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ASYMPTOTIC PROPERTIES OF MULTICOLOR RANDOMLY REINFORCED PÓLYA URNS

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Abstract

The generalized Pólya urn has been extensively studied and is widely applied in many disciplines. An important application of urn models is in the development of randomized treatment allocation schemes in clinical studies. The randomly reinforced urn was recently proposed, but, although the model has some intuitively desirable properties, it lacks theoretical justification. In this paper we obtain important asymptotic properties for multicolor reinforced urn models. We derive results for the rate of convergence of the number of patients assigned to each treatment under a set of minimum required conditions and provide the distributions of the limits. Furthermore, we study the asymptotic behavior for the nonhomogeneous case.

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1. Introduction

Randomization is often a preferred way of assigning patients to different treatments in a clinical trial. Response adaptive designs that link the randomization procedure to the responses of treated patients have proved to be extremely valuable from an ethical perspective, because these designs are able to reduce the expected number of patients receiving the inferior treatments. In an adaptive design, patients enter the experiment sequentially and are randomly allocated to a treatment, according to a rule that depends on the previous allocations and the previous observed responses. A vast number of novel adaptive designs have been proposed in recent years. For a review of these innovations, the reader is referred to [19] and [32]. In the long history of the development of adaptive designs, urn models have remained an influential and popular family of response adaptive-randomization procedures ever since Wei and Durham [33] proposed the randomized play-the-winner rule. Recently, Zhang *et al.* [35] unified the most classical urn models into a general family of urn models (the immigrated urn (IMU) models)

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and obtained their general asymptotic properties. However, they did not include the important urn model that is the focus of this paper. The present paper studies the asymptotic properties of a multicolor, randomized Pólya urn, referred to as a randomly reinforced Pólya urn (RRPU). It is crucial to note that the RRPU is not a special case of the IMU, mainly because in the RRPU the random selection of a ball rewards a random number of balls of the same type and ignores all remaining types. When the number of balls rewarded at each selection is nonrandom and equal, the model reduces to the original Pólya urn (cf. [15] and [30]) which is widely studied in the literature.

The model proposed by Durham and Yu [13] for two treatments was possibly the first RRPU model for adaptive designs in clinical trials. Numerous studies have been conducted in this area, including [10], [11], [14], [23], [27], [28], [29], etc. Although the RRPU model is of fundamental importance in many applied areas, such as economics (cf. [9] and [16]), information science (cf. [24]), and resampling theory, in this paper we focus on clinical trial applications due to the model's important role in the treatment allocation process. However, the results reported in this paper certainly have much wider applications.

For many adaptive designs in the literature, the proportion of patients allocated to each treatment converges to a limit in (0, 1). Besides rare exceptions such as the response adaptive design of Aletti *et al.* [1] that a two-color RRPU can target fixed asymptotic allocations (see also [17]), a design driven by the RRPU usually allocates patients in an optimal manner so that the proportion of patients assigned to the best treatment converges to 1. However, it is important to know the (expected) number of patients in each treatment when the statistical test for the treatment differences and the power of the test are considered.

Recently, for the two-treatment case, May and Flournoy [25] found the exact convergence rate of the allocation proportion of the inferior treatment. They proved that the number of patients allocated to the inferior treatment after being suitably normalized converges to a random limit η. Similar results have also been found by Durham and Yu [13]. However, May and Flournoy [25] proved that the limit η is strictly positive and showed that the power of the test for the treatment difference is a decreasing function of η. As discussed in [25], the distribution of η is of fundamental importance for calculating the exact power of the test. Aletti *et al.* [2], [4] characterized the limiting distribution as the unique continuous solution of a functional equation when the reinforcements have different bounded distribution with the same mean, and Aletti *et al.* [3] proved that the limit distribution has no point mass by establishing a conditional central limit theorem. For the Pólya's original urn, a particular case of the RRPU, it is well known that the limit distribution of the urn proportions is a beta distribution. Janson [22] studied the limit distribution problem for a generalized two-color Pólya urn in which the number of balls added at each time can differ from color to color but should be nonrandom. In the general case, the limit distribution is an open problem. Furthermore, in the literature, including [2], [3], [4], [22], [25], [27], and [29], the results for RRPU models are usually limited to the two-treatment case and are established under the strict condition that the number of balls added at each stage has a distribution on a bounded real set. Berti *et al.* [10] [11] derived central limit theorems for a multicolor RRPU. However, their results were generally limited to treatments with equal reinforcement means or the best treatments.

The main aims of this paper are

- to study the asymptotic properties, including the convergence, the convergence rate, and the asymptotic normalities, of response-adaptive designs generated by a general multicolor RRPU under the minimum requirement of conditions;

- to find the distributions of both the limit of the normalized number of patients allocated to each treatment and the limit of the normalized number of balls of each type; and
- to generalize the model to the nonhomogeneous case in which the updating of the urn may use information from all previous stages.

In Section 2 we illustrate the almost-sure asymptotic properties of the first order for multicolor reinforced Pólya urn models. We show that both the urn proportions and the allocation proportions, after being suitably normalized, converge to a positive random limit. In Section 3, the exact distributions of the limits are obtained by applying the theory of branching processes to the case in which all of the balls have integer numbers. In Section 4, the asymptotic properties of the second order, including the rates of almost-sure convergence and asymptotic normalities, are established for both the urn proportions and the allocation proportions. In Section 5, the nonhomogeneous case is considered. The proofs of the main results are given in Appendix A. Throughout this paper, for two positive sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n = O(b_n)$ if there is a constant C such that $a_n \leq Cb_n$, $a_n \sim b_n$ if $a_n/b_n \to 1$, and $a_n \approx b_n$ if $a_n = O(b_n)$ and $b_n = O(a_n)$.

2. The model and asymptotic properties of the first order

Consider a clinical study with K different treatments. Patients arrive sequentially and respond without delay. Each incoming patient is allocated to one of the K treatments according to a treatment allocation scheme. In urn models, an urn with K types of balls is used to randomize incoming patients. Let $\boldsymbol{Y}_0 = (Y_{0,1}, \dots, Y_{0,K})$ be the initial urn components, where $Y_{0,k}$ is the number of type k-balls. After m stages ($m \geq 0$) and the first m patients are assigned, suppose that the urn components are $\boldsymbol{Y}_m = (Y_{m,1}, \dots, Y_{m,K})$. At the $m+1$ stage, a ball is drawn at random, its label noted, and the ball is *replaced*. If its label was k then the $(m+1)$th patient is assigned to treatment k, i.e. the $(m+1)$th patient is assigned to treatment k with a probability

$$p_{m+1,k} = \frac{Y_{m,k}}{|\boldsymbol{Y}_m|}, \qquad \text{where} \qquad |\boldsymbol{Y}_m| = Y_{m,1} + \cdots + Y_{m,K}.$$

The urn is updated according to the response $\xi_{m+1,k}$ of the $(m+1)$th patient. Different methods of updating produce different types of urn models. Urn models have been widely studied in the literature. Rosenberger [31] traced the historical development of generalized urn models, their properties, and applications in sequential designs. The more recent literature includes, for instance, studies by Bai and Hu [6], [7], Bai *et al.* [8], Chauvin *et al.* [12], Janson [21], and Zhang *et al.* [34]. However, the RRPU that we consider here is not covered by their assumptions. The main reason for this omission is that the mean replacement matrix of an RRPU is not irreducible. In the RRPU, the urn is updated by adding $U_{m+1,k} \geq 0$ balls of type k to the urn when the response $\xi_{m+1,k}$ is observed, and so the mean replacement matrix $\operatorname{diag}(\mathbb{E}U_{m+1,1}, \dots, \mathbb{E}U_{m+1,K})$ is diagonal. Usually, it is assumed that $U_{m+1,k}$ is a function $U(\xi_{m+1,k})$ of the response $\xi_{m+1,k}$. In our setting, unless specifically noted otherwise, $Y_{0,k}$ and $U_{m,k}$, $k = 1, \dots, K$, $m = 1, 2, \dots$, take nonnegative real values and are not necessarily integers. Furthermore, $(U_{m,1}, \dots, U_{m,K})$, $m = 1, 2, \dots$, are assumed to be independent, identically distributed random vectors. We denote by $m_k = \mathbb{E}[U_{m,k}]$ the mean of the number of balls of type k added. Throughout this paper, we assume that the means $m_1, \dots, m_K$ are finite and positive.

Let $X_{m,k}$ be the result of the mth assignment, i.e. $X_{m,k} = 1$ if the mth patient is assigned to treatment k, and 0 otherwise. Let $N_{m,k} = \sum_{j=1}^{m} X_{m,k}$ be the number of patients out of the

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first m patients assigned to treatment k. The asymptotic properties of $N_{n,k}$, $k = 1, \dots, K$, are important in clinical trial studies (cf. [19]).

Suppose that $m_1 > m_k$, $k \neq 1$. It has been shown that $Y_{n,1}/|\mathbf{Y}_n| \to 1$ and $N_{n,1}/n \to 1$ almost surely (a.s.), i.e.

$$\frac{Y_{n,k}}{|\mathbf{Y}_n|} \to 0 \quad \text{and} \quad \frac{N_{n,k}}{n} \to 0 \quad \text{a.s. for } k \neq 1, \tag{2.1}$$

if $U_{n,1}, \dots, U_{n,K}$ are distributed on a nonnegative and bounded set (cf. [9] and [28]). Recently, May and Flournoy [25] studied the convergence rate of (2.1) in the two-treatment case.

Theorem 2.1. *Suppose that $K = 2$, $m_1 > m_2$, and that $U_{n,1}$ and $U_{n,2}$ have distributions on a nonnegative and bounded real set. Then*

$$\frac{Y_{n,2}}{|\mathbf{Y}_n|^{m_2/m_1}} \to \psi \quad \text{a.s.,} \qquad \frac{N_{n,2}}{n^{m_2/m_1}} \to \eta \quad \text{a.s.,} \tag{2.2}$$

where ψ and η are two random variables with support in $(0, \infty)$.

Durham and Yu [13] obtained a similar result:

$$\frac{N_{n,2}^{1/m_2}}{N_{n,1}^{1/m_1}} \quad \text{converges a.s. to a random variable} \tag{2.3}$$

(cf. their Theorem 4). May and Flournoy [25] showed that the limits ψ and η are strictly positive and also proved that the power for testing $\mu_1 = \mu_2$ is a function of η. However, the distributions have not been found.

We refer to convergences of the type (2.2) and (2.3) as first-order convergences because they are related to the classical law of large numbers, and the limits ψ and η are called the first-order limits. The convergence corresponding to the convergence rates of (2.2) and (2.3) is called the second-order convergence. In this section we consider the first-order convergence. The following theorem gives the convergence of both the urn components and the allocation numbers of patients for a general multitreatment design driven by an RRPU. The exact distributions of the limits are derived in the next section for the case in which all of the numbers of balls are integers.

Theorem 2.2. *Suppose that $\mathbb{E}[U_{1,k} \log U_{1,k}] < \infty$. Hereafter, $\log x = \ln(x \vee \mathrm{e})$. Thus, there exist K positive random variables ϖ_i such that*

$$\frac{Y_{n,i}^{1/m_i}}{Y_{n,j}^{1/m_j}} \to \frac{\varpi_i^{1/m_i}}{\varpi_j^{1/m_j}} \quad \text{a.s.,} \tag{2.4}$$

$$\frac{N_{n,i}^{1/m_i}}{N_{n,j}^{1/m_j}} \to \frac{(\varpi_i/m_i)^{1/m_i}}{(\varpi_j/m_j)^{1/m_j}} \quad \text{a.s.} \tag{2.5}$$

As a consequence,

$$\frac{Y_{n,k}}{n^{m_k/m_{\max}}} \to \psi_i := \left(\frac{m_{\max}}{\sum_{\{j\,:\, m_j = m_{\max}\}} \varpi_j}\right)^{m_k/m_{\max}} \varpi_k \quad \text{a.s.,} \tag{2.6}$$

$$\frac{N_{n,k}}{n^{m_k/m_{\max}}} \to \eta_k \widehat{=} \frac{\psi_k}{m_k} \quad \text{a.s.,} \tag{2.7}$$

where $m_{\max} = \max_i m_i$.

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The above theorem implies that $Y_{n,k}^{1/m_k}$ and $N_{n,k}^{1/m_k}$ increase at the same rate for all $k = 1, \dots, K$. In Section 4 we consider the second-order convergence of $Y_{n,k}^{1/m_k}$ and $N_{n,k}^{1/m_k}$, including the almost-sure convergence rate of (2.4)–(2.7) and related asymptotic normalities.

The following theorem indicates that the condition $\mathbb{E}[U_{1,k} \log U_{1,k}] < \infty$ cannot be weakened.

Theorem 2.3. *Suppose that, for each $k = 1, \dots, K$, there is a $\delta_k > 0$ such that $Y_{n,k}/n^{\delta_k}$ (or $N_{n,k}/n^{\delta_k}$) converges in distribution to a finite limit ϖ_k^* with $\mathbb{P}(\varpi_k^* > 0) > 0$. Then we must have $\delta_k = m_k/m_{\max}$, $k = 1, \dots, K$. Furthermore, if one of $\mathbb{E}[U_{1,k} \log U_{1,k}]$, $k = 1, \dots, K$, is finite then all of them are finite.*

The main idea underlying the proofs of Theorems 2.2 and 2.3 is to find a common random normalization factor l_n such that, for each k, both $Y_{n,k}/l_n^{m_k}$ and its reciprocal look like nonnegative supermartingales and converge a.s. to a positive limit if and only if $\mathbb{E}[U_{1,k} \log U_{1,k}] < \infty$. The details of the proofs are postponed to Appendix A.1 as some preparations are required first.

In practice, the constants m_i in the normalization factors are unknown and need to be estimated. The following corollary tells us that they can be replaced by the sample means.

Corollary 2.1. *Under the conditions of Theorem 2.2, (2.4)–(2.7) hold whenever some or all of the means m_i, $i = 1, \dots, K$, on the left-hand side are replaced by the corresponding sample means*

$$\hat{m}_i =: \hat{m}_{n,i} = \frac{Y_{n,i} + 1/2}{N_{n,i} + 1}, \qquad i = 1, \dots, K.$$

Here, 1 *and* $\frac{1}{2}$ *are added to the denominator and numerator, respectively, to avoid the case* $0/0$.

Proof. According to (2.6) and (2.7), we have $\log Y_{n,k} \approx \log N_{n,k} \approx \log n$ a.s. To show that m_k can be replaced by $\hat{m}_k$, it is sufficient to show that

$$\left(\frac{1}{m_k} - \frac{1}{\hat{m}_k}\right) \log Y_{n,k} \to 0 \quad \text{a.s.} \quad \text{and} \quad \left(\frac{1}{m_k} - \frac{1}{\hat{m}_k}\right) \log N_{n,k} \to 0 \quad \text{a.s.},$$

which are equivalent to

$$\frac{\sum_{l=1}^{n} X_{l,k}(U_{l,k} - m_k)}{N_{n,k}/\log N_{n,k}} \to 0 \quad \text{a.s.}$$

if $N_{n,k} \to \infty$. The proof is completed by Lemma A.1 in the appendix.

3. The limit distributions

May and Flournoy [25] pointed out that the distribution of η in (2.2) is of fundamental importance in calculating the exact power of the test of treatment efficacy. For the two-treatment model, Aletti *et al.* [2], [4] characterized the limit distribution of the urn proportions as the unique continuous solution of a function equation for the case of equal reinforcement means, where $U_{n,1}$ and $U_{n,2}$ are assumed to be bounded and have the same distribution, and Janson [22] established a general result for unequal but nonrandom reinforcements. However, the limit distribution remains unknown for general cases. The following theorem characterizes the distributions of all of the limits in (2.4)–(2.7) for the general multitreatment RRPU model, in which the number of random balls is an integer. In the general case, in which a fractional number of balls is possible, the distribution of the limits is still an open problem.

Theorem 3.1. *Suppose that $Y_{0,k}$ and $U_{n,k}$, $k = 1, 2, \ldots, K$, are all integers. Let $f_k(s) = \mathbb{E}[s^{U_{1,k}}]$, $0 \le s \le 1$, be the probability generating function of $U_{n,k}$, and let $m_k = \mathbb{E}U_{1,k}$ be the mean. Suppose that $\mathbb{E}[U_{1,k} \log U_{1,k}] < \infty$. Then $\varpi_1, \ldots, \varpi_K$ in (2.4)–(2.7) can be chosen such that they are independent, positive, continuous random variables, and the distribution of ϖ_k is determined by $\mathbb{E}[\mathrm{e}^{-u\varpi_k}] = [g_k(u)]^{Y_{0,k}}$ with*

$$\operatorname{inv} g_k(u) = (1-u)\exp\biggl\{\int_1^u \biggl(\frac{m_k}{s(f_k(s)-1)} + \frac{1}{1-s}\biggr)\,\mathrm{d}s\biggr\}, \qquad 0 < u \le 1. \tag{3.1}$$

Remark 3.1. According to (3.1), the distribution of ϖ_k is uniquely determined by $Y_{0,k}$ and the distribution of $U_{1,k}$. Therefore, according to (2.6), (2.7), and the independence of $\varpi_1, \ldots, \varpi_K$, the distributions of ψ_k and η_k are uniquely determined by $Y_{0,1}, \ldots, Y_{0,K}$ and the distributions of $U_{1,1}, \ldots, U_{1,K}$.

Remark 3.2. We conjecture that the results in Theorem 3.1 hold for all cases where $Y_{0,k} > 0$ and $U_{n,k} \ge 0$ are real numbers.

Next, we provide an example for illustrative purposes before stating the proof of Theorem 3.1. The example is a generalization of the Pólya urn as well as the randomized Pólya urn proposed in [23].

Example 3.1. For the dichotomous response case in clinical trials, let $\xi_{m,k} = 1$ if the outcome of the mth patient receiving treatment k is a success, and 0 if it is a failure. In addition, let $p_k = \mathbb{P}(\xi_{m,k} = 1)$ be the probability of success. Suppose that we add α_k type-k balls to the urn when we observe that treatment k has been a success, and so $U_{m,k} = \alpha_k \xi_{m,k}$, where α_k is a positive integer. Then $m_k = \alpha_k p_k$ and $f_k(s) = 1 - p_k + s^{\alpha_k} p_k$. From (3.1), it follows that $g_k(u) = (1 + \alpha_k u)^{-1/\alpha_k}$. Therefore, the distribution of ϖ_k is gamma with the parameters $Y_{0,k}/\alpha_k$ and $1/\alpha_k$.

Suppose that $m_1 = \cdots = m_m > m_k$, $k > m$. Then, for $k > m$, η_k is distributed as

$$\frac{m_1^{m_k/m_1}}{m_k} \frac{\alpha_k \Gamma_k(Y_{0,k}/\alpha_k, 1)}{[\sum_{j=1}^m \alpha_j \Gamma_j(Y_{0,j}/\alpha_j, 1)]^{m_k/m_1}}, \tag{3.2}$$

where the $\Gamma_j(Y_{0,j}/\alpha_j, 1)$ are independent gamma-distributed random variables with the parameters given in the parentheses; and, for $k = 1, \ldots, m$, η_k is distributed as

$$\frac{\alpha_k \Gamma_k(Y_{0,k}/\alpha_k, 1)}{\sum_{j=1}^m \alpha_j \Gamma_j(Y_{0,j}/\alpha_j, 1)}. \tag{3.3}$$

The limit in (2.7) for the urn proportions is $\psi_k = m_k \eta_k$.

In particular, when the α_k are equal (to α, say), the random variables in (3.2) and (3.3) are respectively distributed as

$$\frac{p_1^{p_k/p_1}}{p_k} \frac{\Gamma_k(Y_{0,k}/\alpha, 1)}{[\Gamma(\sum_{j=1}^m Y_{0,j}/\alpha, 1)]^{p_k/p_1}} \quad \text{and} \quad \operatorname{Beta}\biggl(\frac{Y_{0,k}}{\alpha}, \sum_{j=1, j \ne k}^{m} \frac{Y_{0,j}}{\alpha}\biggr);$$

when the p_k are equal (to p, say), the random variables in (3.2) and (3.3) are respectively distributed as

$$p^{\alpha_k/\alpha_1 - 1} \frac{\Gamma(Y_{0,k}/\alpha_k, 1)}{[\Gamma(\sum_{j=1}^m Y_{0,j}/\alpha_1, 1)]^{\alpha_k/\alpha_1}} \quad \text{and} \quad \operatorname{Beta}\biggl(\frac{Y_{0,k}}{\alpha_1}, \sum_{j=1, j \ne k}^{m} \frac{Y_{0,j}}{\alpha_1}\biggr). \tag{3.4}$$

This example provides the limit distribution for the randomized Pólya urn proposed in [23] which is a special case with equal α_k. Also, the Pólya urn is a special case of the example with $p_k \equiv 1$. For the Pólya urn, it is well known that the limit distribution of the urn proportion is a beta distribution when the α_k are equal, and Theorem 1.4 of [22] gives a general result for the urn components in the two-treatment case. Our (3.4) provides the distribution for all the cases. Recently, Aletti *et al.* [4] proved that the limiting distribution of the urn proportions for a two-color RRPU with equal reinforcement means is the solution of a function equation, and found several new families of distributions generalizing the beta family. It is easy to show that the distribution in (3.3) is linked to the new distribution in their Section 6.2, with both being members of a generalized beta family.

To conclude this section, we give the proof of Theorem 3.1.

Proof of Theorem 3.1. As the number of balls is an integer, we can use the embedding method of [5] to derive the limit. Let $\{Z(t) = (Z_1(t), \dots, Z_K(t));\ t \geq 0\}$ be a K-type Markov branching process with $Z_k(t)$, $k = 1, 2, \dots, K$ (the K branching processes are independent) and $\boldsymbol{Z}(0) = \boldsymbol{Y}_0$. Assume that (i) each particle lives for a unit exponential random time, and (ii) when a type-k ($k = 1, 2, \dots, K$) particle dies, new type-k particles are born according to the probability generating function $s f_k(s)$, i.e. the random number of born particles has the same distribution as $U_{1,k} + 1$. Let $\tau_0 = 0$ and τ_n be the time of the nth death in the system. Then, following the same argument as that given in Theorem 9.2 of [5], $\{\boldsymbol{Z}(\tau_n); n \geq 0\}$ is equivalent to $\{\boldsymbol{Y}_n; n \geq 0\}$; in other words, these two random sequences have the same distribution. By Theorem 8.3 of [5] and the assumption that $\mathbb{E}[U_{1,k} \log U_{1,k}] < \infty$, for each $Z_k(t)$, there exists a positive, continuous random variable $\widetilde{\varpi}_k$, with $\mathbb{E}[\mathrm{e}^{-u\widetilde{\varpi}_k}] = [g_k(u)]^{Z_k(0)} = [g_k(u)]^{Y_{0,k}}$ satisfying (3.1) such that

$$\mathrm{e}^{-m_k t} Z_k(t) \to \widetilde{\varpi}_k \quad \text{a.s.} \tag{3.5}$$

Furthermore, $\widetilde{\varpi}_k$, $k = 1, \dots, K$, are independent because $\{Z_k(t)\}$, $k = 1, \dots, K$, are K independent processes. Now, it is obvious that $\tau_n \to \infty$ a.s. as $n \to \infty$. From (3.5), we conclude that $\mathrm{e}^{-m_k \tau_n} Z_k(\tau_n) \to \widetilde{\varpi}_k$ a.s. Hence,

$$\frac{Z_i^{1/m_i}(\tau_n)}{Z_j^{1/m_j}(\tau_n)} \to \frac{\widetilde{\varpi}_i^{1/m_i}}{\widetilde{\varpi}_j^{1/m_j}} \quad \text{a.s.}$$

By (2.4), it follows that

$$\widetilde{\varpi}_i^{1/m_i} / \widetilde{\varpi}_j^{1/m_j} = \varpi_i^{1/m_i} / \varpi_j^{1/m_j} \quad \text{a.s.}$$

So, without loss of generality, we can assume that $\widetilde{\varpi}_k = \varpi_k$.

4. The second order of convergence

In this section we consider the convergence rate of (2.5) and (2.6). The first theorem gives the strong convergence rates and the second is the central limit theorem. Although the distributions of the first-order limits are unknown unless the numbers of the balls are integers, the distributions of the second-order limits are mixing normal in general. We use the notation $m_k = \mathbb{E}U_{1,k}$, $m_{\max} = \max_i m_i$, and ψ_i and $\eta_i = \psi_i / m_i$ as defined in Theorem 2.2. Furthermore, let m_{sec} be the second largest value of m_i, $i = 1, 2, \dots, K$, and define $\delta_i = m_i / m_{\max}$ and $\delta_{i \wedge j} = \delta_i \wedge \delta_j$.

Theorem 4.1. *Suppose that* $\mathbb{E}[U_{1,k}^p] < \infty$ *and* $m_k > 0$, $k = 1, \dots, K$, *where* $1 < p < 2$. *Then*

$$\frac{Y_{n,i}^{1/m_i}}{Y_{n,j}^{1/m_j}} - \frac{\psi_i^{1/m_i}}{\psi_j^{1/m_j}} = o(n^{-\delta_{i\wedge j}(1-1/p)}) \quad a.s., \tag{4.1}$$

$$\frac{N_{n,i}^{1/m_i}}{N_{n,j}^{1/m_j}} - \frac{\eta_i^{1/m_i}}{\eta_j^{1/m_j}} = o(n^{-\delta_{i\wedge j}(1-1/p)}) \quad a.s.$$

Theorem 4.2. *Suppose that* $\mathbb{E}[U_{1,k}^2] < \infty$ *and* $m_k > 0$, $k = 1, \dots, K$. *Define* $\sigma_{U,k}^2 = \mathrm{var}(U_{1,k}/m_k)$. *Therefore, there are independent, standard normal random variables* $N_{k1}(0, 1)$, $N_{k2}(0, 1)$, $k = 1, \dots, K$, *which are also independent of* ψ_k, η_k, $k = 1, \dots, K$, *such that*

$$\sqrt{n^{\delta_{i\wedge j}}}\bigg(\frac{(Y_{n,i}/\psi_i)^{1/m_i}}{(Y_{n,j}/\psi_j)^{1/m_j}} - 1, \frac{(N_{n,i}/\eta_i)^{1/m_i}}{(N_{n,j}/\eta_j)^{1/m_j}} - 1\bigg) \xrightarrow{\mathrm{D}} \big(A_{ij}, A_{ij} + B_{ij}\big) \quad \textit{(stably)}\ i, j = 1, \dots, K, \tag{4.2}$$

and

$$\frac{Y_{n,k} - m_k N_{n,k}}{m_k\sqrt{n^{\delta_k}}} \xrightarrow{\mathrm{D}} \sqrt{\eta_k}\sigma_{U,k} N_{k2}(0, 1) \quad \textit{(stably)},$$

where

$$A_{ij} = \frac{\mathbf{1}\{m_i \le m_j\}}{m_i\sqrt{\eta_i}}\sqrt{1+\sigma_{U,i}^2}N_{i1}(0, 1) - \frac{\mathbf{1}\{m_i \ge m_j\}}{m_j\sqrt{\eta_j}}\sqrt{1+\sigma_{U,j}^2}N_{j1}(0, 1),$$

$$B_{ij} = \frac{\mathbf{1}\{m_i \le m_j\}}{m_i\sqrt{\eta_i}}\sigma_{U,i}N_{i2}(0, 1) - \frac{\mathbf{1}\{m_i \ge m_j\}}{m_j\sqrt{\eta_j}}\sigma_{U,j}N_{j2}(0, 1).$$

Hereafter, we simply use the notation $\zeta_{n,j} \xrightarrow{\mathrm{D}} \zeta_j$ *(stably) to denote the vector convergence* $\{\zeta_{n,j}, j = 1, \dots, J\} \xrightarrow{\mathrm{D}} \{\zeta_j, j = 1, \dots, J\}$ *(stably). For the definition of stability, we refer the reader to [18, p. 56].*

By the delta method, from (4.1) we can prove the following corollary, which is consistent with Theorem 4 of [10] (when $m_j = m_{\max}$) and Theorem 1.4 of [3].

Corollary 4.1. *For fixed* j, *define* $\Omega_j = \{i : m_i = m_j\}$, $Z_{(j)} = \eta_j / \sum_{i\in\Omega_j}\eta_j$,

$$\Sigma_{(j)} = \frac{\eta_j^2}{(\sum_{i\in\Omega_j}\eta_i)^4}\sum_{i\in\Omega_j\setminus\{j\}}\eta_i(1+\sigma_{U,i}^2) + \frac{\eta_j}{(\sum_{i\in\Omega_j}\eta_i)^2}(1-Z_{(j)})^2(1+\sigma_{U,j}^2).$$

Under the condition in Theorem 4.1, we have

$$n^{\delta_j/2}\bigg(\frac{Y_{n,j}}{\sum_{i\in\Omega_j}Y_{n,i}} - Z_{(j)}, \frac{N_{n,j}}{\sum_{i\in\Omega_j}N_{n,i}} - \frac{Y_{n,j}}{\sum_{i\in\Omega_j}Y_{n,i}}\bigg) \xrightarrow{\mathrm{D}} N(0, \Sigma_{(j)})N\bigg(0, \Sigma_{(j)} - \frac{1}{\sum_{i\in\Omega_j}\eta_i}Z_{(j)}(1-Z_{(j)})\bigg) \quad \textit{(stably)}.$$

The proofs of Theorems 4.1 and 4.2 are given in Appendix A.2. From these two theorems, we can also derive the following corollary on the convergence rate and asymptotic distribution for $Y_{n,k}/n^{m_k/m_{\max}} - \psi_k$ and $N_{n,k}/n^{m_k/m_{\max}} - \eta_k$.

Corollary 4.2. *Write* $\delta_{\mathrm{sec}} = m_{\mathrm{sec}}/m_{\max}$. *Under the condition in Theorem 4.1, we have*

$$\frac{Y_{n,k}}{n^{\delta_k}} - \psi_k = o((n^{-\delta_k(1-1/p)}) + O(n^{\delta_{\mathrm{sec}}-1}) \quad a.s.,$$
$$\frac{N_{n,k}}{n^{\delta_k}} - \eta_k = o((n^{-\delta_k(1-1/p)}) + O(n^{\delta_{\mathrm{sec}}-1}) \quad a.s.,$$

for all $k = 1, \dots, K$. *Under the condition in Theorem 4.2, we have*

$$\begin{aligned}(n^{\delta_k/2} \wedge n^{1-\delta_{\mathrm{sec}}})\Big(\frac{Y_{n,k}}{n^{\delta_k}} - \psi_k\Big) & \\ \xrightarrow{\mathrm{D}} \mathbf{1}\big\{\tfrac{1}{2}\delta_k + \delta_{\mathrm{sec}} \le 1\big\} m_k \sqrt{\eta_k}\sqrt{1+\sigma_{U,k}^2} N_{k1}(0,1) & \\ - \mathbf{1}\big\{\tfrac{1}{2}\delta_k + \delta_{\mathrm{sec}} \ge 1\big\} m_k \delta_k \eta_k \sum_{\{i\colon m_i = m_{\mathrm{sec}}\}} \eta_i \quad (stably) &\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}(n^{\delta_k/2} \wedge n^{1-\delta_{\mathrm{sec}}})\Big(\frac{N_{n,k}}{n^{\delta_k}} - \eta_k\Big) & \\ \xrightarrow{\mathrm{D}} \mathbf{1}\big\{\tfrac{1}{2}\delta_k + \delta_{\mathrm{sec}} \le 1\big\} \sqrt{\eta_k}(\sqrt{1+\sigma_{U,k}^2} N_{k1}(0,1) + \sigma_{U,k} N_{k2}(0,1)) & \\ - \mathbf{1}\big\{\tfrac{1}{2}\delta_k + \delta_{\mathrm{sec}} \ge 1\big\} \delta_k \eta_k \sum_{\{i\colon m_i = m_{\mathrm{sec}}\}} \eta_i \quad (stably), &\end{aligned} \tag{4.4}$$

whenever $m_k < m_{\max}$. *In particular, when* $m_{\mathrm{sec}} < m_{\max}/2$, *the asymptotic distributions are mixing normal.*

5. Nonhomogeneous case

When $\xi_{1,k}, \xi_{2,k}, \dots$ are not identically distributed, the mean of $U_{n,k}$ will depend on n. In some practical problems, $U_{n,k}$ may depend on previous assignments and the outcomes of previous trials. For example, the current estimators of the unknown distribution parameters may be used to adjust the model. In such a case, $U_{n,k}$ and its mean are functions of the estimators $\hat{\theta}_i$, $i = 1, \dots, K$, so the means of the replacement are not homogeneous. For the nonhomogeneous cases, we still have the following limit results. Suppose that $\{U_{n,k},\ k = 1, 2, \dots, K\}$ is independent of $\boldsymbol{X}_n$ for given $\boldsymbol{Y}_0, \dots, \boldsymbol{Y}_{n-1}$, and let $m_{n,k} = \mathbb{E}[U_{n,k} \mid \boldsymbol{Y}_0, \dots, \boldsymbol{Y}_{n-1}]$.

Theorem 5.1. *Suppose that* $\sup_n \mathbb{E}[U_{n,k} \log^p U_{n,k} \mid \boldsymbol{Y}_0, \dots, \boldsymbol{Y}_{n-1}] < \infty$ *a.s. for some* $p > 1$. *If*

$$m_{n,k} \to m_k > 0 \quad a.s.,\ k = 1, \dots, K, \tag{5.1}$$

then

$$\log Y_{n,k} \sim \log N_{n,k} \sim \frac{m_k}{m_{\max}} \log n \quad a.s. \tag{5.2}$$

If

$$\sum_n \frac{|m_{n,k} - m_k|}{n} < \infty \quad a.s. \quad and \quad m_k > 0, \qquad k = 1, 2 \dots, K, \tag{5.3}$$

then there exist K *positive random variables* ϖ_i *such that (2.4)–(2.7) hold.*

Remark 5.1. For the nonhomogeneous case, the distribution of ϖ_i is also unknown.

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Theorem 5.2. *Suppose that*

$$\sup_n |m_{n,k}| < \infty \quad a.s., \quad \sum_n \frac{|m_{n,k} - m_k|}{\sqrt{n}} < \infty \quad a.s., \quad m_k > 0, \qquad k = 1, 2 \ldots, K, \tag{5.4}$$

$\sup_n \mathbb{E}[U_{n,k}^2 \mathbf{1}\{U_{n,k} \ge x\} \mid \boldsymbol{Y}_0, \ldots, \boldsymbol{Y}_{n-1}] \to 0$ *in probability as* $x \to \infty$*, and* $\mathbb{E}[(U_{n,k} - m_{n,k})^2 \mid \boldsymbol{Y}_0, \ldots, \boldsymbol{Y}_{n-1}] \to m_k^2 \sigma_{U,k}^2$ *a.s. for some constants* $\sigma_{U,k} \ge 0$*,* $k = 1, \ldots, K$*. Then (4.2), (4.3), and (4.4) hold.*

Remark 5.2. Conditions (5.3) and (5.4) are similar to a condition that Bai and Hu [6], [7] used to study a certain kind of nonhomogeneous generalized Pólya urn model in which the expected total number of balls to be added at each stage is the same. The RRPU is not covered by their assumptions because the expectation of the total number of balls to be added differs from stage to stage.

The proof of Theorem 5.1 is given in Appendix A.3; the proof of Theorem 5.2 is very similar to that of the homogeneous case and is therefore omitted.

Appendix A. Proofs of the main results

For the proofs in this section, some trivial steps are omitted, leaving them to the full online version (see www.sta.cuhk.edu.hk/shcheung/supplementary-material.pdf). We first consider the homogeneous case and prove Theorems 2.2 and 2.3 for the first-order convergence. We then prove Theorems 4.1 and 4.2 for second-order convergence. Finally, we consider the results in Section 5 for nonhomogeneous cases. Now, let us define $\mathscr{F}_n = \sigma(\boldsymbol{Y}_m, \boldsymbol{X}_m, U_{m,k}, m = 1, \ldots, n, k = 1, \ldots, K)$, which is the sigma-field that contains the history of the urn process.

A.1. Proofs of the first-order asymptotic properties

Before the proofs, we need two lemmas. The first lemma can be proved using the same argument as that used to prove Lemma A.4 of [20] (see also [26]).

Lemma A.1. *With a probability of* 1*, on the event* $\{N_{n,k} \to \infty\}$*, we have*

$$Y_{n,k} = \sum_{l=1}^{n} X_{l,k} U_{l,k} \sim N_{n,k} \quad \text{if } \mathbb{E}[U_{1,k}] < \infty,$$

$$\sum_{l=1}^{n} X_{l,k}(U_{l,k} - m_k) = \begin{cases} o\Big(\dfrac{N_{n,k}}{\log N_{n,k}}\Big) & \text{if } \mathbb{E}[U_{1,k} \log U_{1,k}] < \infty, \\ o(N_{n,k}^{1/p}) & \text{if } \mathbb{E}[U_{1,k}^p] < \infty,\ 1 \le p < 2, \\ o(\sqrt{N_{n,k} \log\log N_{n,k}}) & \text{if } \mathbb{E}[U_{1,k}^2] < \infty. \end{cases}$$

The following is the key lemma for proving Theorems 2.2 and 2.3.

Lemma A.2. *Suppose that, for each* k*,* $U_{n,k}$*,* $n = 1, 2 \ldots$*, are independent and identically distributed, nonnegative random variables with* $0 < m_k = \mathbb{E}U_{n,k} < \infty$*. Then*

$$\log Y_{n,k} \sim \frac{m_k}{m_{\max}} \log n \quad a.s., k = 1, \ldots, K, \tag{A.1}$$

and there is a random variable ϖ_k *such that*

$$Y_{n,k} \exp\bigg\{-\sum_{l=1}^{n} \frac{m_k}{|\boldsymbol{Y}_{l-1}|}\bigg\} \to \varpi_k \quad a.s. \tag{A.2}$$

Furthermore, we have

$$\textit{either} \quad \mathbb{P}(\varpi_k > 0) = 0 \quad \textit{or} \quad \mathbb{P}(\varpi_k > 0) = 1 \tag{A.3}$$

and

$$\varpi_k > 0 \quad \textit{a.s.} \quad \Longleftrightarrow \quad \mathbb{E}[U_{1,k} \log U_{1,k}] < \infty. \tag{A.4}$$

Proof. First, it is trivial that $|\boldsymbol{Y}_n| \le \sum_{l=1}^{n} \sum_{k=1}^{K} U_{l,k} = O(n)$; hence,

$$\sum_{n=1}^{\infty} \mathbb{P}(X_{n,k} = 1 \mid \mathcal{F}_{n-1})) \ge c \sum_{n=1}^{\infty} \frac{1}{|\boldsymbol{Y}_{n-1}|} = \infty \quad \text{a.s.},$$

which implies that $\mathbb{P}(X_{n,k} = 1$ infinitely often$) = 1$. Furthermore, $Y_{n,k} \sim m_k N_{n,k} \to \infty$ a.s. by Lemma A.1. Write $q_{n-1} = \sum_{l=1}^{n} 1/|\boldsymbol{Y}_{l-1}|$. It is obvious that

$$\mathbb{E}[Y_{n,k} \mid \mathcal{F}_{n-1}] = Y_{n-1,k}\bigg(1 + \frac{m_k}{|\boldsymbol{Y}_{n-1}|}\bigg) \le Y_{n-1,k} \exp\bigg\{\frac{m_k}{|\boldsymbol{Y}_{n-1}|}\bigg\}.$$

It follows that $Y_{n,k} \exp\{-m_k q_{n-1}\}$ is a nonnegative supermartingale and, hence, it converges a.s. to a finite limit according to the fundamental convergence theorem for supermartingales. Therefore, (A.2) is proved.

If we let $H_k(x) = \mathbb{E}[U_{1,k}^2/(x + U_{1,k})]$, then

$$\begin{aligned} \mathbb{E}\bigg[\frac{Y_{n-1,k}}{Y_{n,k}} \;\bigg|\; \mathcal{F}_{n-1}\bigg] &= \mathbb{E}\bigg[1 - \frac{U_{n,k} X_{n,k}}{Y_{n-1,k}} + \frac{X_{n,k}}{Y_{n-1,k}} \frac{U_{n,k}^2}{Y_{n-1,k} + U_{n-1,k}} \;\bigg|\; \mathcal{F}_{n-1}\bigg] \\ &= 1 - \frac{m_k}{|\boldsymbol{Y}_{n-1}|} + \frac{H_k(Y_{n-1,k})}{|\boldsymbol{Y}_{n-1}|} \\ &\le \exp\bigg\{-\frac{m_k}{|\boldsymbol{Y}_{n-1}|} + \frac{H_k(Y_{n-1,k})}{|\boldsymbol{Y}_{n-1}|}\bigg\}. \end{aligned}$$

It follows that

$$Y_{n,k}^{-1} \exp\bigg\{m_k q_{n-1} - \sum_{l=1}^{n} \frac{H_k(Y_{l-1,k})}{|\boldsymbol{Y}_{n-1}|}\bigg\} \quad \text{converges to a finite limit a.s.} \tag{A.5}$$

because it is also a nonnegative supermartingale. In addition, $H_k(Y_{l-1,k}) \to 0$ a.s. because $\mathbb{E}U_{1,k} < \infty$ and $Y_{l-1,k} \to \infty$ a.s. By combining (A.2) and (A.5) we conclude that

$$\log Y_{n,k} \sim m_k q_{n-1} \quad \text{a.s. } k = 1, \ldots, K. \tag{A.6}$$

From (A.6), it is obvious that $Y_{n,k}/|\boldsymbol{Y}_n| \to 0$ a.s. if $m_k < m_{\max}$. As $Y_{n,k} \sim m_k N_{n,k}$ a.s., we have

$$|\boldsymbol{Y}_n| \sim m_{\max} \sum_{\{i \,:\, m_i = m_{\max}\}} N_{n,i} \sim m_{\max} \sum_{i=1}^{K} N_{n,i} = m_{\max} n \quad \text{a.s.}, \tag{A.7}$$

which together with (A.6) implies that

$$\log Y_{n,k} \sim \frac{m_k}{m_{\max}} \sum_{l=1}^{n} \frac{1}{l} \sim \frac{m_k}{m_{\max}} \log n \quad \text{a.s., } k = 1, \ldots, K.$$

Hence, (A.1) is proved.

Finally, we show (A.3) and (A.4). Assume that $\mathbb{E}[U_{1,k}\log U_{1,k}]<\infty$. Then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{E}\bigg[\frac{U_{1,k}^2}{n^{m_k/(2m_{\max})}+U_{1,k}}\bigg]\\ \le \sum_{n=1}^{\infty}\frac{1}{n}\frac{n^{m_k/(4m_{\max})}\mathbb{E}U_{1,k}}{n^{m_k/(2m_{\max})}}+\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{E}[U_{1,k}\,\mathbf{1}\{U_{1,k}\ge n^{m_k/(4m_{\max})}\}]<\infty,\end{aligned}$$

which together with (A.7) and (A.1) implies that

$$\sum_{n=1}^{\infty}\frac{1}{|\boldsymbol{Y}_{n-1}|}H_k(Y_{n-1,k})<\infty\quad\text{a.s.}\tag{A.8}$$

From (A.8), (A.2), and (A.5), it follows that both $Y_{n,k}\exp\{-m_kq_{n-1}\}$ and $Y_{n,k}^{-1}\exp\{m_kq_{n-1}\}$ have finite limits, and so $\varpi_k>0$ a.s.

Now, suppose that $m_k=\mathbb{E}[U_{1,k}]<\infty$ and $\mathbb{E}[U_{1,k}\log U_{1,k}]=\infty$. We will show that $\varpi_k=0$ a.s. Define $U_{n,k}^{(1)}=U_{n,k}\,\mathbf{1}\{U_{n,k}<n\}$, $Y_{n+1,k}^{(1)}=Y_{n,k}^{(1)}+X_{n+1,k}U_{n+1,k}^{(1)}$ with $Y_{0,k}^{(1)}=Y_{0,k}$, $U_{n,k}^{(2)}=U_{n,k}-U_{n,k}^{(1)}$, and $Y_{n,k}^{(2)}=Y_{n,k}-Y_{n,k}^{(1)}$. Then $Y_{n+1,k}^{(2)}=Y_{n,k}^{(2)}+X_{n+1,k}U_{n+1,k}^{(2)}$ with $Y_{0,k}^{(2)}=0$. Define $m_{n,k}^{(1)}=\mathbb{E}[U_{1,k}\,\mathbf{1}\{U_{1,k}<n\}]$. Then

$$\mathbb{E}[Y_{n,k}^{(1)}\mid\mathcal{F}_{n-1}]=Y_{n-1,k}^{(1)}+\frac{Y_{n-1,k}}{|\boldsymbol{Y}_{n-1}|}m_{n,k}^{(1)}=Y_{n-1,k}^{(1)}\bigg(1+\frac{m_{n,k}^{(1)}}{|\boldsymbol{Y}_{n-1}|}+\frac{m_{n,k}^{(1)}Y_{n-1,k}^{(2)}}{|\boldsymbol{Y}_{n-1}|Y_{n-1,k}^{(1)}}\bigg).$$

Following the same argument as in the proof of (A.2), we find that

$$Y_{n,k}^{(1)}\exp\bigg\{-\sum_{l=1}^{n}\frac{m_{l,k}^{(1)}}{|\boldsymbol{Y}_{l-1}|}-\sum_{l=1}^{n}\frac{m_{l,k}^{(1)}Y_{l-1,k}^{(2)}}{|\boldsymbol{Y}_{l-1}|Y_{l-1,k}^{(1)}}\bigg\}$$

converges to a finite limit a.s. However, as $\mathbb{E}[U_{1,k}]<\infty$, $\sum_{n=1}^{\infty}\mathbb{P}(U_{n,k}\ge n)<\infty$. According to the Borel–Cantelli lemma, $\mathbb{P}(U_{n,k}^{(2)}\neq 0\text{ infinitely often})=0$. It follows that $Y_{n,k}^{(2)}=O(1)$ and $Y_{n,k}=Y_{n,k}^{(1)}+O(1)$ a.s. Hence,

$$\sum_{l=1}^{\infty}\frac{m_{l,k}^{(1)}Y_{l-1,k}^{(2)}}{|\boldsymbol{Y}_{l-1}|Y_{l-1,k}^{(1)}}\le C\sum_{l=1}^{\infty}\frac{m_k}{|\boldsymbol{Y}_{l-1}|Y_{l-1,k}}\le C\sum_{l=1}^{\infty}\frac{1}{l^{1+m_k/(2m_{\max})}}<\infty\quad\text{a.s.}$$

by (A.7) and (A.1). It follows that

$$Y_{n,k}\exp\bigg\{-m_kq_{n-1}+\sum_{l=1}^{n}\frac{m_{l,k}^{(2)}}{|\boldsymbol{Y}_{l-1}|}\bigg\}=Y_{n,k}\exp\bigg\{-\sum_{l=1}^{n}\frac{m_{l,k}^{(1)}}{|\boldsymbol{Y}_{l-1}|}\bigg\}\to\zeta$$

for some $0\le\zeta<\infty$, where $m_{l,k}^{(2)}=\mathbb{E}[U_{1,k}\,\mathbf{1}\{U_{1,k}\ge l\}]$. It can be shown that

$$\begin{aligned}\sum_{l=1}^{\infty}\frac{m_{l,k}^{(2)}}{|\boldsymbol{Y}_{l-1}|}&\ge c\sum_{l=1}^{\infty}\frac{m_{l,k}^{(2)}}{l}\\&\ge c\mathbb{E}\bigg[\int_{\mathrm{e}}^{\infty}\frac{U_{1,k}\,\mathbf{1}\{U_{1,k}\ge x\}}{x}\,\mathrm{d}x\bigg]\\&=c\mathbb{E}[U_{1,k}(\log U_{1,k}-1)]\\&=\infty\quad\text{a.s.}\end{aligned}$$

Hence, $Y_{n,k}\exp\{-m_kq_{n-1}\}\to 0$ a.s. This completes the proof.

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Proof of Theorem 2.2. Equation (2.4) follows immediately from (A.2) and (A.4). By noting that $Y_{n,k} = \sum_{m=1}^{n} X_{m,k} U_{m,k} \sim m_k N_{n,k}$ a.s. due to Lemma A.1, (2.5) is also proven. To prove (2.6) and (2.7), without loss of generality, we suppose that $m_1 = m_2 = \cdots = m_{k_0} = m_{\max} > m_k > 0,\ k = k_0 + 1, \ldots, K$. Owing to (2.5),

$$\frac{N_{n,k}}{N_{n,1}} \to \frac{\varpi_k}{\varpi_1} \quad \text{a.s., } k \le k_0 \quad \text{and} \quad \frac{N_{n,k}}{N_{n,1}} \to 0 \quad \text{a.s., } k > k_0 + 1.$$

Note that $N_{n,1} + \cdots + N_{n,K} = n$. It follows that

$$\frac{N_{n,k}}{n} \to 0 \quad \text{a.s., } k > k_0 + 1 \quad \text{and} \quad \frac{N_{n,k}}{n} \to \frac{\varpi_k}{\varpi_1 + \cdots + \varpi_{k_0}} \quad \text{a.s., } k \le k_0,$$

which, together with (2.5), imply (2.7). Finally, (2.6) follows from (2.7) because $Y_{n,k} \sim m_k N_{n,k}$ a.s.

Proof of Theorem 2.3. Note that $\log Y_{n,k} \sim m_k / m_{\max} \log n$ a.s. due to Lemma A.2. Hence, if $Y_{n,k}/n^{\delta_k}$ converges in distribution to a finite limit ϖ_k^* with $\mathbb{P}(\varpi_k^* > 0) > 0$, then $\delta_k = m_k / m_{\max}$. Whereas if $N_{n,k}/n^{\delta_k}$ converges in distribution to a finite limit φ_k^*, then $Y_{n,k}/n^{\delta_k}$ converges in distribution to $m_k \varphi_k^*$ by the fact that $Y_{n,k} \sim m_k N_{n,k}$ a.s. The first part of the theorem is proven.

Now, suppose that

$$\frac{Y_{n,k}}{n^{m_k/m_{\max}}} \xrightarrow{\text{D}} \varpi_k^* \quad \text{with} \quad \mathbb{P}(\varpi_k^* > 0) > 0. \tag{A.9}$$

By (A.2), we have

$$\sum_{\{j:\, m_j = m_{\max}\}} Y_{n,j} \exp\bigg\{ -\sum_{l=1}^{n} \frac{m_{\max}}{|\boldsymbol{Y}_{l-1}|} \bigg\} \to \sum_{\{j:\, m_j = m_{\max}\}} \varpi_k \quad \text{a.s.}$$

and $\sum_{\{j:\, m_j = m_{\max}\}} Y_{n,j} \sim m_{\max} \sum_{\{j:\, m_j = m_{\max}\}} N_{n,j} \sim m_{\max} n$ a.s. We conclude that

$$n^{1/m_{\max}} \exp\bigg\{ -\sum_{l=1}^{n} \frac{1}{|\boldsymbol{Y}_{l-1}|} \bigg\} \to \bigg(\frac{\sum_{\{j:\, m_j = m_{\max}\}} \varpi_k}{m_{\max}} \bigg)^{1/m_{\max}} := \widetilde{\varpi}^* \quad \text{a.s.,} \tag{A.10}$$

where $\mathbb{P}(\widetilde{\varpi}^* > 0) = 0$ or $\mathbb{P}(\widetilde{\varpi}^* > 0) = 1$ by (A.3).

If $\mathbb{P}(\widetilde{\varpi}^* > 0) = 0$ then, by (A.9) and (A.10), we have $Y_{n,k} \exp\{-\sum_{l=1}^{n} m_k / |\boldsymbol{Y}_{l-1}|\} \xrightarrow{\text{D}} 0$. It follows that $\mathbb{P}(\varpi_k > 0) = 0$ by (A.2).

If $\mathbb{P}(\widetilde{\varpi}^* > 0) = 1$ then (A.10) and (A.2) imply that $Y_{n,k}/n^{m_k/m_{\max}} \to \varpi_k / (\widetilde{\varpi}^*)^{m_k}$ a.s. It follows that $\mathbb{P}(\varpi_k > 0) = \mathbb{P}(\varpi_k^* > 0) > 0$ by (A.9). Hence, $\mathbb{P}(\varpi_k > 0) = 1$ by (A.3). We conclude that if one of the $\varpi_k,\ k = 1, \ldots, K$, is positive, all of them are positive, while, if one of $\varpi_k,\ k = 1, \ldots, K$, is 0, all of them are 0. By (A.4), the proof is complete.

A.2. Proofs of the second-order asymptotic properties

To prove the second-order convergence, we need the following central limit theorem for martingale vectors which is a multidimensional version of Corollary 3.1 of [18, p. 58] and can be obtained using the Cramér-Wold device (cf. Lemma A.3 of [25]).

Lemma A.3. *Let* $\{\boldsymbol{\zeta}_{n,i} = (\zeta_{n,i}^{(1)}, \dots, \zeta_{n,i}^{(K)}), \mathscr{A}_{n,0}, \mathscr{A}_{n,i}; 1 \le i \le k_n\}$ *be an array of martingale differences with* $\mathscr{A}_{n,i} \subset \mathscr{A}_{n+1,i}$, $0 \le i \le k_n$ *and* $n \ge 1$,

$$\sum_i \mathbb{E}[\|\boldsymbol{\zeta}_{n,i}\|^2 \mathbf{1}\{\|\boldsymbol{\zeta}_{n,i}\| \ge \varepsilon\} \mid \mathscr{A}_{n,i-1}] \xrightarrow{\mathbb{P}} 0 \quad \text{for all } \varepsilon > 0,$$

$$\boldsymbol{V}_n =: \sum_i \mathbb{E}[(\boldsymbol{\zeta}_{n,i})' \boldsymbol{\zeta}_{n,i} \mid \mathscr{A}_{n,i-1}] \xrightarrow{\mathbb{P}} \boldsymbol{V} := (V_{ij}).$$

Then $\sum_{i=1}^{k_n} \boldsymbol{\zeta}_{n,i} \xrightarrow{\mathrm{D}} N(\mathbf{0}, \boldsymbol{V})$ *stably, where* $N(\mathbf{0}, \boldsymbol{V})$ *is a multidimensional mixing normal distribution with the characteristic function* $\mathbb{E}[\exp\{-\frac{1}{2}\sum_{i,j} t_i t_j V_{ij}\}]$.

Proof of Theorem 4.1. Recall that $\delta_k = m_k/m_{\max}$. Let

$$q_{n-1} = \sum_{l=1}^{n} \frac{1}{|\boldsymbol{Y}_{l-1}|} \quad \text{and} \quad Q_{n,k} = \frac{1}{m_k} \log Y_{n,k} - q_{n-1}.$$

In addition, $N_{n,k} \approx Y_{n,k} \approx n^{\delta_k}$ a.s. due to Theorem 2.2. Also,

$$\begin{aligned}\frac{1}{m_k}\log(m_k N_{n,k}) - \frac{1}{m_k}\log(Y_{n,k}) &= -\frac{1}{m_k}\log\left(1 + \frac{\sum_{l=1}^{n} X_{l,k}(U_{l,k} - m_k)}{m_k N_{n,k}}\right) \\ &= o(N_{n,k}^{1/p-1}) \\ &= o(Y_{n-1,k}^{1/p-1}) \quad \text{a.s.}\end{aligned}$$

due to Lemma A.1. So, according to the Taylor expansion, it is sufficient to prove that

$$Q_{n,k} - \log \varpi_k = o(Y_{n-1,k}^{1/p-1}) \quad \text{a.s.}$$

Now, we let $U_{n,k}^{(\delta_k)} = U_{n,k} \mathbf{1}\{U_{n,k} \le n^{\delta_k/p}\}$, $\bar{U}_{n,k}^{(\delta_k)} = U_{n,k} - U_{n,k}^{(\delta_k)}$, and $f(x) = x - \log(1 + x)$. Then $0 \le f(x) \le x^2/(1+x)$ $(x \ge 0)$, $Q_{n,k} - \log \varpi_k = -\sum_{l=n+1}^{\infty} \Delta Q_{l,k}$, and

$$\begin{aligned}\Delta Q_{n,k} =: Q_{n,k} - Q_{n-1,k} &= \frac{1}{m_k} X_{n,k} \log\left(1 + \frac{U_{n,k}}{Y_{n-1,k}}\right) - \frac{1}{|\boldsymbol{Y}_{n-1}|} \\ &= \frac{1}{m_k}\left[\frac{U_{n,k}}{Y_{n-1,k}} X_{n,k} - \frac{m_k}{|\boldsymbol{Y}_{n-1}|} - X_{n,k} f\left(\frac{U_{n,k}}{Y_{n-1,k}}\right)\right] \\ &= \frac{1}{m_k}\left[\left(\frac{U_{n,k}^{(\delta_k)}}{Y_{n-1,k}} X_{n,k} - \frac{\mathbb{E}[U_{n,k}^{(\delta_k)}]}{|\boldsymbol{Y}_{n-1}|}\right) + \left(\frac{\bar{U}_{n,k}^{(\delta_k)}}{Y_{n-1,k}} X_{n,k} - \frac{\mathbb{E}[\bar{U}_{n,k}^{(\delta_k)}]}{|\boldsymbol{Y}_{n-1}|}\right) \right. \\ &\qquad \left. - X_{n,k} f\left(\frac{U_{n,k}}{Y_{n-1,k}}\right)\right] \\ &:= \frac{\Delta Q_{n,k}^{(11)} + \Delta Q_{n,k}^{(12)} - \Delta Q_{n,k}^{(2)}}{m_k}.\end{aligned}$$

It is obvious that

$$\begin{aligned}\sum_{l=1}^{\infty} Y_{l-1,k}^{1-1/p} \mathbb{E}[|\Delta Q_{l,k}^{(2)}| \mid \mathscr{F}_{l-1}] &\le \sum_{l=1}^{\infty} \frac{Y_{l-1,k}^{1-1/p}}{|\boldsymbol{Y}_{l-1}|} \mathbb{E}\left[\frac{U_{l,k}^2}{Y_{l-1,k} + U_{l,k}} \,\middle|\, \mathscr{F}_{l-1}\right] \\ &\le \sum_{l=1}^{\infty} \frac{Y_{l-1,k}^{1-1/p}}{|\boldsymbol{Y}_{l-1}|} \frac{\mathbb{E}[U_{l,k}^p]}{Y_{l-1,k}^{p-1}}\end{aligned}$$

$$\leq C \sum_{l=1}^{\infty} l^{-1-\delta_k(p+1/p-2)}$$
$$< \infty \quad \text{a.s.} \tag{A.11}$$

It follows that $\sum_{l=1}^{\infty} Y_{l-1,k}^{1-1/p} |\Delta Q_{l,k}^{(2)}| < \infty$ a.s., and, hence,

$$\sum_{l=n+1}^{\infty} |\Delta Q_{l,k}^{(2)}| = o(Y_{n-1,k}^{1/p-1}) \quad \text{a.s.} \tag{A.12}$$

On the other hand, it can be shown that $\{\Delta Q_{n,k}^{(11)}\}$ and $\{\Delta Q_{n,k}^{(12)}\}$ are both martingale differences with

$$\sum_{l=1}^{\infty} Y_{l-1,k}^{(1-1/p)} \mathbb{E}[|\Delta Q_{l,k}^{(12)}| \mid \mathcal{F}_{l-1}] < \infty, \qquad \sum_{l=1}^{\infty} Y_{l-1,k}^{2(1-1/p)} \mathbb{E}[|\Delta Q_{l,k}^{(11)}|^2 \mid \mathcal{F}_{l-1}] < \infty.$$

It follows that $\sum_{l=1}^{\infty} Y_{l-1}^{(1-1/p)} \Delta Q_{l,k}^{(1i)}$, $i = 1, 2$, converges a.s., and, hence,

$$\sum_{l=n+1}^{\infty} (\Delta Q_{l,k}^{(11)} + \Delta Q_{l,k}^{(12)}) = o(Y_{n-1}^{1/p-1}) \quad \text{a.s.}$$

Therefore, $\log \varpi_k - Q_{n,k} = \sum_{l=n+1}^{\infty} \Delta Q_{l,k} = o(Y_{n-1}^{1/p-1})$ a.s. This completes the proof.

Proof of Theorem 4.2. Let δ_k, q_n, and $Q_{n,k}$ be defined as in the proof of Theorem 4.1. Without loss of generality, we assume that $m_1 = \cdots = m_{k_0} > m_k$, $k = k_0+1, \dots, K$. Define $I_k = \mathbf{1}\{m_k = m_{\max}\}$. Let $N_0(0, 1)$ be a standard normal variable which is independent of all other variables, and let $\varsigma = \sum_{k=1}^{k_0} \sqrt{\eta_k/(1+\sigma_{U,k}^2)} N_{k1}(0, 1) + \sqrt{1 - \sum_{k=1}^{k_0} \eta_k/(1+\sigma_{U,k}^2)} N_0(0, 1)$. Then ς is a standard normal variable such that $\mathbb{E}[\varsigma N_{k1}(0, 1) \mid \eta_k] = \sqrt{\eta_k/(1+\sigma_{U,k}^2)}$ for a given η_k, $k = 1, \dots, k_0$.

According to the delta method, it is sufficient to show that

$$\sqrt{n^{\delta_k}}(Q_{n,k} - \log \varpi_k) \xrightarrow{\text{D}} \frac{1}{m_k\sqrt{\eta_k}} \sqrt{1+\sigma_{U,k}^2} N_{k1}(0, 1) - \frac{\varsigma I_k}{m_k} \quad \text{(stably)},$$
$$\sqrt{n^{\delta_k}}\big(\log(m_k N_{n,k}) - q_{n-1} - Q_{n,k}\big) \xrightarrow{\text{D}} \frac{1}{m_k\sqrt{\eta_k}} \sigma_{U,k} N_{k2}(0, 1) \quad \text{(stably)},$$

for $k = 1, \dots, K$. Note that (A.12) and (A.11) also hold for $p = 2$. It follows that

$$Q_{n,k} - \log \varpi_k = -\sum_{l=n+1}^{\infty} \frac{1}{m_k} \Delta Q_{l,k}^{(1)} + o(n^{-\delta_k/2}) \quad \text{a.s.},$$

where $\Delta Q_{n,k}^{(1)} = \Delta Q_{n,k}^{(11)} + \Delta Q_{n,k}^{(12)} = X_{n,k}U_{n,k}/Y_{n-1,k} - m_k/|\mathbf{Y}_{n-1}|$, and

$$\frac{1}{m_k}\log(m_k N_{n,k}) - q_{n-1} = Q_{n,k} - \frac{1}{m_k}\log\bigg(1 + \frac{\sum_{l=1}^{n} X_{l,k}(U_{l,k} - m_k)}{m_k N_{n,k}}\bigg)$$
$$= Q_{n,k} - \frac{1}{m_k} \frac{\sum_{l=1}^{n} X_{l,k}(U_{l,k}/m_k - 1)}{\eta_k n^{\delta_k}} + o(n^{-\delta_k/2}) \quad \text{a.s.}$$

Hence, it is sufficient to show that

$$\frac{\sum_{l=1}^{n} X_{l,k}(U_{l,k}-m_k)}{\sqrt{n^{\delta_k}}} \xrightarrow{D} \sqrt{\eta_k} m_k \sigma_{U,k} N_{k2}(0,1) \quad \text{(stably)}, \tag{A.13}$$

$$\sqrt{n^{\delta_k}} \sum_{l=n+1}^{\infty} \Delta Q_{l,k}^{(1)} \xrightarrow{D} \frac{1}{\sqrt{\eta_k}} \sqrt{1+\sigma_{U,k}^2} N_{k1}(0,1) - \varsigma I_k \quad \text{(stably)}. \tag{A.14}$$

It can be checked that

$$n^{\delta_k} \sum_{l=n+1}^{\infty} \mathbb{E}[(\Delta Q_{n,k}^{(1)})^2 \mid \mathcal{F}_{l-1}] \to \begin{cases} \dfrac{\sigma_{U,k}^2+1}{\eta_k} & \text{if } m_k \neq m_{\max}, \\ \dfrac{\sigma_{U,k}^2+1}{\eta_k} - 1 & \text{if } m_k = m_{\max}, \end{cases} \quad \text{a.s.},$$

$$\sqrt{n^{\delta_k} n^{\delta_j}} \sum_{l=n+1}^{\infty} \mathbb{E}[(\Delta Q_{n,k}^{(1)})(\Delta Q_{n,j}^{(1)}) \mid \mathcal{F}_{l-1}] \to \begin{cases} -1 & \text{if } m_k = m_j = m_{\max}, \\ 0 & \text{otherwise}, \end{cases} \quad \text{a.s.},$$

for $k \neq j$, and

$$\sum_{l=n+1}^{\infty} \mathbb{E}[(\sqrt{n^{\delta_k}} \Delta Q_{l,k}^{(1)})^2 \mathbf{1}\{(\sqrt{n^{\delta_k}} \Delta Q_{l,k}^{(1)})^2 \geq \varepsilon\} \mid \mathcal{F}_{l-1}] \to 0 \quad \text{a.s.}$$

On the other hand, it is obvious that the martingales $\sum_{l=1}^{n} X_{l,k}(U_{l,k}-m_k)$, $k=1,\ldots,K$, are uncorrelated among themselves, and also uncorrelated with all $\sum_{l=n+1}^{\infty} \Delta Q_{l,j}^{(1)}$, $j=1,\ldots,K$. Furthermore,

$$n^{-\delta_k} \sum_{l=1}^{n} \mathbb{E}[X_{l,k}(U_{l,k}-m_k)^2 \mid \mathcal{F}_{l-1}] \to \eta_k m_k^2 \sigma_{U,k}^2 \quad \text{a.s.},$$

$$n^{-\delta_k} \sum_{l=1}^{n} \mathbb{E}[X_{l,k}(U_{l,k}-m_k)^2 \mathbf{1}\{(U_{l,k}-m_k)^2 \geq \varepsilon n^{\delta_k}\} \mid \mathcal{F}_{l-1}] \to 0 \quad \text{a.s.}$$

Then, by Lemma A.3, (A.13) and (A.14) hold.

A.3. Proofs for the nonhomogeneous case

To prove Theorem 5.1, we need the following lemma. Since its proof utilizes similar arguments as those given in Lemma A.2, it is only given in the online supplement.

Lemma A.4. *Suppose that* $\sup_n \mathbb{E}[U_{n,k} \log^p U_{n,k} \mid \mathbf{Y}_0,\ldots,\mathbf{Y}_{n-1}] < \infty$ *a.s. for some* $p > 1$. *Under (5.1) or (5.3), we have* $N_{n,k} \to \infty$ *a.s.,* $Y_{n,k} \sim m_k N_{n,k}$ *a.s.,*

$$\min_k m_k \leq \liminf_{n\to\infty} \frac{|\mathbf{Y}_n|}{n} \leq \limsup_{n\to\infty} \frac{|\mathbf{Y}_n|}{n} \leq \max_k m_k \quad \textit{a.s.}, \tag{A.15}$$

and

$$Y_{n,k} \exp\left\{-\sum_{l=1}^{n} \frac{m_{l,k}}{|\mathbf{Y}_{l-1}|}\right\} \quad \textit{converges a.s. to a positive finite limit.} \tag{A.16}$$

Proof of Theorem 5.1. If (5.1) is satisfied then

$$\log Y_{n,k} \sim \sum_{l=1}^{n} \frac{m_{l,k}}{|\boldsymbol{Y}_{l-1}|} \sim \sum_{l=1}^{n} \frac{m_k}{|\boldsymbol{Y}_{l-1}|} \quad \text{a.s.} \tag{A.17}$$

Hence, (A.6) remains true, which together with $Y_{n,k} \sim m_k N_{n,k}$, implies (A.1). So (5.2) is proven. Finally, (5.3) and the fact that $|\boldsymbol{Y}_n| \approx n$ imply that $\sum_{l=1}^{\infty} |m_{l,k} - m_k|/|\boldsymbol{Y}_{l-1}| < \infty$ a.s. It follows that $Y_{n,k} \exp\{-\sum_{l=1}^{n} m_k/|\boldsymbol{Y}_{l-1}|\}$ converges a.s. to a positive finite limit by (A.16). Then (2.4)–(2.7) follow by the same argument as used in the proof of Theorem 2.2.

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