Appendices

A

Symbols and SI units [1, 2]

Symbol	Quantity	Unit	Dimension
I	current	A	$Q\ T^{-1}$
K	sheet current density	A/m	$Q\ T^{-1}\ L^{-1}$
J	volume current density	A/m^2	$Q\ T^{-1}\ L^{-2}$
B	magnetic flux density	T	$M\ T^{-1}\ Q^{-1}$
Φ_B	magnetic flux	$Wb = T\ m^2$	$M\ L^2\ T^{-1}\ Q^{-1}$
μ_0	permeability of free space	$=4\pi\ 10^{-7}$ T m/A	$M\ L\ Q^{-2}$
M	magnetization	A/m	$Q\ T^{-1}\ L^{-1}$
H	magnetic intensity	A/m	$Q\ T^{-1}\ L^{-1}$
A	vector potential	Wb/m = T m	$M\ L\ T^{-1}\ Q^{-1}$
V_m	scalar potential	A	$Q\ T^{-1}$
L, M	self, mutual inductance	H = Wb/A	$M\ L^2\ Q^{-2}$
ρ	electric charge density	C/m^3	$Q\ L^{-3}$
V	potential difference	V	$M\ L^2\ T^{-2}\ Q^{-1}$
E	electric field intensity	V/m	$M\ L\ T^{-2}\ Q^{-1}$
σ	conductivity	$(\Omega\ m)^{-1}$	$T\ Q^2\ M^{-1}\ L^{-3}$
ε	permittivity	farad/m	$T^2\ Q^2\ M^{-1}\ L^{-3}$
D	electric flux density	coulomb m^2	$Q\ M^{-2}$
F	force	N = J/m	$M\ L\ T^{-2}$
W	stored energy	J = N m	$M\ L^2\ T^{-2}$
P	power	W = J/s	$M\ L^2\ T^{-3}$

References

[1] E. R. Cohen (ed.), *The Physics Quick Reference Guide*, American Institute of Physics, 1996, p. 37–47.

[2] D. Halliday & R. Resnick, *Physics for Students of Science and Engineering*, Wiley, 1962, appendix G.

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B

Vector analysis

Vector analysis plays an essential role in describing the theory of magnetic phenomena.[1, 2] A vector V is a quantity that has both a magnitude and a direction. A scalar S is a quantity that only has an associated magnitude. Vector fields are functions that describe a physical quantity at every point in space.

The vector differential operator (*del*) is

$$\nabla = \frac{\partial}{\partial x}\,\hat{x} + \frac{\partial}{\partial y}\,\hat{y} + \frac{\partial}{\partial z}\,\hat{z}.$$

When ∇ is applied to a scalar function, it results in a vector known as the *gradient*.

$$\nabla S = \frac{\partial S}{\partial x}\,\hat{x} + \frac{\partial S}{\partial y}\,\hat{y} + \frac{\partial S}{\partial z}\,\hat{z}.$$

The gradient gives a measure of the rate of change of a vector. The dot product of ∇ with a vector forms a scalar known as the *divergence*.

$$\nabla \cdot \overrightarrow{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}.$$

Roughly speaking, the divergence gives a measure for the spreading out of a function away from a localized source. The *Laplacian* is an important operator that describes the second derivative of a scalar function and is given by

$$\nabla^2 S = \nabla \cdot \nabla S = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}.$$

It is also useful to define the Laplacian of a vector function, which is given in Cartesian coordinates as

$$\nabla^2 \overrightarrow{V} = \nabla^2 V_x\,\hat{x} + \nabla^2 V_y\,\hat{y} + \nabla^2 V_z\,\hat{z}.$$

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The cross product of ∇ with a vector forms another vector known as the *curl*.

$$\nabla \times \overrightarrow{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ V_x & V_y & V_z \end{vmatrix}.$$

The curl gives a measure of the tendency of the vector to circulate around some source. According to *Helmholtz's theorem*,[3] a vector function that is bounded at infinity can be uniquely defined by specifying its divergence and its curl.

If we consider a volume of space V enclosed by a surface S, then we find that any changes in a vector W inside the volume must match the flux of W through the boundary surface. This is the basis for an important result known as Gauss's *divergence theorem*.[4]

$$\int \nabla \cdot \overrightarrow{W}\, dV = \int \overrightarrow{W} \cdot \hat{n}\, dS,$$

where n is the normal vector to the surface. If, on the other hand, we break up the surface S into a number of smaller areas and look at the net result of the circulation in all the subareas, we find that the circulations cancel in the interior of the region and only give a net result around the perimeter of S. The result is known as *Stokes's theorem*.[5]

$$\int (\nabla \times \overrightarrow{W}) \cdot \hat{n}\, dS = \oint \overrightarrow{W} \cdot \overrightarrow{dl}$$

Some other integral relations involving the gradient, divergence, and curl are less common, but still useful.[6]

$$\int \nabla P\, dV = \int P\, \hat{n}\, dS$$

$$\int \hat{n} \times \nabla P\, dS = \oint P\, \overrightarrow{dl}$$

$$\int \nabla \times \overrightarrow{W}\, dV = -\int \overrightarrow{W} \times \hat{n}\, dS,$$

where P is a scalar function and S is the surface that bounds the volume V.

The differential vector operators for cylindrical and spherical coordinate systems are given in Table B1.

Some important vector identities are

$$\overrightarrow{A} \times (\overrightarrow{B} \times \overrightarrow{C}) = \overrightarrow{B}\, (\overrightarrow{A} \cdot \overrightarrow{C}) - \overrightarrow{C}\, (\overrightarrow{A} \cdot \overrightarrow{B}) \tag{B.1}$$

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Table B1 *Vector operators in cylindrical and spherical coordinates [6]*

	Cylindrical	Spherical
∇S	$\partial_\rho S\,\hat{\rho} + \frac{1}{\rho}\partial_\phi S\,\hat{\phi} + \partial_z S\,\hat{z}$	$\partial_r S\,\hat{r} + \frac{1}{r}\partial_\theta S\,\hat{\theta} + \frac{1}{r\sin\theta}\partial_\phi S\,\hat{\phi}$
$\nabla\cdot\vec{V}$	$\frac{1}{\rho}\partial_\rho(\rho\,V_\rho) + \frac{1}{\rho}\partial_\phi V_\phi + \partial_z V_z$	$\frac{1}{r^2}\,\partial_r(r^2 V_r) + \frac{1}{r\sin\theta}\,[\partial_\theta(V_\theta \sin\theta) + \partial_\phi V_\phi]$
$\nabla\times\vec{V}$	$\left(\frac{1}{\rho}\partial_\phi V_z - \partial_z V_\phi\right)\hat{\rho} + (\partial_z V_\rho - \partial_\rho V_z)\,\hat{\phi} + \frac{1}{\rho}[\partial_\rho(\rho\,V_\phi) - \partial_\phi V_\rho]\;\hat{z}$	$\frac{1}{r\sin\theta}[\partial_\theta(V_\phi \sin\theta) - \partial_\phi V_\theta]\hat{r} + \frac{1}{r\sin\theta}[\partial_\phi V_r - \sin\theta\partial_r(r\,V_\phi)]\hat{\theta} + \frac{1}{r}[\partial_r(r\,V_\theta) - \partial_\theta V_r]\hat{\phi}$
$\nabla^2 S$	$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho S) + \frac{1}{\rho^2}\partial_\phi^2 S + \partial_z^2 S$	$\frac{1}{r^2}\,\partial_r(r^2\partial_r S) + \frac{1}{r^2\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta S) + \frac{1}{r^2\sin^2\theta}\,\partial_\phi^2 S$

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\,\vec{B} + (\vec{B}\cdot\nabla)\,\vec{A} \tag{B.2}$$

$$\nabla\cdot(S\,\vec{V}) = \vec{V}\cdot\nabla S + S\,\nabla\cdot\vec{V} \tag{B.3}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \tag{B.4}$$

$$\nabla\cdot(\nabla\times\vec{V}) = 0 \tag{B.5}$$

$$\nabla\times(S\,\vec{V}) = \nabla S\times\vec{V} + S\,\nabla\times\vec{V} \tag{B.6}$$

$$\nabla\times\nabla\times\vec{V} = \nabla(\nabla\cdot\vec{V}) - \nabla^2\vec{V} \tag{B.7}$$

$$\nabla\times\nabla S = 0 \tag{B.8}$$

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}\,(\nabla\cdot\vec{B}) - \vec{B}\,(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\,\vec{A} - (\vec{A}\cdot\nabla)\,\vec{B} \tag{B.9}$$

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[1] G. Harnwell, *Principles of Electricity and Magnetism*, 2nd ed., McGraw-Hill, 1949, p. 636–649.
[2] J. Marion, *Classical Electromagnetic Radiation*, Academic Press, 1965, p. 447–456.
[3] L. Eyges, *The Classical Electromagetic Field*, Dover, 1980, p. 387.
[4] P. Lorrain & D. Corson, *Electromagnetic Fields and Waves*, 2nd ed., Freeman, 1970, p. 13–16.
[5] Ibid., p. 21–22.
[6] E. R. Cohen (ed.), *The Physics Quick Reference Guide*, American Institute of Physics, 1996, p. 162.

C
Bessel functions

Using the method of separation of variables for the Laplace equation in cylindrical coordinates gives rise to Bessel's equation.[1, 2]

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + (k^2\rho^2 - n^2)\ R = 0.$$

In this equation, $R = R(\rho)$ and k and n are separation constants. The parameter n must be an integer to keep the azimuthal dependence of the solution single-valued, i.e., we must have

$$\Phi(\phi) = \Phi(\phi + 2\pi n).$$

Bessel's equation is a second order differential equation that has two independent classes of solution. One class involves Bessel functions of the first kind,[3] $R(\rho) = J_n(k\rho)$. The behavior of the first three Bessel functions J_n are shown as a function of $k\rho$ in Figure C1. All functions of this type are well-behaved at $\rho = 0$. They are oscillatory with a decreasing amplitude that approaches zero as $k\rho \to \infty$. The first root of the function $J_0(x)$ occurs at $x = 2.405$, where $x = k\rho$. The first root of $J_1(x)$ occurs at $x = 3.832$. The series expansion is

$$J_n(x) = \sum_{k\ =\ 0}^{\infty} \frac{(-1)^k}{k!\ (n+k)!} \left(\frac{x}{2}\right)^{n+2k}.$$

The Bessel functions satisfy the recurrence relation

$$\frac{2n}{x}\ J_n(x) = J_{n-1}(x) + J_{n+1}(x),$$

while the derivatives satisfy the relation

$$2\ J'_n(x) = J_{n-1}(x) - J_{n+1}(x).$$

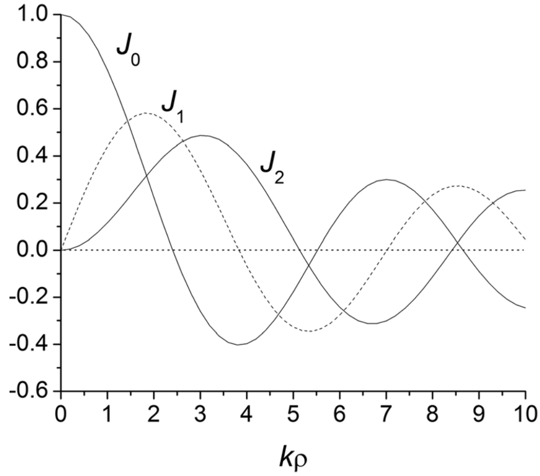

Figure C1 Bessel functions of the first kind for $n = 0, 1, 2$.

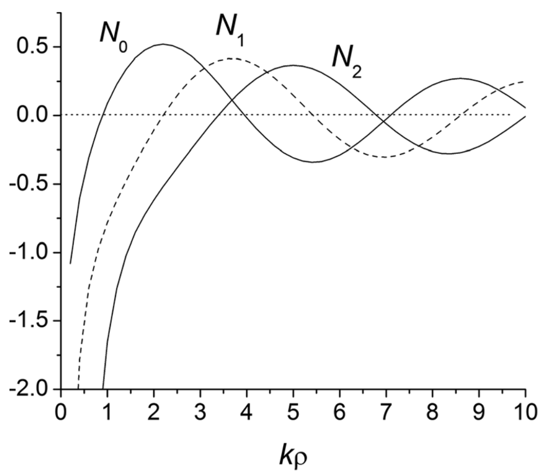

Figure C2 Bessel functions of the second kind for $n = 0, 1, 2$.

The derivative of J_0 is given by

$$\frac{dJ_0(x)}{dx} = -J_1(x).$$

The other class of solutions to Bessel's equation are the Bessel functions of the second kind,[4] $R(\rho) = N_n(k\rho)$. The behavior of the first three Bessel functions N_n are shown as a function of $k\rho$ in Figure C2. These solutions are also oscillatory with decreasing amplitude that approach zero as $k\rho \to \infty$. However, they diverge at $\rho = 0$, so they cannot be used in magnetostatics for any region that contains the

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origin. The $N_n(x)$ functions satisfy the same recurrence relations as $J_n(x)$. The derivative of N_0 is given by

$$N'_0(x) = -N_1(x).$$

If in applying the method of separation of variables for the Laplace equation in cylindrical coordinates, we require that the solution along z is oscillatory, then the separation parameter for the axial and radial terms must have the opposite sign from that used in deriving the Bessel differential equation. This leads to the radial equation

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) - (k^2\rho^2 + n^2)\, R = 0.$$

The solutions of this equation are known as *modified Bessel functions*. This same equation can be produced by replacing k with $i\,k$ in the ordinary Bessel equation. One class of radial solutions involves the modified Bessel function $I_n(k\rho)$.[5] The behavior of the first three modified Bessel functions I_n are shown in Figure C3. All functions of this type are well behaved at $\rho = 0$. They are related to the ordinary Bessel functions by

$$I_v(x) = i^{-v}\, J_v(i\,x).$$

The series expansion is

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!\ (n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

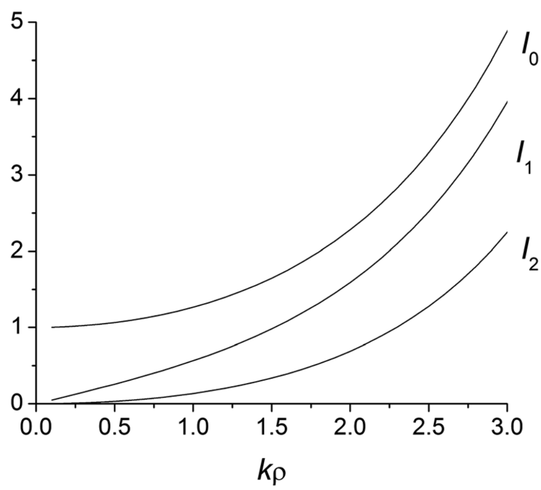

Figure C3 Modified Bessel functions $I_n(k\rho)$ for $n = 0, 1, 2$.

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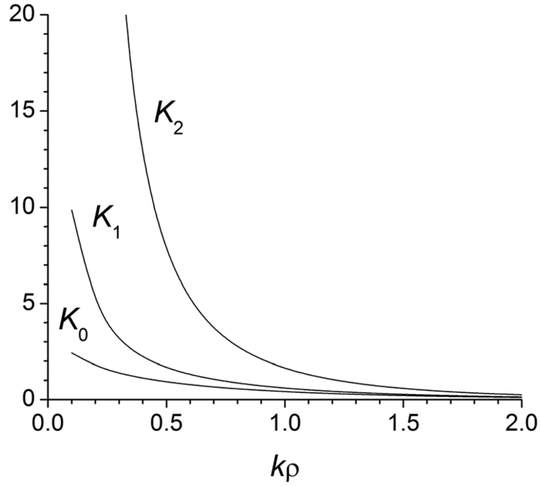

Figure C4 Modified Bessel functions $K_n(k\rho)$ for $n = 0, 1, 2$.

and it satisfies the recursion relations[1]

$$\frac{2n}{x} I_n(x) = I_{n-1}(x) - I_{n+1}(x)$$

$$2I'_n(x) = I_{n-1}(x) + I_{n+1}(x).$$

The other class of solutions for the modified Bessel's equation are the functions $K_n(k\rho)$. The behavior of the first three modified Bessel functions K_n are shown in Figure C4. These solutions diverge at $\rho = 0$, so they cannot be used in any region that contains the origin. The functions K_n satisfy the recursion relations[2]

$$-\frac{2n}{x} K_n(x) = K_{n-1}(x) - K_{n+1}(x)$$

$$-2K'_n(x) = K_{n-1}(x) + K_{n+1}(x).$$

References

[1] F. Bowman, *Introduction to Bessel Functions*, Dover Publications, 1958.
[2] M. Abramowitz & I. Stegun (eds.), *Handbook of Mathematical Functions*, Dover Publications, 1972, chapter 9.
[3] G. Arfken, *Mathematical Methods for Physicists*, 3rd ed., Academic Press, 1985, p. 573–584.
[4] Ibid., p. 596–601.
[5] Ibid., p. 610–619.

[1] GR 8.486.1,8.486.2. [2] GR 8.486.10,8.486.11.

D
Legendre functions

Separation of variables for the Laplace equation in spherical coordinates gives the partial differential equation

$$\frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta Y) + \frac{1}{\sin^2\theta}\,\partial_\phi^2 Y + l(l+1)\,Y = 0$$

for the angular dependence. The solution of this equation is given in terms of the spherical harmonic functions Y_{lm} [1, 2]

$$Y_{l\,m}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi},$$

where l and m are integers and P_l^m is an associated Legendre function. Allowed values of m are all integers in the range $-l \le m \le l$. Values of the spherical harmonics for negative m are given by

$$Y_{l,-m} = (-1)^m\,Y^*_{l,m},$$

where the asterisk denotes complex conjugation. The spherical harmonics for $l \le 2$ are given in Table D1.

The polar angle part $\Theta(\theta)$ of the solution to the Laplace equation has to satisfy the second order, ordinary differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]\Theta = 0$$

where $x = \cos\theta$. The solutions of this equation are called associated Legendre functions of the first and second kind,

$$\{P_l^m(x), Q_l^m(x)\}.$$

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Table D1 *Spherical harmonics*

l	m	Y_{lm}
0	0	$\dfrac{1}{\sqrt{4\pi}}$
1	0	$\sqrt{\dfrac{3}{4\pi}}\cos\theta$
1	1	$-\sqrt{\dfrac{3}{8\pi}}\sin\theta\, e^{i\,\phi}$
2	0	$\sqrt{\dfrac{5}{16\pi}}\,(3\cos^2\theta - 1)$
2	1	$-\sqrt{\dfrac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\,\phi}$
2	2	$\sqrt{\dfrac{15}{32\pi}}\sin^2\theta\, e^{2i\,\phi}$

Only the functions of the first kind have convergent power series over the complete range $0 \le x \le 1$, so we choose

$$\Theta(\theta) = P_l^m(\cos\theta).$$

The associated Legendre functions can be calculated from

$$P_l^m(x) = \frac{(-1)^m}{2^l\, l!}\,(1 - x^2)^{m/2}\,\frac{d^{l+m}}{dx^{l+m}}(x^2 - 1)^l.$$

Associated Legendre functions with negative m are related to functions with positive m by

$$P_l^{-m}(x) = (-1)^m\,\frac{(l-m)!}{(l+m)!}\,P_l^m(x)\,.$$

The associated Legendre functions for $l \le 3$ and $m > 0$ are given in Table D2.

In problems with azimuthal symmetry, we have $m = 0$. In this case, the associated Legendre functions reduce to the ordinary Legendre polynomials.

$$P_l^0(\cos\theta) = P_l(\cos\theta)$$

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Table D2 *Associated Legendre functions*

l	m	P_l^m
1	1	$\sin\theta$
2	1	$3\cos\theta\sin\theta$
2	2	$3\sin^2\theta$
3	1	$\frac{3}{2}(5\cos^2\theta-1)\sin\theta$
3	2	$15\cos\theta\sin^2\theta$
3	3	$15\sin^3\theta$

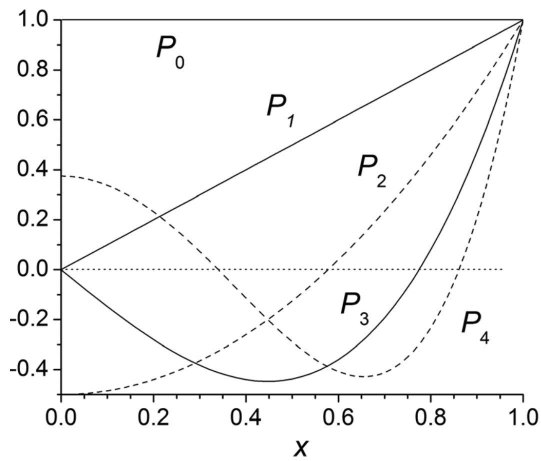

Figure D1 Legendre polynomials for $l \leq 4$.

The Legendre polynomials form a complete set of orthogonal functions over the interval $-1 \leq \cos\theta \leq 1$. The behavior of the Legendre polynomials for $l \leq 4$ are shown in Figure D1 as a function of x. Legendre polynomials satisfy the recurrence relation

$$(l+1)\,P_{l+1}(x) = (2l+1)\,x\,P_l(x) - l\,P_{l-1}(x)$$

and their derivatives satisfy the recurrence relation

$$(x^2-1)\,P_l'(x) = l\,x\,P_l(x) - l\,P_{l-1}(x).$$

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References

[1] G. Arfken, *Mathematical Methods for Physicists*, 3rd ed., Academic Press, 1985, chapter 12.
[2] M. Abramowitz & I. Stegun (eds.), *Handbook of Mathematical Functions*, Dover Publications, 1972, chapter 8.

E

Complex variable analysis

We present here a brief summary without proofs of some of the important results from complex analysis that are relevant to the material covered in this book.[1, 2]

Complex variables

In a Cartesian coordinate system, the complex variable z is given by

$$z = x + i\,y,$$

where x is called the *real* part of z, y is called the *imaginary* part of z, and $i = \sqrt{-1}$. In polar coordinates, z can be written in the form

$$\begin{aligned} z &= r\,e^{i\theta} \\ &= r\,(\cos\theta + i\sin\theta), \end{aligned}$$

where r is called the *modulus* of z and θ is the *argument* of z. The *De Moivre formula* is useful for evaluating powers of z.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

The *complex conjugate* of a complex variable z is

$$z^* = x - i\,y.$$

The real and imaginary parts of a complex number can be written as

$$\begin{aligned} \mathbb{R}e(z) &= \frac{z + z^*}{2} \\ \mathbb{I}m(z) &= \frac{z - z^*}{2\,i}. \end{aligned}$$

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Care is required in working with the complex counterparts of some real functions. An important function in magnetostatics is the complex logarithm function. This is defined as

$$\begin{aligned} w &= \ln z \\ &= \ln(r\, e^{i\theta}) \\ &= \ln r + i\,(\theta + 2\pi n), \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \ldots$. This function has multiple *branches* of angular width 2π, depending on the value of n.[3] We customarily compute this function using the principal branch where $n = 0$ and where θ is in the range $-\pi < \theta \leq \pi$. In this case, the function changes discontinuously when crossing the negative x axis, which is called a *branch cut*.

Complex differentiation

The derivative of the complex function F is defined as [4]

$$F'(z) = \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z},$$

provided that the limit exists and is independent of the manner in which Δz approaches 0. If the derivative of F exists at all points throughout some planar region R, we say that the function is *analytic* in the region.[5] Examples of analytic functions include polynomials, exponentials, trigonometric, and hyperbolic functions. The real and imaginary parts of an analytic function are harmonic, i.e., they satisfy the Laplace equation.

Points where a function $F(z)$ is not analytic are called *singularities*. A singularity in $F(z)$ at a point z_0 is called a *pole* of order n if [6]

$$\lim_{z \to z_0} (z - z_0)^n F(z)$$

exists and is not 0.

An important property of analytic functions is that constraints exist between their real and imaginary parts.

Theorem E.1 (Cauchy-Riemann) [7] (Necessity) *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in some domain D, then u and v have continuous first partial derivatives in D and satisfy the Cauchy-Riemann equations*

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

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(Sufficiency) *If a function* $f(z) = u(x,y) + iv(x,y)$ *is defined in* D*, if* u *and* v *have continuous first partial derivatives in* D *and if the Cauchy-Riemann equations hold in* D*, then* f*(z) is analytic in* D.

For a region not including the origin, the Cauchy-Riemann equations can be written in polar coordinates as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Series

Theorem E.2 (power series) [8] *Let f(z) be analytic on a domain* G *and let* z_o *be an arbitrary point of* G*. Let* $d = d(z_0)$ *be the distance between* z_o *and the boundary of* G*. Then there exists a power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_o)^n$$

that converges to f(z) on the disk $|z - z_o| < d$.

A power series can be differentiated or integrated term-by-term within its radius of convergence.

Theorem E.3 (Taylor series) [9] *Let f(z) be analytic and single-valued in an open region* G*. Let* a *be any point in* G *and let* C *be a circle with center at* a*, which together with its interior lies entirely in* G*. Then at every point* z *in* C*, the series*

$$f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots$$

converges to f(z)*.*

In other words, *f*(*z*) can be written as a Taylor series that converges in the region $|z - a| < R$, where R is the radius of convergence.

Theorem E.4 (Laurent series) [10] *Let* f*(z) be analytic for the annular region*

$$G\colon R_1 < |z - z_o| < R_2$$

and let C *be any simple closed contour lying inside* G *and having* z_o *in its interior. Then for points* z *in* G*, the function* f*(z) may be expanded in the series*

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$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_o)^k,$$

where

$$c_k = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{k+1}} \, dz$$

and the integration is along the contour C.

The Laurent series is valid in a region surrounding, but not including, a singularity. Note that this series includes negative values of k. The coefficient c_{-1} has special significance and is known as the *residue*.

Complex integration

Theorem E.5 [11] *If* $f(z) = u(x,y) + iv(x,y)$ *is continuous on a simple smooth arc from points* a *to* b*, then the integral exists and is given by*

$$\int f(z) \, dz = \int_a^b (u + iv) \, (dx + i \, dy).$$

Theorem E.6 (Cauchy integral theorem) [12] *If* f*(z) is analytic in a simply connected domain* D*, then*

$$\oint f(z) \, dz = 0$$

on every simple closed path in D.

If instead of $f(z)$, we consider the contour integral of $f(z) / (z - z_o)$, then we have the following theorem.

Theorem E.7 (Cauchy's Integral Formula) [13] *Let* f*(z) be analytic within and on a simple closed contour* C*. Then, if* z_o *is a point inside* C,

$$f(z_o) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_o)} \, dz.$$

This gives the value of $f(z_o)$ at the singularity z_o inside a region in terms of the contour integral around the boundary C.

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Theorem E.8 [14] *Let* f*(z) be analytic within and on a simple closed contour* C*. Then all derivatives of* $f(z_o)$ *exist at a point* z_o *inside* C *and are given by*

$$f^{(n)}(z_o) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z - z_o)^{n+1}}\,dz.$$

Theorem E.9 (Residue theorem) [15] *Let* f*(z) be analytic within and on a simple closed contour* C*, except for a finite number of isolated singularities inside* C*. Let* σ *be the sum of the residues at the singular points of* f*(z) that lie inside* C*. Then*

$$\frac{1}{2\pi i}\oint f(z)\,dz = \sigma.$$

In other words, the value of the contour integral is $2\pi i$ times the sum of the residues for the enclosed singularities. For a pole of order n, the residue can be found as [16]

$$a_{-1} = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right].$$

In the case of a simple pole ($n = 1$), the residue is given by

$$a_{-1} = \lim_{z\to a}\,(z-a)\,f(z).$$

Conformal mapping

We can define a function F that maps a complex variable z into a variable w in another two-dimensional space.

$$w = F(z).$$

Assume that two curves that cross at a point z_o in the z space are separated by an angle θ. A mapping $w = F(z)$ is *conformal*, or angle preserving, if the mapped curves in the w space cross at the point $w_o = F(z_o)$ with the same angle θ.

Theorem E.10 [17] *A mapping defined by an analytic function* F*(z) is conformal, except at points where the derivative* $F'(z)$ *is zero.*

Theorem E.11 (Riemann mapping theorem) [18] *Let* D *be a simply connected domain with at least two boundary points. Then there exists a simple function* $w = F(z)$ *which maps* D *onto the unit disk* $|w| < 1$*. If we specify that a given point* z_o *in* D *maps into the origin and a given direction at* z_o *is mapped into a given direction at the origin, then the mapping is unique.*

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Theorem E.12 (Schwarz-Christoffel transformation) [19] *Let* R *be a polygon in the* w *plane having vertices at* $w_1, w_2, \ldots, w_n$ *with corresponding interior angles* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *respectively. Let the points* $w_1, w_2, \ldots, w_n$ *map into the points* $x_1, x_2, \ldots, x_n$ *on the real axis of the* z *plane. Then the transformation*

$$\frac{dw}{dz} = A(z - x_1)^{\alpha_1/\pi - 1}(z - x_2)^{\alpha_2/\pi - 1} \cdots (z - x_n)^{\alpha_n/\pi - 1},$$

where A *is a complex constant, maps the interior of the polygon in the* w *plane onto the upper half of the* z *plane and maps the boundary of the polygon onto the real axis of the* z *plane.*

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F
Complete elliptic integrals

An elliptic integral is an integral that can be written in the form [1]

$$\int R(x, \sqrt{f(x)})dx,$$

where R is a rational function and f is a third- or fourth-order polynomial in x. All integrals of this type can be written in terms of the three standard forms.

$$F(k, \theta') = \int_0^{\theta'} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

$$E(k, \theta') = \int_0^{\theta'} \sqrt{1 - k^2 \sin^2\theta}\, d\theta$$

$$\prod(k, n, \theta') = \int_0^{\theta'} \frac{d\theta}{(1 + n\, \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}}.$$

Each of these integrals depends on a parameter k called the *modulus* that satisfies $k^2 \leq 1$. The third type of integral also depends on a second parameter n called the *characteristic*.[2] When the upper limit of integration is

$$\theta' = \frac{\pi}{2},$$

these functions define the *complete elliptic integrals* of the first, second, and third[1] kinds.

[1] One should be aware that the complete elliptic integral of the third kind is sometimes defined with a negative sign before the factor n in the denominator.

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, d\theta$$

$$\prod(k, n) = \int_0^{\pi/2} \frac{d\theta}{(1 + n \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}}$$

Moreover, these functions can alternatively be defined in polynomial form as

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

$$E(k) = \int_0^1 \frac{\sqrt{1 - k^2x^2}}{\sqrt{(1 - x^2)}} dx$$

$$\prod(k, n) = \int_0^1 \frac{dx}{(1 + nx^2)\sqrt{(1 - x^2)(1 - k^2x^2)}}.$$

It is important to emphasize that, despite the awkward nomenclature, the complete elliptic integrals are functions of k, and in the case of the third kind, also a function of n.

The complete elliptic integrals K and E can be expressed in terms of the infinite series

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \cdots\right]$$

$$E(k) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} + \cdots\right],$$

where $k^2 < 1$.[2] Efficient numerical algorithms have been developed to calculate the complete elliptic integrals.[3]

The dependences of the complete elliptic integrals of the first and second kinds are shown as a function of k in Figure F1. Both functions have the value $\pi/2$ for $k = 0$. The function $E(k)$ has the value 1 for $k = 1$, while $K(k)$ approaches ∞ as $k \to 1$. The behavior of the complete elliptic integral of the third kind for several values of n is shown as a function of k in Figure F2. The function $\prod(k,\ n)$ increases as k increases for all values of n. For a given value of k, the function increases as n becomes more negative.

If the vector potential is defined in terms of complete elliptic integrals, we need to take derivatives to find the magnetic field. In this case, we need to know

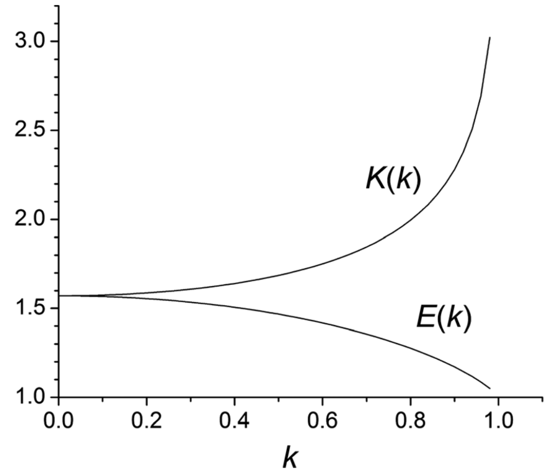

Figure F1 Dependence of the functions $K(k)$ and $E(k)$ on the modulus k.

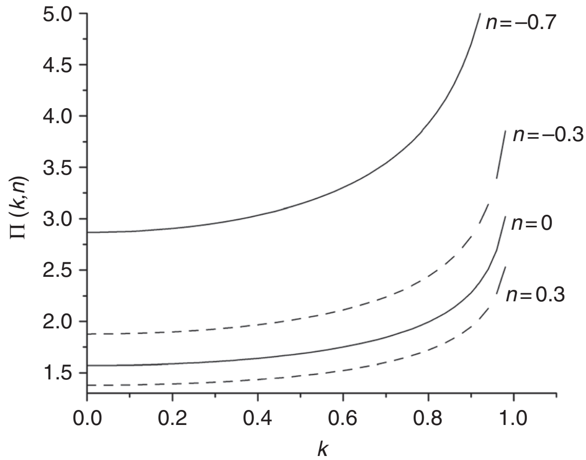

Figure F2 Behavior of the complete elliptic integral of the third kind.

the derivatives of the complete elliptic integrals with respect to their arguments.[2]

$$\frac{\partial K(k)}{\partial k} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}$$
$$\frac{\partial E(k)}{\partial k} = \frac{E(k)}{k} - \frac{K(k)}{k}$$

[2] GR 8.123.2, GR 8.123.4.

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For the complete elliptic integral of the third kind, the derivatives are given by [4, 5]

$$\frac{\partial\prod(k,n)}{\partial k} = \frac{k}{(1-k^2)(k^2-n)}\left[E(k)-(1-k^2)\prod(k,n)\right]$$
$$\frac{\partial\prod(k,n)}{\partial n} = \frac{1}{2(n-1)(k^2-n)}\left[E(k)+\frac{k^2-n}{n}K(k)-\frac{k^2-n^2}{n}\prod(k,n)\right].$$

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