Canad. Math. Bull. Vol. **20** (1), 1977

ACCUMULATION POINTS OF CONTINUOUS REAL-VALUED FUNCTIONS AND COMPACTIFICATIONS

BY
ENG UNG CHOO

All topological spaces are assumed to be completely regular. $C(X)$ (resp. $C^*(X)$) will denote the ring of all (resp. all bounded) continuous real-valued functions on X. βX is the Stone–Cech compactification of X. A real number t is said to be an accumulation point of a function $f \in C(X)$ if and only if $f^{-1}[[t-\varepsilon, t+\varepsilon]]$ is not compact for every $\varepsilon > 0$. The set of all accumulation points of f will be denoted by $\Delta(f)$. For any positive integer n, a topological criterion for the existence of a function $f \in C(X)$ such that $|\Delta(f)| = n$ is given. It is proved that for every function $g \in C(X)$ with finite $\Delta(g)$, there exists a function $f \in C^*(X)$ which has finite range on every discrete closed subset of X such that $|\Delta(f)| = |\Delta(g)|$. Peter A. Loeb [5] has constructed the minimal compactification X^f of X in which f has a continuous extension which is one-one on $X^f - X$. It is shown that every n-point compactification [6] of X (if it exists) is of this type. Finally, an equivalent condition for the existence of a homeomorphism h from X^f onto X^g such that $h(x) = x$ for each $x \in X$ is given for any any two functions $f, g \in C^*(X)$. All notations are referred to [3].

DEFINITION 1. Let $f \in C(X)$. A real number t is said to be an accumulation point of f if and only if $f^{-1}[[t-\varepsilon, t+\varepsilon]]$ is not compact for every $\varepsilon > 0$. The set of all accumulation points of f is denoted by $\Delta(f)$.

Intuitively, $\Delta(f)$ gives the 'dense' portion of $f[X]$. If $f^{-1}(t)$ is not compact, then $t \in \Delta(f)$. The converse may not be true.

EXAMPLE 1. Let $X = \{(a, \sin(1/a)) : a > 0\} \cup \{(0, 0)\}$ and f be the function defined by $f((a, \sin(1/a))) = a$ for each $a > 0$ and $f((0, 0)) = 0$. Then $f \in C(X)$. It is easily seen that $0 \in \Delta(f)$ even though $f^{-1}(0)$ is compact.

We can always restrict ourselves to $C^*(X)$ in the study of $|\Delta(f)|$ since for every $f \in C(x)$, there exists $g \in C^*(X)$ such that $|\Delta(g)| = |\Delta(f)|$.

LEMMA 1. *Let $f \in C^*(R)$ and $t \in R$. Then*

$$\Delta(f) = \bigcap \{Cl_R f[X-K] : K \quad \textit{is a compact subset of } X\}.$$

Thus $\Delta(f)$ is closed in R.

Received by the editor Sept 25, 1975.

Proof. Suppose $t\in\Delta(f)$. Let K be any compact subset of X. Then $f^{-1}[[t-\varepsilon, t+\varepsilon]]-K\neq\phi$ for every $\varepsilon>0$.

Thus $[t-\varepsilon, t+\varepsilon]\cap f[X-K]\neq\phi$ for every $\varepsilon>0$. Hence $t\in Cl_R f[X-K]$. Since K is any arbitrary compact subset of X, we have

$$t\in\bigcap\{Cl_R f[X-K[:K \quad \text{is a compact subset of } X\}.$$

Conversely, if $t\notin\Delta(f)$, then $K=f^{-1}[[t-\delta, t+\delta]]$ is compact for some $\delta>0$. Thus $(t-\delta, t+\delta)\bigcap f[X-K]=\phi$ and $t\notin Cl_R f[X-K]$.

LEMMA 2. *Let $f\in C^*(X)$ and $t\in R$. Then $t\in\Delta(f)$ if and only if $(f^\beta)^{-1}[[t-\varepsilon, t+\varepsilon]]-X\neq\phi$ for every $\varepsilon>0$, where f^β is the continuous extension of f over βX.*

Proof. If $t\in\Delta(f)$, then $f^{-1}[[t-\varepsilon, t+\varepsilon]]$ is not compact for every $\varepsilon>0$. Hence $(f^\beta)^{-1}[[t-\varepsilon, t+\varepsilon]]\neq f^{-1}[[t-\varepsilon, t+\varepsilon]]$ for every $\varepsilon>0$. It follows that $(f^\beta)^{-1}[[t-\varepsilon, t+\varepsilon]]-x\neq\phi$ for every $\varepsilon>0$.

Suppose $t\notin\Delta(f)$. Then $f^{-1}[[t-\delta, t+\delta]]$ is compact for some $\delta>0$. If there exists $z\in(f^\beta)^{-1}[[t-\delta/3, t+\delta/3]]-X$, then there exists a neighbourhood O_1 of z in βX such that $f^\beta[O_1]\subset(t-\delta/2, t+\delta/2)$. Since $f^{-1}[[t-\delta, t+\delta]]$ is compact, $O_2=\beta X-f^{-1}[[t-\delta, t+\delta]]$ is also a neighborhood of z. Thus $O_1\cap O_2$ is a neighborhood of z. But $(O_1\cap O_2)\cap X=\phi$. This is impossible since X is dense in βX. Hence $(f^\beta)^{-1}[[t-\delta/3, t+\delta/3]]-X=\phi$. Consequently, if $(f^\beta)^{-1}[[t-\varepsilon, t+\varepsilon]]-X\neq\phi$ for every $\varepsilon>0$, then $t\in\Delta(f)$.

COROLLARY 1. *Let $f\in C^*(X)$. If X is locally compact, then $\Delta(f)=f^\beta[\beta X-X]$.*

Proof. If $t\in f^\beta[\beta X-X]$, then $(f^\beta)^{-1}[[t-\varepsilon, t+\varepsilon]]-X\neq\phi$ for every $\varepsilon>0$. By Lemma 2, $t\in\Delta(t)$.

Conversely, suppose $t\in\Delta(t)$. Then $(f^\beta)^{-1}[[t-\varepsilon, t+\varepsilon]]-X\neq\phi$ for every $\varepsilon>0$. Since X is locally compact, $\beta X-X$ is compact and $f^\beta[\beta X-X]$ is closed. If $t\notin f^\beta[\beta X-X]$, then there exists $\delta>0$ such that $[t-\delta, t+\delta]\cap f^\beta[\beta X-X]=\phi$. Thus $(f^\beta)^{-1}[[t-\delta, t+\delta]]-X=\phi$, which is a contradiction. Hence $t\in f^\beta[\beta X-X]$.

LEMMA 3. *Let $f\in C(X)$ such that $|\Delta(f)|=n$ where n is a positive integer. Then there exists $g\in C^*(X)$ such that $\Delta(g)=\{1, 2, \ldots, n\}$.*

Proof. Let $\Delta(f)=\{a_1, a_2, \ldots, a_n\}$ where $a_1<a_2<\cdots<a_n$. Let h be a function in $C^*(R)$ defined by

$$h(x)=\begin{cases}\exp(x-a_1) & \text{if} \quad x\le a_1\\ \left(\dfrac{x-a_i}{a_{i+1}-a_i}\right)(i+1)+\left(\dfrac{a_{i+1}-x}{a_{i+1}-a_i}\right)i, & \text{if} \quad a_i<x\le a_{i+1}, \quad i=1,2,\ldots,n-1,\\ n+2-\exp(a_n-x) & \text{if} \quad x>a_n.\end{cases}$$

https://doi.org/10.4153/CMB-1977-009-7 Published online by Cambridge University Press

Then h is a homeomorphism from R onto the open interval $(0, n+2)$. Let $g = h \cdot f$. Then $g \in C^*(X)$ and $\Delta(g) = \{1, 2, \ldots, n\}$.

There may not exist any function $f \in C^*(X)$ such that $\Delta(f)$ is finite. The following theorem gives a topological criterion for the existence of $f \in C^*(X)$ satisfying $|\Delta(f)| = n$.

THEOREM 1. *Let n be a positive integer. There exists $f \in C^*(X)$ such that $|\Delta(f)| = n$ if and only if there exist n mutually disjoint closed non-compact subsets $A_1, A_2, \ldots, A_n$ of X such that $X - \bigcup_{i=1}^n A_i$ has compact closure.*

Proof. ($\Rightarrow$) Let $f \in C^*(X)$ and $|\Delta(f)| = n$. By Lemma 3, we may assume that $\Delta(f) = \{1, 2, \ldots, n\}$. The sets $A_i = f^{-1}[[i-\frac{1}{3}, i+\frac{1}{3}]]$, $i = 1, 2, \ldots, n$ are mutually disjoint closed non-compact subsets of X. For each $\gamma \in Cl_R f[X] - \bigcup_{i=1}^n (i-\frac{1}{3}, i+\frac{1}{3})$, there exists a real number $\varepsilon(\gamma) > 0$ such that $f^{-1}[[\gamma - \varepsilon(\gamma), \gamma + \varepsilon(\gamma)]]$ is compact. By the compactness of the set $Cl_R f[X] - \bigcup_{i=1}^n (i-\frac{1}{3}, i+\frac{1}{3})$, there exist $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that $Cl_R f[x] - \bigcup_{i=1}^n (i-\frac{1}{3}, i+\frac{1}{3})$ is contained in $\bigcup_{i=1}^k (\gamma_i - \varepsilon(\gamma_i), \gamma_i + \varepsilon(\gamma_i))$. Thus $X - \bigcup_{i=1}^n f^{-1}[(i-\frac{1}{3}, i+\frac{1}{3})]$ is contained in $\bigcup_{i=1}^k f^{-1}[[\gamma_i - \varepsilon(\gamma_i), \gamma_i + \varepsilon(\gamma_i)]]$ which is a compact set, being a finite union of compact sets. Hence $X - \bigcup_{i=1}^n A_i$ has compact closure.

($\Leftarrow$) Let $A_1, A_2, \ldots, A_n$ be n mutually disjoint closed non-compact subsets of X and $X - \bigcup_{i=1}^n A_i \subset K$ where K is a compact subset of X. For each $i = 1, 2, \ldots, n$, let $g(x) = i$ for each $x \in A_i \cap K$. Then g is a continuous function on $(\bigcup_{i=1}^n A_i) \cap K$. K is compact and thus is a normal space. Since $(\bigcup_{i=1}^n A_i) \cap K$ is a closed subset of K, hence g has a continuous extension $h \in C^*(K)$. Let $f(x) = h(x)$ for each $x \in K$ and $f(x) = i$ for each $x \in A_i$, $i = 1, 2, \ldots, n$. Then $f \in C^*(X)$ and $|\Delta(f)| = n$.

THEOREM 2. *Suppose $g \in C^*(X)$ and $\Delta(g)$ is finite. Then there exists $h \in C^*(X)$ such that $\Delta(g) = \Delta(h) = \{\gamma \in R : h^{-1}(\gamma)$ is not compact$\}$*

Proof. By Lemma 3, we may assume that $\Delta(g) = \{1, 2, \ldots, m\}$. Let f be a function in $C(R)$ defined by

$$f(x) = \begin{cases} x + \frac{1}{3} & \text{if } x < \frac{2}{3} \\ i & \text{if } i - \frac{1}{3} \le x \le i + \frac{1}{3}, \quad i = 1, 2, \ldots, m \\ 3x - 2i - 1 & \text{if } i + \frac{1}{3} < x < i + \frac{2}{3}, \quad i = 1, 2, \ldots, m-1 \\ x - \frac{1}{3} & \text{if } x > m + \frac{1}{3} \end{cases}$$

It is easily seen that f is a homeomorphism from $(\bigcup_{i=1}^{m-1} (i+\frac{1}{3}, i+\frac{2}{3})) \cup (-\infty, \frac{2}{3}) \cup (m+\frac{1}{3}, \infty)$ onto $R - \{1, 2, \ldots, m\}$. Then $h = f \cdot g \in C^*(X)$. For each $i = 1, 2, \ldots, m$, $h^{-1}(i) = g^{-1}[f^{-1}(i)] = g^{-1}[[i-\frac{1}{3}, i+\frac{1}{3}]]$ is not compact since $i \in \Delta(g)$. Then $\Delta(g) = \{1,2, \ldots, m\} \subset \Delta(h)$. Let $\gamma \in R - \Delta(g)$. Then there exists a real number $\varepsilon > 0$ such that $[\gamma - \varepsilon, \gamma + \varepsilon] \cap \Delta(g) = \phi$. Thus

$f^{-1}[[\gamma-\varepsilon, \gamma+\varepsilon]]\cap[i-\frac{1}{3}, i+\frac{1}{3}]=\phi$ for each $i=1,2,\ldots,m$. Hence $h^{-1}[[\gamma-\varepsilon, \gamma+\varepsilon]]\subset X-\bigcup_{i=1}^m g^{-1}[[i-\frac{1}{3}, i+\frac{1}{3}]]$. From the proof of Theorem 1, we know that $X-\bigcup_{i=1}^m g^{-1}[(i-\frac{1}{3}, i+\frac{1}{3})]$ is compact. Therefore, $h^{-1}[[\gamma-\varepsilon, \gamma+\varepsilon]]$ being a closed subset of a compact set is itself compact. Hence $\gamma\notin\Delta(h)$. Consequently, $\Delta(h)=\Delta(g)=\{\gamma\in R: h^{-1}(\gamma)$ is not compact$\}$.

COROLLARY 2. *If $f\in C^*(X)$ and $\Delta(f)$ is finite, then there exists an opening covering $\{O_i: i\in I\}$ of X such that*

$$\Delta(f)=\bigcap\{f[X-\bigcup_{i\in F} O_i]: F \text{ is a finite non-empty subset of } I\}$$

Proof. By previous theorem, there exists $g\in C^*(X)$ such that $\Delta(f)=\{\gamma\in R: g^{-1}[\gamma]$ is not compact$\}$. For each $\gamma\in\Delta(f)$, by the non-compactness of $g^{-1}[\gamma]$, there exists an open covering $\{O_i: i\in I_\gamma\}$ such that $g^{-1}[\gamma]$ has no finite subcover. Let $O_\tau=X-\bigcup_{\gamma\in\Delta(f)} g^{-1}[\gamma]$ and $I=\{\tau\}[\bigcup_{\gamma\in\Delta(f)} I_\gamma]$. Then $\{O_i: i\in I\}$ is an open covering of X and

$$\Delta(f)=\bigcap\{f[X-\bigcup_{i\in F} O_i]: F \text{ is a finite non-empty subset of } I\}.$$

In [1], it is proved that the set $D(X)$ of all functions $f\in C(X)$, where $f[A]$ is finite for every closed discrete subset of A of X, is a subring of $C^*(X)$. The next theorem shows that we can restrict ourselves to $D(X)$ in searching for functions with a finite set of accumulation points.

THEOREM 3. *Let $g\in C^*(X)$ where $\Delta(g)$ is finite. There exists $h\in D(X)$ such that $|\Delta(h)|=|\Delta(g)|$.*

Proof. By Lemma 3, we may assume that $\Delta(g)=\{1,2,\ldots,m\}$. Let f be the function defined in the proof of Theorem 2 and let $h=f\cdot g$. Then $h\in C^*(X)$ and $|\Delta(h)|=|\Delta(g)|$. It follows from the proof of Theorem 1 that $X-\bigcup_{i=1}^m g^{-1}[(i-\frac{1}{3}, i+\frac{1}{3})]$ is compact. For every closed discrete subset A of X, since $X-\bigcup_{i=1}^m g^{-1}[(i-\frac{1}{3}, i+\frac{1}{3})]$ is compact, $A-\bigcup_{i=1}^m g^{-1}[(i-\frac{1}{3}, i+\frac{1}{3})]$ is a finite set. Thus $h[A]$ is finite. Hence $h\in D(X)$.

From here onwards, X is assumed to be a locally compact space. The proofs of the following two theorems can be found in [5]. For any two compactifications X_1, X_2, of X, we write $X_1\cong X_2$ if there exists a homeomorphism h from X_1 onto X_2 such that $h(x)=x$ for every $x\in X$.

THEOREM 4. *Let $f\in C^*(X)$. For every open set Q in R and compact subset K of X, let $Q_k=[Q\cap\Delta(f)]\cup[f^{-1}[Q]-K]$. If the disjoint union $X^f=X\cup\Delta(f)$ has the topology generated by the base consisting of all open sets of X and all sets Q_K, then X^f is a Hausdorff compactification of X in which f has a continuous extension which is one–one on $\Delta(f)$.*

THEOREM 5. *Let $f\in C^*(X)$. If $\bar{X}$ is a Hausdorff compactification of X such that f has a continuous extension which is one–one on $\bar{X}-X$, then $\bar{X}\cong X^f$.*

https://doi.org/10.4153/CMB-1977-009-7 Published online by Cambridge University Press

Magill [6] has proved some necessary and sufficient conditions for a space X to have an n-point compactification. We will show that such compactifications are of type X^f.

THEOREM 6. *If $\bar{X}$ is an n-point compactification of X, then $\bar{X} \cong X^f$ for some $f \in C^*(X)$.*

Proof. Let $\bar{f} \in C(\bar{X})$ such that $\bar{f}$ is one–one on $\bar{X} - X$. If f is the restriction of $\bar{f}$ on X, then $f \in C^*(X)$. By Theorem 4 and Theorem 5, we conclude that $X^f \cong \bar{X}$.

The following corollary follows immediately from Theorem 1 and Theorem 6.

COROLLARY 3. *A space X has an n-point compactification if and only if there exist n mutually disjoint closed non-compact subsets $A_1, A_2, \ldots, A_n$ of X such that $X - \bigcup_{i=1}^{n} A_i$ has compact closure.*

THEOREM 7. *Let $f \in C^*(X)$ and $|\Delta(f)| \leq \aleph_0$. For every positive $n \leq |\Delta(f)|$, there exists $g \in C^*(X)$ such that $|\Delta(g)| = n$.*

Proof. Suppose $\Delta(f)$ is finite. By Lemma 3, we may assume that $\Delta(f) = \{1, 2, \ldots, m\}$. Given any positive integer $n \leq m$, let $\phi(\gamma) = \gamma$ for each $\gamma \leq n$ and $\phi(\gamma)$ for $\gamma > n$. Then $\phi \in C(R)$. Let $g = \phi \cdot f$. Then $g \in C^*(X)$ and $|\Delta(g)| = n$.

Suppose now that $\Delta(f)$ is countably infinite. Then X has a countably infinite compactification. It follows from Theorem 2.1 in [9] that X has an n-point compactification, for each positive integer n. Thus by Theorem 6, there exists $g \in C^*(X)$ such that $|\Delta(g)| = n$, for each positive integer n.

It follows from Theorem 4.3.2 in [2] that there is no n-point compactification of R for $n \geq 3$. Thus there is no function $f \in C^*(R)$ such that $\Delta(f)$ is finite and $|\Delta(f)| \geq 3$. In [1], it is shown that for $n \geq 2$, $D(R^n) = \{f \in C(R^n)$: there exists a positive integer k such that f is constant on $\{x \in R^n : \|x\| \geq k\}\}$. Therefore there is no function $f \in C^*(R^n)$, such that $\Delta(f)$ is finite and $|\Delta(f)| \geq 2$. We note that the continuous function $g(x) = \sin x$ for each $x \in R$ satisfies $\Delta(g) = [-1, 1]$. Thus $|\Delta(g)| = c$. This shows that the condition $|\Delta(f)| \leq \aleph_0$ in Theorem 7 is essential.

EXAMPLE 2. The space N of positive integers has the discrete topology. Let $f(4n) = 0$ for each $n \in N$ and $f(n) = 1/n$ for each n which is not a multiple of 4. Let $g(n) = 0$ for each even $n \in N$ and $g(n) = 1/n$ for each odd $n \in N$. Then $f, g \in C^*(N)$ and $\Delta(f) = \Delta(g) = \{0\}$. The open set $\Delta(f) \cup \{4n : n \in N\}$ in X^f is not open in X^g. Therefore, $X^f \not\cong X^g$.

Finally, an equivalent condition for $X^f \cong X^g$ is given where we use only the function values of f and g.

THEOREM 8. *Let f, $g \in C^*(X)$. $X^f = X^g$ if and only if there is an one–one* *correspondence* Φ *between* $\Delta(f)$ *and* $\Delta(g)$ *satisfying*

(1) $\forall\epsilon>0_\exists\quad\delta>0$ s.t. $g^{-1}[[\Phi(\gamma)-\delta,\quad\Phi(\gamma)+\delta]]-f^{-1}[\gamma-\varepsilon,\quad\gamma+\varepsilon)]$

is compact and

(2) $\forall\epsilon>0_\exists\quad\delta>0$ s.t. $f^{-1}[[\gamma-\delta,\gamma+\delta]]-g^{-1}[(\phi(\gamma)-\varepsilon.\ \phi(\gamma)+\varepsilon)]$

is compact for every $r\in\Delta(f)$.

Proof. ($\Rightarrow$) Let h be a homeomorphism from X^f onto X^g such that $h(x)=x$ for each $x\in X$. Then the correspondence defined by h satifies (1) and (2) for every $\gamma\in\Delta(f)$.

($\Leftarrow$) Suppose there is an one–one correspondence Φ which satisfies (1) and (2). Let $h(x)=x$ for each $x\in X$ and $h(\gamma)=\Phi(\gamma)$ for each $\gamma\in\Delta(f)$. Then h is an one–one function from X^f onto X^g. Obviously, h is continuous at each $x\in X$. Let $\gamma\in\Delta(f)$ and $t=h(\gamma)\in\Delta(g)$. Given $\varepsilon>0$ and a compact subset K of X, $[(t-\varepsilon,\ t+\varepsilon)\cap\Delta(g)]\cup(g^{-1}[(t-\varepsilon,\ t+\varepsilon)]-K)$ is a basic neighborhood of t in X^g. By (2), there exists $\delta>0$ such that $f^{-1}[[\gamma-\delta,\ \gamma+\delta]]-g^{-1}[(t-\varepsilon/2,\ t+\varepsilon/2)]$ is compact. Let $u\in(\gamma-\delta,\ \gamma+\delta)\cap\Delta(f)$. Suppose $h(u)\notin(t-\varepsilon,\ t+\varepsilon)\cap\Delta(g)$. There exists $\beta>0$ such that $[h(u)-\beta,\ h(u)+\beta]\cap[t-\varepsilon/2,\ t+\varepsilon/2]=\phi$. By (2) again, there exists $\eta>0$ such that $f^{-1}[[u-\eta,\ u+\eta]]-g^{-1}[(h(u)-\beta,\ h(u)+\beta)]$ is compact. Let $\alpha>0$ be sufficiently small so that $[u-\alpha,\ u+\alpha]\subset(\gamma-\delta,\ \gamma+\delta)\cap(u-n,\ u+\eta)$. Now, $f^{-1}[[u-\alpha,\ u+\alpha]]-g^{-1}[(t-\varepsilon/2,\ t+\varepsilon/2)]$ and $f^{-1}[[u-\alpha,\ u+\alpha]]-g^{-1}[(h(u)-\beta,\ h(u)+\beta)]$ are compact and $[h(u)-\beta,\ h(u)+\beta]\cap[t-\varepsilon/2,\ t+\varepsilon/2]=\phi$. Hence $f^{-1}[[u-\alpha,\ u+\alpha]]$ is compact. But this contradicts the assumption that $u\in\Delta(f)$. Thus $h(u)\in(t-\varepsilon,t+\varepsilon)\cap\Delta(g)$ for each $u\in(\gamma-\delta,\gamma+\delta)\cap\Delta(f)$. This means that h maps the basic neighborhood $[(\gamma-\delta,\ \gamma+\delta)\cap\Delta(f)]\cup(f^{-1}[\gamma-\delta,\ \gamma+\delta)]-K)$ of γ into $[(t-\varepsilon,\ t+\varepsilon)\cap\Delta(g)]\cup(g^{-1}[(t-\varepsilon,\ t-\varepsilon)]-K)$. Therefore h is continuous. Since X^f is compact and h is one–one, onto and continuous, hence h is a homeomorphism and $X^f\cong X^g$.

References

1. E. U. Choo, *Admissible Subrings of Real-valued Functions*, Doctoral Thesis, The University of British Columbia, Canada. 1971.

2. P. A. Firby, *Lattice and Compactifications, III.* London Mathematical Society Proceedings Ser. **3**, Vol. **27** (1973), 61–68.

3. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrant, N.Y., 1960.

4. P. A. Loeb, *Compactifications of Hausdorff Spaces*, Proc. Amer. Math. Soc. **22** (1969), 627–634.

5. P. A. Loeb, *A Minimal Compactification for Extending Functions*, Proc. Amer. Math. Soc. **18** (1967), 282–283.

6. K. D. Magill, Jr., *N-point Compactifications*, Amer. Math. Monthly **72** (1965), 1075–1081.

7. K. D. Magill, Jr., *Countable Compactifications*, Canad. J. Math. **18** (1966), 616–620.

Department of Mathematics,
University of British Columbia,
Vancouver, B.C.