

REMARKS ON THE SEMIVARIATION OF VECTOR MEASURES
 WITH RESPECT TO BANACH SPACES.

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Suppose that $L^q(\nu) \widehat{\otimes}_{\gamma_q} Y = L^q(\nu, Y)$ and $X \widehat{\otimes}_{\Delta_p} L^p(\mu) = L^p(\mu, X)$. It is shown that any $L^p(\mu)$ -valued measure has finite $L^2(\nu)$ -semivariation with respect to the tensor norm $L^2(\nu) \widehat{\otimes}_{\Delta_p} L^p(\mu)$ for $1 \leq p < \infty$ and finite $L^q(\nu)$ -semivariation with respect to the tensor norm $L^q(\nu) \widehat{\otimes}_{\gamma_q} L^p(\mu)$ whenever either $q = 2$ and $1 \leq p \leq 2$ or $q > \max\{p, 2\}$. However there exist measures with infinite L^q -semivariation with respect to the tensor norm $L^q(\nu) \widehat{\otimes}_{\gamma_q} L^p(\mu)$ for any $1 \leq q < 2$. It is also shown that the measure $m(A) = \chi_A$ has infinite L^q -semivariation with respect to the tensor norm $L^q(\nu) \widehat{\otimes}_{\gamma_q} L^p(\mu)$ if $q < p$.

1. INTRODUCTION

Let Z be a Banach space and let $m : \Sigma \rightarrow Z$ be a vector measure defined on a σ -algebra Σ of subsets of Ω . We write $|m|$ for the variation of the measure

$$|m|(A) = \sup \left\{ \sum_{j=1}^k \|m(A_j \cap A)\| : A_j \text{ pairwise disjoint, } k \in \mathbb{N} \right\}$$

and denote, for $1 \leq p < \infty$, the p -variation of the measure

$$\|m\|_p = \sup \left\{ \left(\sum_{j=1}^k \|m(A_j)\|^p \right)^{1/p} : A_j \text{ pairwise disjoint, } k \in \mathbb{N} \right\}.$$

We also write $\|m\| = \sup_{A \in \Sigma} \|m(A)\|$, which is equivalent to the semivariation of the vector measure m , that is

$$\|m\| \approx \sup \left\{ |\langle z^*, m \rangle|(\Omega) : \|z^*\| = 1 \right\}.$$

Suppose that X, Y is a Banach spaces and let τ be a norm on $X \otimes Y$ such that $\|x \otimes y\|_\tau \leq C \|x\| \|y\|$ for $x \in X, y \in Y$ and denote $X \widehat{\otimes}_\tau Y$ the completion under such a

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norm. Given a vector measure $m : \Sigma \rightarrow Y$ defined on a σ -algebra Σ of subsets of Ω , R. Bartle (see [2, 7]) introduced the notion of X -semivariation of m in $X \otimes Y$ given by

$$\beta_X(m, \tau, Y)(A) = \sup \left\{ \left\| \sum_{j=1}^k x_j \otimes m(A \cap A_j) \right\|_{\tau} \right\}$$

for every $A \in \Sigma$ where the supremum is taken over $\|x_j\| \leq 1$, A_j pairwise disjoint sets in Σ and $k \in \mathbb{N}$. We shall denote

$$\beta_X(m, \tau, Y) = \sup_{A \in \Sigma} \beta_X(m, \tau, Y)(A).$$

It is clear that

$$\|m\| \leq \beta_X(m, \tau, Y) \leq \|m\|_1.$$

If $X \widehat{\otimes}_{\varepsilon} Y$ and $X \widehat{\otimes}_{\pi} Y$ stand for the injective and projective tensor norms respectively, then one always has

$$\|m\| \leq \beta_X(m, \varepsilon, Y) \leq \beta_X(m, \tau, Y) \leq \beta_X(m, \pi, Y) \leq \|m\|_1.$$

It is well-known and easy to see that actually $\beta_X(m, \varepsilon, Y) = \|m\|$.

In [7] Jefferies and Okada developed a theory of integration of X -valued functions with respect to Y -valued measures of bounded X -semivariation in the case of completely separated tensor norms.

We shall be concerned with some interesting examples of norms coming from the theory of vector-valued functions: Throughout the paper $(\Omega_1, \Sigma_1, \mu)$ and $(\Omega_2, \Sigma_2, \nu)$ are finite measure spaces, $1 \leq p, q < \infty$ and the Banach spaces will be either $Y = L^p(\mu)$ or $X = L^q(\nu)$. We define γ_q and Δ_p the norms on $L^q(\nu) \otimes Y$ and $X \otimes L^p(\mu)$ identified as subspace of $L^q(\nu, Y)$ and $L^p(\mu, X)$, that is to say

$$L^q(\nu) \widehat{\otimes}_{\gamma_q} Y = L^q(\nu, Y), \quad X \widehat{\otimes}_{\Delta_p} L^p(\mu) = L^p(\mu, X).$$

In the case $p = q$ the $L^p(\nu)$ -semivariation of $L^p(\mu)$ -valued measures with respect to the topology τ_p such that $L^p(\mu) \widehat{\otimes}_{\tau_p} L^p(\nu)$ becomes $L^p(\mu \times \nu)$ for the product measure was studied in [8, 9].

In particular, if both $X = L^q(\nu)$ and $Y = L^p(\mu)$ then $L^q(\nu) \widehat{\otimes}_{\Delta_p} L^p(\mu)$ and $L^q(\nu) \widehat{\otimes}_{\gamma_q} L^p(\mu)$ coincide with the spaces of measurable functions $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ such that

$$\left(\int_{\Omega_1} \left(\int_{\Omega_2} |f(x, y)|^q d\nu(y) \right)^{p/q} d\mu(x) \right)^{1/p} < \infty$$

and

$$\left(\int_{\Omega_2} \left(\int_{\Omega_1} |f(x, y)|^p d\mu(x) \right)^{q/p} d\nu(y) \right)^{1/q} < \infty.$$

In this paper we shall try to understand better the difference between the classical semivariation or variation of a $L^p(\mu)$ -valued measure m and the $L^q(\nu)$ -semivariation with respect to the norms Δ_p , γ_q and π .

Let us establish the main results of the paper. Our first result establishes the following descriptions of the L^q -semivariation of L^p -valued measures with respect to the projective tensor norm, where we denote $L^p = L^p([0, 1])$ for $1 \leq p \leq \infty$.

THEOREM 1.1. *Let $1 \leq p, q \leq \infty$ and let $m : \Sigma \rightarrow L^p([0, 1])$ be a vector measure. Then*

- (i) $\beta_{L^p}(m, \pi, L^p) \approx \|m\|_1 \quad 1 \leq p \leq \infty.$
- (ii) $\beta_{L^2}(m, \pi, L^p) \approx \|m\|_1, \quad 1 < p < \infty.$
- (iii) $\beta_{L^2}(m, \pi, L^1) \approx \|m\|.$

This result shows that L^2 -valued measures are of finite L^2 -semivariation on $L^2 \otimes_{\pi} L^2$ if and only if they are of finite variation.

It was noticed in [8] that any L^2 -valued measure is of bounded L^2 -semivariation with respect to $L^2([0, 1]) \widehat{\otimes}_{\tau_2} L^2([0, 1])$, in other words $\beta_{L^2}(m, \Delta_2, L^2) \approx \|m\|.$

On the other hand $\beta_{L^q}(m, \pi, L^1) = \beta_{L^q}(m, \Delta_1, L^1)$. Hence Theorem 1.1 shows that $\beta_{L^2}(m, \Delta_1, L^1) = \|m\|.$

Let us just point out that this implies

$$(1) \quad \beta_{L^2}(m, \Delta_p, L^p) \approx \|m\|, 1 \leq p \leq 2$$

due to the simple observation

$$(2) \quad \beta_{L^q(\nu)}(m, \Delta_{p_1}, L^{p_1}(\mu)) \leq C \beta_{L^q(\nu)}(m, \Delta_{p_2}, L^{p_2}(\mu)) \quad p_1 \leq p_2.$$

We shall present another alternative proof that cover all the cases and gives an alternative proof of the known case $p = q = 2$ and extend (1) as follows.

THEOREM 1.2. *Let $1 \leq p < \infty$ and let $m : \Sigma \rightarrow L^p([0, 1])$ be a vector measure. Then*

$$\beta_{L^2}(m, \Delta_p, L^p) \approx \|m\|.$$

The question which now arises is whether or not there exist L^p -valued measures with $\beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)) = \infty$ if $q \neq 2$. In [7] examples of $L^p([0, 1])$ -valued measures of infinite $L^p([0, 1])$ -semivariation in $L^p([0, 1]) \widehat{\otimes}_{\tau_p} L^p([0, 1])$ were obtained for the values $p \neq 2$. For $1 \leq p < 2$ the approach was much simpler than for $p > 2$ and the example in this case relies on the existence of a non absolutely summing operator from $\ell^1 \rightarrow \ell^p$ for $p > 2$ (see [8, 9]).

We shall use the relationship between the tensor norms γ_q and Δ_p to get other examples. Recall that Minkowski's inequality gives $L^p(\mu, L^q(\nu)) \subseteq L^q(\nu, L^p(\mu))$ for $p \leq q$

and $L^q(\nu, L^p(\mu)) \subseteq L^p(\mu, L^q(\nu))$ for $q \leq p$. Hence

$$(3) \quad \beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)) \leq \beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)), \quad p \leq q,$$

$$(4) \quad \beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)) \leq \beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)), \quad q \leq p.$$

Also using general techniques, similar to those used in [8] one can show that for $1 \leq p \leq \infty$ and $1 \leq q < 2$ there exist $L^p(\mu)$ -valued measures m such that $\beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)) = \infty$. This, in particular, using the estimate (3), shows the existence of measures for which $\beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)) = \infty$ if $1 \leq q < 2, p \leq q$, completing and extending the case $p = q$.

THEOREM 1.3. *Let $1 \leq p \leq \infty$ and let $m : \Sigma \rightarrow L^p([0, 1])$ be a vector measure. Then*

- (i) $\beta_{L^2}(m, \gamma_2, L^p) \approx \|m\|, \quad 1 \leq p \leq 2.$
- (ii) $\beta_{L^q}(m, \gamma_q, L^p) \approx \|m\|, \quad \max\{p, 2\} < q.$

This gives that any measure has $\beta_{L^q}(m, \gamma_q, L^p) < \infty$ for $q > p \geq 2$. However in the last section it is shown that the $L^p([0, 1])$ -valued measure $m_p(A) = \chi_A$ has infinite $L^q([0, 1])$ -semivariation in $L^q([0, 1]) \widehat{\otimes}_{\gamma_q} L^p([0, 1])$ for $q < p$.

2. BOUNDED X-SEMIVARIATION

We start by the following characterisation of the bounded X -semivariation.

Taking into account that $X \widehat{\otimes}_{\pi} Y \subset X \widehat{\otimes}_{\tau} Y$, then $(X \widehat{\otimes}_{\tau} Y)^*$ can be regarded as a subspace of the space of bounded operators $\mathcal{L}(Y, X^*)$. Moreover $\|u\| \leq \|u\|_{(X \widehat{\otimes}_{\tau} Y)^*}$ for any $u \in (X \widehat{\otimes}_{\tau} Y)^*$, where the duality is given by

$$\left\langle u, \sum_{j=1}^k x_j \otimes y_j \right\rangle = \sum_{j=1}^k \langle u(y_j), x_j \rangle.$$

THEOREM 2.1. *Let $m : \Sigma \rightarrow Y$ be a vector measure. Then*

$$\beta_X(m, \tau, Y) \approx \sup\{\|u \circ m\|_1 : u \in \mathcal{L}(Y, X^*), \|u\|_{(X \widehat{\otimes}_{\tau} Y)^*} \leq 1\}.$$

PROOF: Let (x_j) be a bounded sequence in X and (A_j) be a sequence of pairwise disjoint sets in Σ . Consider, for $k \in \mathbb{N}$, the X -valued simple function $\phi = \sum_{j=1}^k x_j \chi_{A_j}$ and denote

$$\phi \widehat{\otimes}_{\tau} m(A) = \sum_{j=1}^k x_j \otimes m(A \cap A_j) \in X \otimes Y.$$

Clearly this defines a new $X \widehat{\otimes}_{\tau} Y$ -valued measure and one can rewrite

$$\beta_X(m, \tau, Y) = \sup\{\|\phi \widehat{\otimes}_{\tau} m\| : \phi \in \mathcal{S}(X), \|\phi\|_{\infty} \leq 1\}.$$

We now write the semivariation of $\phi \otimes_{\tau} m$ using duality, that is to say

$$\begin{aligned} \|\phi \otimes_{\tau} m\| &\approx \sup\{|\langle u, \phi \otimes m \rangle|(\Omega) : \|u\|_{(X \widehat{\otimes}_{\tau} Y)^*} \leq 1\} \\ &= \sup\left\{\sum_{j=1}^k |\langle u \circ m(A_j), x_j \rangle| : (A_j) \text{ pairwise disjoint, } \|u\|_{(X \widehat{\otimes}_{\tau} Y)^*} \leq 1\right\}, \end{aligned}$$

which, taking supremum over $\|x_j\| \leq 1$, gives

$$\begin{aligned} \beta_X(m, \tau, Y) &\approx \sup\left\{\sum_{j=1}^k \|u \circ m(A_j)\| : (A_j) \text{ pairwise disjoint, } \|u\|_{(X \widehat{\otimes}_{\tau} Y)^*} \leq 1\right\} \\ &\approx \sup\{\|u \circ m\|_1 : u \in \mathcal{L}(Y, X^*), \|u\|_{(X \widehat{\otimes}_{\tau} Y)^*} \leq 1\}. \end{aligned}$$

□

Let us see the formulation of Theorem 2.1 in the case $\tau = \Delta_p$ or $\tau = \gamma_q$.

It is well known that for $1 < p, q < \infty$ and $1/p' + 1/p = 1, 1/q + 1/q' = 1$ and for X, Y such that X^* and Y^* have the Radon-Nikodym property (see [6]) then

$$\left(L^q(\nu) \widehat{\otimes}_{\gamma_q} Y\right)^* = L^{q'}(\nu) \widehat{\otimes}_{\gamma_{q'}} Y^*$$

and

$$\left(X \widehat{\otimes}_{\Delta_p} L^p(\mu)\right)^* = X^* \widehat{\otimes}_{\Delta_{p'}} L^{p'}(\mu).$$

Now for each $f \in L^{p'}(\mu, X^*)$ we can define the operators $u_f : L^p(\mu) \rightarrow X^*$ and $v_f : X \rightarrow L^{p'}(\mu)$ given by

$$\langle u_f(\phi), x \rangle = \int_{\Omega} \langle f(t), x \rangle \phi(t) d\mu(t)$$

and

$$v_f(x) = \langle f, x \rangle.$$

Of course $(v_f)^* = u_f$ and $(u_f)^* = v_f$ if X is reflexive.

THEOREM 2.2. *Let $1 < p, q < \infty, X = L^q(\nu)$ and $Y = L^p(\mu)$. If $m : \Sigma \rightarrow L^p(\mu)$ is a vector measure then*

$$(5) \quad \beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)) = \sup\{\|u_f \circ m\|_1 : \|f\|_{L^{p'}(\mu, L^{q'}(\nu))} \leq 1\},$$

$$(6) \quad \beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)) = \sup\{\|v_g \circ m\|_1 : \|g\|_{L^{q'}(\nu, L^{p'}(\mu))} \leq 1\}.$$

PROOF: In the case $Y = L^p(\mu)$ and $X = L^q(\nu)$ for $1 < q, p < \infty$ the elements $u : L^p(\mu) \rightarrow L^{q'}(\nu)$ such that $u \in \left(L^q(\nu) \widehat{\otimes}_{\Delta_p} L^p(\mu)\right)^*$ can be seen as $u = u_f$ for some $f \in L^{p'}(\mu, L^{q'}(\nu))$, that is $u : L^p(\mu) \rightarrow L^{q'}(\nu)$ is given by

$$u(\phi)(y) = \int_{\Omega_1} f(x, y) \phi(x) d\mu(x).$$

Then (6) follows from Theorem 2.1 in this case.

Similarly the elements $u : L^p(\mu) \rightarrow L^q(\nu)$ such that $u \in (L^q(\nu) \widehat{\otimes}_{\gamma_q} L^p(\mu))^*$ can be seen as $u = v_g$ for some $g \in L^q(\nu, L^{p'}(\mu))$ and now

$$u(\psi)(y) = \langle g, \psi \rangle = \int_{\Omega_1} g(y, x)\psi(x)d\mu(x).$$

Again (6) follows from Theorem 2.1. □

3. PROOF OF THE MAIN THEOREMS

We use first the characterisation in Theorem 2.1 to get the following corollaries.

COROLLARY 3.1. *Let $m : \Sigma \rightarrow Y$ be a vector measure and X a Banach space.*

Then

$$\beta_X(m, \pi, Y) \approx \sup\{\|u \circ m\|_1 : u \in \mathcal{L}(Y, X^*), \|u\| \leq 1\}.$$

We use the notation $\Pi_p(X, Y)$ for the space of p -summing operators from X into Y and write $\pi_p(u)$ for the p -summing norm. The reader is referred to [5] for the basics in the theory of summing operators.

COROLLARY 3.2. *Let Y be a Grothendieck space, that is, $\Pi_1(Y, H) = \mathcal{L}(Y, H)$ for any Hilbert space H . Then*

$$(7) \quad \beta_H(m, \pi, Y) \approx \|m\|.$$

PROOF: Note that $\sum m(A_j)$ is an unconditionally convergent series in Y for any sequence of pairwise disjoint sets A_j . Now for any operator from $u : Y \rightarrow H$ one has $\sum \|u(m(A_j))\| \leq K_G \|u\| \|m\|$, where K_G is the Grothendieck constant. Now use Corollary 3.1. □

PROOF OF THEOREM 1.1: (i) Let $Y = L^p$ and $X = L^{p'}$ then choosing $u = Id : L^p \rightarrow (L^{p'})^*$, one concludes that $\|u \circ m\|_1 = \|m\|_1$. This shows $\beta_{L^{p'}}(m, \pi, L^p) = \|m\|_1$

(ii) follows from the following observation: If X^* is isomorphic to a complemented subspace of Y then $\beta_X(m, \pi, Y) \approx \|m\|_1$.

Indeed, assume $id : Y \rightarrow Y$ factors through X^* as $id = u_1 \circ u_2$ where $u_2 : Y \rightarrow X^*$ and $u_1 : X^* \rightarrow Y$ are bounded operators. Now observe that $\|m\|_1 \leq \|u_1\| \|u_2 \circ m\|_1$ and use Corollary 3.1.

Now use that the space Rad is complemented in $L^p([0, 1])$ and isomorphic to ℓ^2 (see [5, Theorem 1.12]) and therefore to L^2 , to conclude that

$$(8) \quad \beta_{L^2}(m, \pi, L^p([0, 1])) \approx \|m\|_1, 1 < p < \infty.$$

(iii) follows from Corollary 3.2. □

We now recall a lemma that we shall need in the sequel.

LEMMA 3.3. (i) Suppose that $1 < q < \infty$ and let Y be a Banach space such that $Y^* \in RNP$. If $u : Y \rightarrow L^q(\nu)$ belongs to $(L^q(\nu) \widehat{\otimes}_{\gamma_q} Y)^*$ then $\pi_{q'}(u) \leq \|u\|_{(L^q(\nu) \widehat{\otimes}_{\gamma_q} Y)^*}$.

(ii) Let $1 < p < \infty$ and let X be a Banach space such that $X^* \in RNP$. If $u : L^p(\mu) \rightarrow X^*$ belongs to $(X \widehat{\otimes}_{\Delta_p} L^p(\mu))^*$ then $\pi_{p'}(u^*) \leq \|u^*\|_{(X \widehat{\otimes}_{\Delta_p} L^p(\mu))^*}$.

PROOF: (i) It is well known (see [5, Example 2.11]) that if $g \in L^{q'}(\nu, Y^*)$ then $v_g : Y \rightarrow L^q(\nu)$ given by $v_g(y) = \langle g, y \rangle$ is q' -summing and $\pi_{q'}(v_g) \leq \|g\|_{L^{q'}(\nu, Y^*)}$. Now use that, under the assumptions, $(L^q(\nu) \widehat{\otimes}_{\gamma_q} Y)^* = L^{q'}(\nu, Y^*)$ and $u = v_g$ for certain $g \in L^{q'}(\nu, Y^*)$.

(ii) Note that $u = u_f$ for some $f \in L^{p'}(\mu, X^*)$. Hence $v_f = u^* : X^{**} \rightarrow L^p(\mu)$ is p' -summing and $\pi_{p'}(u^*) \leq \|f\|_{L^{p'}(\mu, X^*)} = \|u\|_{(L^q(\nu) \widehat{\otimes}_{\gamma_q} Y)^*}$. □

PROOF OF THEOREM 1.2: The case $p = 1$ is included in (iii) Theorem 1.1.

Assume now $1 < p < \infty$ and let $m : \Sigma \rightarrow L^p$ be a vector measure. Given $u : L^p \rightarrow L^2$ with $u \in (L^2 \widehat{\otimes}_{\Delta_p} L^p)^*$ we can use (ii) in Lemma 3.3 to conclude that there exist $f \in L^{p'}([0, 1], L^2)$ such that $v_f : L^2 \rightarrow L^p$ given by $\phi \rightarrow \int_0^1 \phi(y) f(x, y) dy$ is p' -summing and $u = u_f = (v_f)^*$. Hence, using [5, Theorem 2.21], one has that $(v_f)^* = u : L^p \rightarrow L^2$ is 1-summing. Therefore

$$\|u_f \circ m\|_1 \leq C \|u_f\| \|m\| \leq C \|f\|_{L^{p'}([0,1], L^2)} \|m\|.$$

□

Let us mention another useful lemma.

LEMMA 3.4. ([1, Proposition 6]) Suppose that Y is a Banach space of finite cotype r and let $\sum_j y_j$ be an unconditionally convergent series in Y .

(i) If $r = 2$ then there exist $(\alpha_j) \in \ell^2$ and a sequence in $(y'_j) \subset Y$ such that $y_j = \alpha_j y'_j$ and

$$\begin{aligned} \sum_j |\alpha_j|^2 &\leq \sup_{\|y^*\|=1} \sum_j |\langle y_j, y^* \rangle|, \\ \sup_{\|y^*\|=1} \sum_j |\langle y'_j, y^* \rangle|^2 &\leq \sup_{\|y^*\|=1} \sum_j |\langle y_j, y^* \rangle|. \end{aligned}$$

(ii) If $r > 2$ then for any $q > r$ there exist $(\alpha_j) \in \ell^q$ and a sequence in $(y'_j) \subset Y$ such that $y_j = \alpha_j y'_j$ and

$$\begin{aligned} \left(\sum_j |\alpha_j|^q\right)^{1/q} &\leq \left(\sup_{\|y^*\|=1} \sum_j |\langle y_j, y^* \rangle|\right)^{1/q}, \\ \left(\sup_{\|y^*\|=1} \sum_j |\langle y'_j, y^* \rangle|^{q'}\right)^{1/q'} &\leq \left(\sup_{\|y^*\|=1} \sum_j |\langle y_j, y^* \rangle|\right)^{1/q'}. \end{aligned}$$

PROOF: (i) Let $T : c_0 \rightarrow Y$ such that $T(e_j) = y_j$. Note that $\mathcal{L}(c_0, Y) = \Pi_2(c_0, Y)$ for any cotype 2 space Y . Now apply [5, Lemma 2.23] to the sequence (e_j) which satisfies

$\sup \left\{ \sum_j |\langle e_j, z \rangle| : \|z\|_{\ell^1} = 1 \right\}$ to conclude that $T(e_j) = y_j = \alpha_j y'_j$ with the desired properties.

(ii) Repeat the proof using now $L(c_0, Y) = \Pi_q(c_0, Y)$ for any $q > r$ (see [5, Theorem 11.14]). □

PROOF OF THEOREM 1.3: Note that Theorem 1.2 and (4) give

$$(9) \quad \beta_{L^2}(m, \gamma_2, L^p) \approx \|m\|, \quad 1 \leq p \leq 2.$$

To obtain (ii) we simply use the following more general result.

THEOREM 3.5. *If Y has cotype $r < \infty$ and Y^* has the RNP then*

$$(10) \quad \beta_{L^2(\nu)}(m, \gamma_2, Y) \approx \|m\|, \quad r = 2.$$

$$(11) \quad \beta_{L^q(\nu)}(m, \gamma_q, Y) \approx \|m\|, \quad q > r > 2.$$

PROOF: We only prove (11). The other is exactly the same.

Suppose that (A_j) is a sequence of pairwise disjoint sets. Since $m(A_j)$ is unconditionally convergent in Y , Lemma 3.4 implies that there exist $(\alpha_j) \in \ell^q$ and a sequence in $(y_j) \subset Y$ with $m(A_j) = \alpha_j y_j$ and

$$\begin{aligned} \left(\sum_j |\alpha_j|^q \right)^{1/q} &\leq \left(\sup_{\|y^*\|=1} \sum_j |\langle m(A_j), y^* \rangle| \right)^{1/q}. \\ \left(\sup_{\|y^*\|=1} \sum_j |\langle y_j, y^* \rangle|^{q'} \right)^{1/q'} &\leq \left(\sup_{\|y^*\|=1} \sum_j |\langle m(A_j), y^* \rangle| \right)^{1/q'}. \end{aligned}$$

On the other hand if $u \in (L^q(\nu) \widehat{\otimes} Y)^*$, using (i) in Lemma 3.3, one has $u \in \Pi_{q'}(Y, L^{q'})$. Therefore

$$\begin{aligned} \sum_j \|u(m(A_j))\| &= \sum_j |\alpha_j| \|u(y_j)\| \\ &\leq \left(\sum_j |\alpha_j|^q \right)^{1/q} \left(\sum_j \|u(y_j)\|^{q'} \right)^{1/q'} \\ &\leq \pi_{q'}(u) \left(\sum_j |\alpha_j|^q \right)^{1/q} \left(\sup_{\|y^*\|=1} \sum_j |\langle y_j, y^* \rangle|^{q'} \right)^{1/q'} \\ &\leq C \|u\|_{(L^q(\nu) \widehat{\otimes} Y)^*} \|m\|. \end{aligned}$$

□

4. MEASURES OF INFINITE X -SEMIVARIATION

We shall present now some necessary conditions to have bounded X -semivariation.

PROPOSITION 4.1. (i) *Assume that $X \widehat{\otimes}_\tau Y$ is of finite cotype q . If $m : \Sigma \rightarrow Y$ be a vector measure then*

$$\|m\|_q \leq C_q \beta_X(m, \tau, Y)$$

for some constant C_q independent of m .

In particular, if X has finite cotype q and $1 \leq p < \infty$ then

$$\|m\|_{\max\{q,2,p\}} \leq C\beta_X(m, \Delta_p, L^p(\mu)).$$

(ii) Let $1 \leq q < \infty$, let ν be a finite measure for which there exists a sequence of pairwise disjoint sets with $\nu(B_j) > 0$ and let $m : \Sigma \rightarrow Y$ be a vector measure. Then

$$\|m\|_q \leq C_q\beta_{L^q(\nu)}(m, \gamma_q, Y)$$

PROOF: (i) Suppose that (x_j) is a sequence in the unit ball of X and a sequence of pairwise disjoint sets A_j . Hence, for $0 \leq t \leq 1$, one has

$$\left\| \sum_{j=1}^k \tau_j(t)x_k \otimes m(A_j) \right\|_{X \otimes_r Y} \leq \beta_X(m, \tau, Y)$$

where τ_j stands for the Rademacher sequence. Now integrate over $[0, 1]$ and use the cotype estimate to get

$$\left(\sum_{j=1}^k \|x_k\|^q \|m(A_j)\|^q \right)^{1/q} \leq C_q\beta_X(m, \tau, Y).$$

Taking the sup over (x_j) and (A_j) one obtains the desired result.

Note that $L^p(\mu, X)$ has cotype equals $\max\{p, q, 2\}$.

(ii) Take $x_j = (\chi_{B_j})/(\nu(B_j)^{1/q})$, $\phi = \sum_{j=1}^k x_j \chi_{A_j}$, for some sequence of pairwise disjoint sets in Σ and notice that, for any $A \in \Sigma$,

$$\|\phi \otimes m(A)\|_{L^q(\nu, Y)} = \left(\sum_{j=1}^k \|m(A \cap A_j)\|^q \right)^{1/q}.$$

This gives the result. □

COROLLARY 4.2. Let Y be infinite dimensional Banach space, $1 \leq q < 2$ and ν be a finite measure for which there exists a sequence of pairwise disjoint sets with $\nu(E_n) > 0$.

- (i) There exist Y -valued measure such that $\beta_{L^q(\nu)}(m, \gamma_q, Y) = \infty$.
- (ii) If $L^p(\mu)$ is infinite dimensional then there exist $L^p(\mu)$ -valued measures m such that $\beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)) = \infty$ for $1 \leq q < 2$ and $q \geq p$.

PROOF: (i) Select an unconditionally convergent series (y_n) with $\sum_k \|y_k\|^q = \infty$ (this can be done for $1 \leq q < 2$, see, for instance [5]).

Now we define the measure over \mathbb{N} given by $m(\{k\}) = y_k$. Clearly $\|m\|_q = \infty$ and therefore $\beta_{L^q(\nu)}(m, \gamma_q, Y) = \infty$ from (ii) in Proposition 4.1.

(ii) follows from (i) and the estimate (3). □

A very important example to analyse is $m_p : \Sigma \rightarrow L^p(\mu)$ given by $m_p(A) = \chi_A$. We shall see that these measures are enough to produce examples with $\beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)) = \infty$ for $q < p$.

THEOREM 4.3. *Let $\mu(\Omega_1) < \infty, \nu(\Omega_2) < \infty, X = L^q(\nu)$ and $Y = L^p(\mu)$. Then the $L^p(\mu)$ -valued measure $m_p(A) = \chi_A$ has finite $L^q(\nu)$ -semivariation in $L^q(\nu) \widehat{\otimes}_{\gamma_q} L^p(\mu)$ if and only if $L^{q'}(\nu, L^{p'}(\mu)) \subseteq L^1(\mu, L^{q'}(\nu))$.*

PROOF: Let $g : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ be such that

$$\|g\|_{L^{q'}(\nu, L^{p'}(\mu))} = \left(\int_{\Omega_2} \left(\int_{\Omega_1} |g(y, x)|^{p'} d\mu(x) \right)^{q'/p'} d\nu(y) \right)^{1/q'} < \infty.$$

Note that the operator $v_g : L^p(\mu) \rightarrow L^{q'}(\nu)$ becomes

$$v_g(\psi)(y) = \int_{\Omega_1} g(y, x)\psi(x)d\mu(x),$$

hence, we have $v_g \circ m_p(A) = \int_A g(y, x)d\mu(x)$ for all $A \in \Sigma_1$. This shows that $v_g \circ m_p$ is the $L^{q'}(\nu)$ -valued measure with Radon–Nikodym derivative $g(y, \cdot)$. Therefore

$$\|v_g \circ m_p\|_1 = \int_{\Omega_1} \left(\int_{\Omega_2} |g(y, x)|^{q'} d\nu(y) \right)^{1/q'} d\mu(x).$$

Now Theorem 2.2 shows that m_p is of bounded $L^q(\nu)$ -semivariation in $L^q(\nu) \widehat{\otimes}_{\gamma_q} L^p(\mu)$ if and only if there exists $C > 0$ such that

$$\int_{\Omega_1} \left(\int_{\Omega_2} |g(y, x)|^{q'} d\nu(y) \right)^{1/q'} d\mu(x) \leq C \left(\int_{\Omega_2} \left(\int_{\Omega_1} |g(y, x)|^{p'} d\mu(x) \right)^{q'/p'} d\nu(y) \right)^{1/q'}.$$

That is to say $L^{q'}(\nu, L^{p'}(\mu)) \subseteq L^1(\mu, L^{q'}(\nu))$. □

COROLLARY 4.4. *Let $1 \leq p < \infty$ and $m_p : \Sigma \rightarrow L^p(\mu)$ given by $m_p(A) = \chi_A$. Then $\beta_{L^q(\nu)}(m_p, \gamma_q, L^p(\mu)) < \infty$ for $p \leq q$.*

PROOF: Note that for $p \leq q$ one obviously has

$$L^{q'}(\nu, L^{p'}(\mu)) \subseteq L^{q'}(\nu, L^{q'}(\mu)) = L^{q'}(\mu, L^{q'}(\nu)) \subseteq L^1(\mu, L^{q'}(\nu)).$$

Apply now Theorem 4.3. □

Actually the previous result is also a consequence of the following general fact.

PROPOSITION 4.5. *Let $1 \leq p < \infty, X$ a Banach space and let $m : \Sigma \rightarrow L^p(\mu)$ be a positive vector measure, that is $m(A) \geq 0$ for all $A \in \Sigma$. Then*

$$\beta_X(m, \Delta_p, L^p(\mu)) = \|m\|.$$

In particular, if m is positive and $p \leq q$ then

$$\beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)) = \|m\|.$$

PROOF: It is well-known that $(L^p(\mu, X))^* = (L^p(\mu) \widehat{\otimes} X)^*$ can be identified with the space of X^* -valued measures in $V^{p'}(\mu, X^*)$ (see [4]). In particular, if $u \in (L^p(\mu) \widehat{\otimes} X)^* \subset L(L^p(\mu), X^*)$ (see for instance [3]) there exists $\phi \in L^{p'}(\mu)$ such that $\|\phi\|_{p'} \leq \|u\|_{(L^p(\mu) \widehat{\otimes} X)^*}$ and satisfies that

$$\|u(\psi)\| \leq \int_{\Omega} \phi(t)\psi(t)d\mu(t)$$

for any positive function $\psi \in L^p(\mu)$. Therefore, if $\|u\|_{(L^p(\mu) \widehat{\otimes} X)^*} = 1$ then

$$\begin{aligned} \sum_{j=1}^k \|u(m(A_j))\| &\leq \|\phi\|_{p'} \int_{\Omega} \sum_{j=1}^k \frac{|\phi(t)|}{\|\phi\|_{p'}} m(A_j)(t) d\mu(t) \\ &\leq \sup \left\{ \sum_{j=1}^k |\langle \phi', m(A_j) \rangle| : \|\phi'\|_{L^{p'}} = 1 \right\} \end{aligned}$$

Hence $\|u_f \circ m\|_1 \leq \|m\|$. Apply now Theorem 2.2.

In the case $X = L^q(\nu)$ and $p \leq q$ (4) allows us to conclude the proof. □

We shall now see that the range of values in Theorem 4.3 is sharp.

LEMMA 4.6. *If $p > q$ then there exists $f : [0, 1]^2 \rightarrow \mathbb{R}^+$ such that*

$$\int_0^1 \left(\int_0^1 f(x, y)^q dy \right)^{p/q} dx < \infty$$

and

$$\int_0^1 \left(\int_0^1 f(x, y)^p dx \right)^{1/p} dy = \infty.$$

PROOF: Denoting $\beta = p/q > 1$ and $g(x, y) = f(x, y)^q$ it suffices to find $g : [0, 1]^2 \rightarrow \mathbb{R}^+$ such that

$$\int_0^1 \left(\int_0^1 g(x, y) dy \right)^{\beta} dx < \infty$$

and

$$\int_0^1 \left(\int_0^1 g(x, y)^{\beta} dx \right)^{1/\beta} dy = \infty.$$

Recall that the Hardy operator $T(\phi)(x) = (1/x) \int_0^x \phi(y) dy$ is bounded on $L^{\beta}([0, 1])$ for $\beta > 1$ and define

$$g(x, y) = \frac{1}{x} \chi_{[0, x]}(y) \phi(y)$$

for a function $\phi \in L^{\beta}([0, 1])$ to be chosen later.

Clearly

$$\int_0^1 \left(\int_0^1 g(x, y) dy \right)^{\beta} dx = \|T(\phi)\|_{\beta}^{\beta} \leq \|T\|^{\beta} \|\phi\|_{\beta}^{\beta}$$

On the other hand

$$\begin{aligned} \int_0^1 \left(\int_0^1 g(x, y)^\beta dx \right)^{1/p} dy &= \int_0^1 \phi(y)^{\beta/p} \left(\int_y^1 \frac{dx}{x^\beta} \right)^{1/p} dy \\ &\geq C \int_0^1 \phi(y)^{\beta/p} \frac{1}{y^{(\beta-1)/p}} dy \\ &= C \left(\int_0^1 \left(\frac{\phi(y)}{y^{1/\beta'}} \right)^{\beta/p} dy \right)^{1/p} \\ &\geq C \left(\int_0^1 \frac{\phi(y)}{y^{1/\beta'}} dy \right)^{\beta/p}. \end{aligned}$$

Now select $\phi(y) = 1/(y^{1/\beta}) \log(1/y)$ to have $\phi \in L^\beta([0, 1])$ and

$$\int_0^1 \frac{\phi(y)}{y^{1/\beta'}} dy = \int_0^1 \frac{dy}{y \log(1/y)} = \infty. \quad \square$$

COROLLARY 4.7. For $q < p$ the $L^p([0, 1])$ -valued measure $m_p(A) = \chi_A$ has infinite $L^q([0, 1])$ -semivariation in $L^q([0, 1]) \widehat{\otimes}_{\gamma_q} L^p([0, 1])$.

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