the $V \times \{i\}, i \in Z$ are the triangle-connected components of H_m and so they have to be mapped by h into the triangle-connected components of H_n. Therefore, for each $i \in Z$ there exists a $\phi(i) \in Z$ such that

$$h(V \times \{i\}) \subseteq V \times \{\phi(i)\}.$$

Since $\langle V; E\rangle$ is rigid we conclude that

$$h\langle v, i\rangle = \langle v, \phi(i)\rangle,$$

so that $\phi : Z \to Z$ determines h. To show $h = g_{mn}$ it will be sufficient to verify that $\phi = f_{mn}$.

Claim 1. *Let $i, j \in Z$, $m \in N$. Then there exist $i', j' \in Z$ and $m' \in N$ such that $i' - j'$ is odd, and*

$$f_{m'm}(i') = i \text{ and } f_{m'm}(j') = j.$$

Proof. If $i - j$ is odd we set $i' = i, j' = j$, and $m' = m$. Now let $i - j = 2^k \cdot t$, when t is odd. We proceed by induction on k. If for smaller exponents the statement has been verified, then if i is even, set $m^* = 2m$ and if i is odd, set $m^* = 2m + 1$. In both cases there are i^* and $j^* \in Z$ such that $i^* - j^* = 2^{k-1} \cdot t$, $f_{m^*m}(i^*) = i$, and $f_{m^*m}(j^*) = j$. Applying the induction hypothesis to i^*, j^*, m^* we get the i', j', and m' as required.

Claim 2. *ϕ is one-to-one.*

Proof. Indeed, if $\phi(i) = \phi(j)$ for $i, j \in Z$ and $i \neq j$, then we choose i', j', and m' as in Claim 1. Consider the map $g = hg_{m'm}$ of $H_{m'}$ into H_n. Since $g_{m'm}$ and h are compatible maps so is g. Consider the path

$$\langle m', i'\rangle, \langle m', i' + 1\rangle, \cdots, \langle m', j'\rangle$$

in $H_{m'}$. Any two adjacent vertices are connected by an edge since this is the level m' in $H_{m'}$. Therefore,

$$g\langle m', i'\rangle, g\langle m', i' + 1\rangle, \cdots, g\langle m', j'\rangle$$

has the same property. Moreover,

$$\begin{aligned} g\langle m', i'\rangle &= h(g_{m'm}\langle m', i'\rangle) = h\langle m', f_{m'n}(i')\rangle \\ &= \langle m', \phi(i)\rangle = \langle m', \phi(j)\rangle = h\langle m', f_{m'm}(j')\rangle = g\langle m', j'\rangle. \end{aligned}$$

We found a cycle of an odd length on the level m' of H_n, which is impossible since any level is two-colorable.

Claim 3. $\phi(0) = f_{mn}(0)$.

Proof. $V \times \{0\}$ is distinguished in H_m by the fact that on each level $2^k m$ there is a mixed edge with one vertex in $V \times \{0\}$. Therefore, $V \times \{\phi(0)\}$ has the same property in H_n and so

$$\phi(0) \in f_{,k\ldots}(C_{,k}) \text{ for } k = 1, 2, \cdots.$$

So $\phi(0) = f_{mn}(x)$ where x is divisible by 2^k for all $k \geqq 1$. Thus $x = 0$ as claimed.

CLAIM 4. $\phi(1) = f_{mn}(1)$.

PROOF. Same proof as that of Claim 3; use the levels $2^k(2m+1)$.

Now we prove that $\phi(x) = f_{mn}(x)$ for all $x \in Z$. We already know that ϕ is one-to-one and $\phi(x) = f_{mn}(x)$ for $x = 0$ and $x = 1$. There is a mixed edge between $\langle m, 1\rangle$ and $\langle m, 2\rangle$ in H_m; therefore there is a mixed edge between $\langle m, 1\rangle$ and $\langle m, \phi(2)\rangle$ in H_n. Consequently, $\phi(2) = 0$ or $\phi(2) = 2$. But $\phi(2) = 0$ contradicts that ϕ is one-to-one, hence $\phi(2) = 2$. Proceeding thus, we conclude that $\phi(x) = f_{mn}(x)$ for all $x \in Z$, concluding the proof of Lemma 12.

Define G_0 as the disjoint union of the $H_{2n}, n = 1, 2, \cdots$ and G_1 as the disjoint union of the H_{2n+1}, $n = 0, 1, 2, \cdots$. Let G_2 be the disjoint union of G_0 and G_1. Since H_k are components of G_i, Lemma 10 and Lemma 12 combined yield that G_0 and G_1 are hopfian but G_2 is not. So if we use these graphs to construct the lattices $M(G_i)$ $i=0, 1, 2$, by Lemmas 4 and 5 we have finished the proof of Theorem 1.

Combining the results of [8] and [9] yields the existence of $2^{\mathfrak{m}}$ pairwise non-isomorphic rigid triangle-connected graphs of any cardinality $\mathfrak{m} \geqq \aleph_0$. Since the correspondence $\langle X; R\rangle \to M(X, R)$ is, in fact, a full embedding of the category of graphs into the category of bounded lattices, it follows that for every infinite $\mathfrak{m}$ there are $2^{\mathfrak{m}}$ pairs of hopfian lattices of cardinality $\mathfrak{m}$ whose $\{0, 1\}$-free product is not hopfian and any two such pairs are non isomorphic. This is obviously best possible.

4. Free products

In this section we prove Theorem 2. Throughout the proof let $\langle V; E\rangle$ be a fixed triangle-connected rigid graph with $|V| > \aleph_0$.

Let $K_n = \langle U_n; T_n\rangle$ be defined by

$$U_n = (V \times Z) \times \{n\},$$

$$\{\langle\langle v, z\rangle, n\rangle, \langle\langle v', z'\rangle, n\rangle\} \in T_n \text{ iff } \{\langle v, z\rangle, \langle v', z'\rangle\} \in S_n.$$

Let h_{mn} be the compatible map between K_m and K_n corresponding to g_{mn}, that is, $h_{mn}\langle\langle v, z\rangle, m\rangle = \langle g_{mn}\langle v, z\rangle, n\rangle$. For $i \in N$ set

$$B_i = M(U_i, T_i).$$

Let L_0 be the free product of the B_{2i}, $i = 1, 2, \cdots$ and let L_1 be the free product of the B_{2i+1}, $i = 0, 1, \cdots$. Finally, let L be the free product of L_0 and L_1, or equivalently, of all the B_i, $i = 1, 2, \cdots$.

LEMMA 13. *L is not hopfian.*

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Proof. Observe that L is generated by $\bigcup(U_i \mid i = 1,2,\cdots)$. Now we define a map on L patterned after the map g of Lemma 10. Let g_1 be the identity on B_1 viewed as an embedding of B_1 into L; let g_{2i} be the extension of $g_{2i,i}$ into a homomorphism of B_{2i} into $B_i \subseteq L$; let g_{2i+1} be the extension of $g_{2i+1,i}$ into a homomorphism of B_{2i+1} into $B_i \subseteq L$. Since L is a free product there is a homomorphism g of L into L extending all the g_i, $i = 1,2,\cdots$. The endomorphism g is onto but not one-to-one and so L is not hopfian.

Lemma 14. *L_0 is hopfian.*

Proof. Let ψ be an onto endomorphism of L_0. We are going to show that ψ is the identity map which yields in particular that L_0 is hopfian.

L_0 is generated by $U = \bigcup(U_{2i} \mid i = 1,2,\cdots)$. U_{2i} is a set of irreducibles in B_{2i} and so by Lemma 8, U_{2i} is a set of irreducibles of L_0. Therefore

$$\psi(U) \supseteq U.$$

Every $u \in U_{2i}$ is contained in a triangle of U_{2i}. If this triangle is collapsed by ψ the B_{2i} is mapped into a singleton by ψ; otherwise, the image of the triangle yields a sublattice isomorphic to M_5 and so, by Corollary 7, $\psi(u) \in U_{2i}$ is mapped by ψ into U_{2j}, then $\psi(U_{2i}) \subseteq U_{2j}$ since the $\langle U_{2k}; T_{2k}\rangle$ are all connected. If ψ is not the identity map on L_0, then ψ is not the identity map on some U_{2i}. Since B_{2i} has no nontrivial non-constant endomorphisms, $\psi(U_{2i})$ is either disjoint from U_{2i} or is a singleton. Thus $|((V \times \{-1\}) \times \{2i\}) \cap \psi(U_{2i})| \leqq 1$ and it follows from the statements in the proof of Lemma 10 that $(V \times \{-1\}) \times \{2i\}$ is disjoint from any $\psi(U_{2j})$ (except if $\psi(U_{2j})$ is a singleton). Consequently, $(V \times \{-1\}) \times \{2i\}$ would have to be covered by those $\psi(U_{2j})$ which are singletons. This is impossible however since there are only finite or countably infinite such $\psi(U_{2j})$ and $(V \times \{-1\}) \times \{2i\}$ is uncountable.

A similar proof shows that L_1 is hopfian. This completes the proof of Theorem 2.

It can be observed that if $\langle X_1; R_1\rangle$ and $\langle X_2; R_2\rangle$ are hopfian graphs, then a free product of $M(X_1, R_1)$ and $M(X_2, R_2)$ is always hopfian. Therefore, there are $2^{\mathfrak{m}}$ pairs of pairwise nonisomorphic lattices of cardinality $\mathfrak{m}$ such that in any pair the lattices are hopfian and also their free product is hopfian. Also, for $\mathfrak{m} > \aleph_0$ we also have $2^{\mathfrak{m}}$ pairs such that the lattices are hopfian but their free product is not. Both results are best possible. It would be interesting to prove the last statement also for $\mathfrak{m} = \aleph_0$.

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