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SPECTRAL THEORY FOR THE NEUMANN LAPLACIAN ON PLANAR DOMAINS WITH HORN-LIKE ENDS

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ABSTRACT. The spectral theory for the Neumann Laplacian on planar domains with symmetric, horn-like ends is studied. For a large class of such domains, it is proven that the Neumann Laplacian has no singular continuous spectrum, and that the pure point spectrum consists of eigenvalues of finite multiplicity which can accumulate only at 0 or ∞. The proof uses Mourre theory.

1. **Introduction.** Given an unbounded domain Ω in R^2 obeying the segment condition ([11]), the Neumann Laplacian is defined as the unique self-adjoint operator whose quadratic form q is given by

$$q(u,u) = \int_\Omega |\bigtriangledown u|^2$$

on the domain $\{u \in L^2(\Omega) | \bigtriangledown u \in L^2(\Omega)\}$. One can show that the Neumann Laplacian is a differential operator with expression

$$-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2},$$

and with domain

$$\Big\{u \in L^2(\Omega) \Big| \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \in L^2(\Omega), \frac{\partial u}{\partial \eta}\Big|_{\partial\Omega} = 0\Big\}.$$

Here $\partial/\partial\eta$ denotes the unit outward normal derivative.

Let Ω be a connected planar domain which obeys the segment condition and assume Ω has the following form:

$$\Omega = \{(x,y), x \geq 0, |y| < f(x)\} \cup \kappa, \tag{1}$$

where κ is a domain with compact closure.

Denote the j-th order derivative of f by $f^{(j)}$. Then $f(x)$ will be assumed to satisfy the following conditions:

$$f > 0,\ f^{(1)} < 0; \tag{2}$$

$$f = O(x^{-1}),\ f^{(1)} = O(x^{-2}),\ f^{(2)} = O(x^{-2}),\ \text{and } f^{(3)},\ f^{(4)},\ f^{(5)}, f^{(6)} \text{ all bounded;} \tag{3}$$

$$(f^{(1)}/f)^{(j)} = O(x^{-1/2-j}), \quad j = 0,1,2,3,4; \tag{4}$$

$$(f/f^{(1)})^{(j)} \text{ bounded}, j = 1,2,3,4; \tag{5}$$

$$(f^{(1)})^2/f = O(x^{-2}). \tag{6}$$

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The purpose of this paper is to prove the following:

THEOREM 1. *Suppose* Ω *is a planar domain obeying the segment condition and of the form given by Equation 1, with f(x) satisfying Equations 2–5 above. Then the spectrum of the Neumann Laplacian is the set* $[0,\infty)$. *Furthermore,*

1) There is no singular continuous spectrum.

2) The pure point spectrum consists of embedded eigenvalues of finite multiplicity, which can accumulate only at 0 and ∞.

That the assumption that Ω satisfies the segment condition is necessary is made clear by the examples presented in [7].

Although the conditions 2–5 on the function f are restrictive, they are satisfied if $f = (x+1)^{-p}$, $p \geq 1$, or if f is of the form $f(x) = \exp(-x^{\alpha})$, $\alpha \in (0, 1/2]$. It should also be remarked that the proof of the theorem is adaptable to a more general class of functions, but we assume Equations 2–5 for simplicity of the proof.

Theorem 1 in some ways extends the following results due to Davis-Simon [3] and Jaksic [8]:

THEOREM 2. *Suppose* Ω *has the form*

$$\Omega = \{(x,y), x > 0, |y| < f(x)\}.$$

Let $\epsilon > 0$, *and let*

$$V = \frac{1}{4}\Big(\frac{f^{(1)}}{f}\Big)^2 + \frac{1}{2}\Big(\frac{f^{(1)}}{f}\Big)^{(1)}, \quad k(x) = |f^{(1)}(x)| + \frac{(f^{(1)})^2}{f(x)}.$$

[3]: Suppose $V = O(x^{-1-\epsilon})$ *and* $k = O(x^{-1-\epsilon})$. *Then the conclusions of Theorem 1 hold.*

[8]: Suppose $V^{(1)} = O(x^{-1-\epsilon})$ *and* $k = O(x^{-1-\epsilon})$, *and suppose further that V is dilation-analytic. Then the conclusions of Theorem 1 hold.*

(Actually in [3] and [8] it is further stated that the eigenvalues cannot accumulate at zero, but their argument does not actually prove this. See in [3]: p. 115, line 9. The claim "we can make $\|g(H)u_n - u_n\|$ uniformly small for all n" will be false if u_n is a sequence of normalised eigenfunctions corresponding to eigenvalues converging to zero because g is supported away from zero).

We remark that [3] contains a number of other results on the spectrum of the Neumann Laplacian on domains with horn-like ends. Also see [1], [5], [9].

The domains in Theorem 2 are called by its authors "horn-like domains", and thus the domains studied in this paper will be called domains with horn-like ends.

The basic idea behind the proof of Theorem 2 is the following. Let S be the set of functions in Ω dependent on x alone. It is proved that resolvent of Neumann Laplacian, restricted to the orthogonal complement of S, is compact. To study the resolvent restricted to S the authors proceed as follows. Consider the inclusion operator

$$J\colon L^2\big((0,\infty), 2f(x)dx\big) \longrightarrow L^2(\Omega, dx\,dy)$$

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given by

$$Ju(x,y) = u(x).$$

This inclusion is a unitary operator onto S. The quadratic form associated to the Neumann Laplacian restricted to S induces, via J, the quadratic form

$$q_1(u,u) = 2\int_{x=0}^{\infty} |u_x|^2 f(x)\,dx - \frac{f'(0)}{\sqrt{2}f'(0)}|u(0)|^2$$

on $L^2\big((0,\infty), 2f(x)dx\big)$. Let H_1 be the operator associated to q_1, and let Δ be the Neumann Laplacian on Ω. Then it is shown that the operator

$$(\Delta + z)^{-1}J - J(H_1 + z)^{-1}$$

is compact for $z \notin [0,\infty)$. The operator H_1 is then shown to be unitarily equivalent to the Schrodinger operator on the $L^2\big((0,\infty), dx\big)$:

$$u(t) \longrightarrow -u''(t) + V(t)u(t) \quad \text{for t>0, and } u'(0) = 0,$$

with V is the potential given in Theorem 2. Theorem 2 is then proven by applying the Enss Theory.

The methods used to prove Theorem 2 cannot be extended to domains with non-trivial compact part κ (although it should be possible to extend the results of Theorem 2 to such domains by methods other than those exhibited in this paper). Theorem 1 is also interesting because it applies to some horn-like domains not covered by [3], [8]. For instance, non-analytic perturbations of $f(x) = \exp(-x^{1/2})$ will be covered by Theorem 1, provided the derivatives of f satisfy Equations 2–5. However, we believe the main interest of this paper are the methods used. It is possible, for instance, to extend the methods of this paper to domains with ends having positive thickness at infinity. This will be done in a companion paper ([4]).

The proof of Theorem 1 is structured along the lines of the proof of an analogous result by Froese and Hislop in [6]. In that work, the spectrum of the Laplace-Beltrami operator Δ on boundaryless manifolds with ends is studied. The ends are diffeomorphic $N \times \mathbf{R}^+$, where N is a compact manifold without boundary, and $\mathbf{R}^+$ denotes the strictly positive reals. The metric is assumed to be such that, on the end, the Laplace-Beltrami operator is a perturbation of

$$\Delta_0 = -\frac{\partial^2}{\partial r^2} - h^2(r)\Delta_N,$$

with Δ_N the Laplace-Beltrami operator on N induced by the restriction of the metric to N, and r the unit parametrization of $\mathbf{R}^+$. The end pinching at infinity is equivalent to $h(r) \longrightarrow \infty$ as $r \longrightarrow \infty$ (the authors also consider the cases $h \sim$ const. and $h \longrightarrow 0$). Assuming that the coefficients of $\Delta - \Delta_0$ are of order $O(r^{-2})$ in the pinching case, the conclusions of Theorem 1 are proven to hold for the Laplace-Beltrami operator.

The proofs of Theorem 1 and the analogous result in [6] follow from applying Mourre theory to the operators in question. For a background on Mourre theory, see [10], [2]. We shall simply state the hypotheses for the Mourre theory, and the conclusion that follows.

We denote the domain of an operator A by $\mathrm{Dom}(A)$. The Mourre hypotheses presuppose the existence of three of operators H, H_0, and A, with H, H_0 self-adjoint and A skew-adjoint. We define a scale of spaces associated to H as follows. For $s \geq 0$ let $W^s = \mathrm{Dom}\big((1+|H|)^{s/2}\big)$, with norm

$$\|u\|_s = \|(1+|H|)^{s/2}u\|_{L^2}.$$

HYPOTHESIS 1. $\mathrm{Dom}(A) \cap W^2$ is dense in W^2.

HYPOTHESIS 2. The form $[H, A]$, defined on $\mathrm{Dom}(A) \cap W^2$, extends to a bounded operator from W^2 to W^{-1}.

HYPOTHESIS 3. $\mathrm{Dom}(H_0) = \mathrm{Dom}(H)$, the form $[H_0, A]$ extends to a bounded map from W^2 to W^0, and $\mathrm{Dom}(A) \cap \mathrm{Dom}(H_0 A)$ is a core for H_0.

HYPOTHESIS 4. The form $\big[[H, A], A\big]$, where $[H, A]$ is as in Hypothesis 2, extends from $W^2 \cap \mathrm{Dom}(HA)$ to a bounded operator from W^2 to W^{-2}.

The key estimate is the following. Given an interval I, let E_I be the spectral projection for H associated with the interval I.

HYPOTHESIS 5. Suppose there exist a number α, $\alpha > 0$, and a compact operator K such that the following quadratic form inequality holds:

$$E_I[H, A]E_I \geq \alpha E_I + K.$$

THEOREM 3. *Suppose the operators H, H_0, and A satisfy hypotheses 1–5. Then H has finitely many eigenvalues, of finite multiplicity, in I. Furthermore H has no singular continuous spectrum in I.*

In this paper we will prove

PROPOSITION 1. *There exists an operator H, unitarily equivalent to the Neumann Laplacian, and there exist operators H_0 and A such that*

i) Mourre Hypotheses 1–4 hold,

ii) for every real number $z > 0$, there exists an interval I containing z such that Hypothesis 5 holds.

Theorem 1 follows from this proposition, along with the observation that the dimension of the zero-eigenspace for the Neumann Laplacian can be at most one dimensional. Part ii) of the proposition fails at $z = 0$, and this explains why this paper does not exclude the possibility of eigenvalues accumulating at 0.

To define the appropriate spaces and operators for our case we proceed as follows.

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We first construct coordinates r, s on the non-compact part of Ω, so that the non-compact part of Ω consists of the strip

$$\{(r,s); r \in [0,\infty), s \in (-1,1)\}.$$

We will define r so that near infinity, x is sufficiently close to r that in the estimates in Equations 3–5, x can be replaced by r.

The Euclidean metric, in the coordinates r, s, can be written as $\alpha^2 dr^2 + \beta^2 ds^2$, with the functions α, β satisfying

$$\alpha(r,s) \sim 1, \ \beta(r,s) \sim f(r), \quad \text{as } r \longrightarrow \infty.$$

(The precise order of the asymptotics will be calculated later.) The Neumann boundary conditions in the r, s coordinates are

$$\frac{\partial u}{\partial s}\Big|_{s=\pm 1} = 0.$$

Note that the measure associated to the metric above is $\alpha\beta\, dr\, ds$. If ω is a positive, smooth function defined by

$$\omega^{-1} = \begin{cases} 1 & \text{on } \{r < 0\} \cup \kappa \\ \alpha\beta & \text{for } r > 1 \end{cases}$$

then one can define a unitary transformation from $L^2(\Omega, dx\, dy)$ to $L^2(\Omega, \omega dx\, dy)$ by $Uv = (\omega)^{-1/2} v$. The main reason this transformation is useful is that for $r > 1$,

$$\omega dx\, dy = dr\, ds.$$

The operator H is now defined as the Neumann Laplacian transformed under U. One calculates that H, as a differential operator, has the following expression for $r > 1$:

$$-\frac{1}{\alpha^2} D_r^2 - \frac{1}{\beta^2} D_s^2 + \text{ lower order terms.}$$

Here D_r^j denotes $\partial^j / \partial r^j$, *etc*.

The boundary conditions transform under U to

$$\partial u / \partial \eta = 0 \quad \text{on } \overline{\kappa \cup \{r < 0\}} \cap \partial\Omega$$

$$(au + u_s)|_{s=\pm 1} = 0, \quad r > 0. \tag{7}$$

Here $a = a(r,s)$ is a function whose formula at $s = \pm 1$ we give explicitly in Equation 15. Because f satisfies Equations 3–5, the function a can be shown to vanish as $r \longrightarrow \infty$. Thus the boundary conditions in Equation 7 can be viewed as asymptotically Neumann.

The operator H_0 is defined to be a perturbation of H. We define H_0 to coincide with H as a differential operator on $\kappa \cup \{r < N\}$, with N a large number. For $r > N + 1$, H_0 is given by

$$H_0 = -D_r^2 - f(r)^{-2} D_s^2.$$

For $N < r < N + 1$ the coefficients of H_0 will be determined so that H_0 is an elliptic differential operator, whose closure as an operator in $L^2(\Omega, \omega dx\, dy)$ under the boundary conditions of Equation 7 is self-adjoint. It is possible to choose N and H_0 so that furthermore, H and H_0 are mutually relatively bounded and the difference of their resolvents is compact.

Because the resolvents of H and H_0 differ by a compact operator, it follows from a generalisation of Weyl's theorem that the essential spectra of H and H_0 coincide. The essential spectrum of H_0 is then proven to be $[0, \infty)$ by constructing a Weyl sequence for an arbitrary positive real number. The first part of Theorem 1 then follows.

The Sobolev spaces W^s are defined as those induced by H_0. Since H and H_0 are relatively bounded, this scale of spaces is equivalent to the scale of spaces induced by H.

The operator A is defined as follows. Consider the Sturm-Liouville problem

$$u''(s) + \lambda u(s) = 0, \quad au(-1) + u_s(-1) = au(1) + u_s(1) = 0, \tag{8}$$

with a as in Equation 7. Define P to be the orthogonal projection of $L^2\big((-1, 1), ds\big)$ onto the eigenspace associated with the unique, smallest eigenvalue of Equation 8. Let χ be a cutoff function which localises to a neighbourhood of infinity. Define the operator A by

$$A = D_r P r\chi + \chi r P D_r.$$

Formally, this is the same operator as used by Froese and Hislop to prove Mourre estimates in [6]. The proof of the Mourre estimates now in fact proceeds along the lines of [6], except for four significant complications.

The first complication is that unlike in [6], the operator P given here is r-dependent (since the boundary conditions in Equation 8 are r-dependent). Thus the operators D_r^j (j a positive integer) and P do not commute. It will be proven that the coefficients associated with the commutators vanish sufficiently rapidly that the commutators make negligible contributions to the bounds found in Mourre Hypotheses.

The second complication is that smallest eigenvalue of Equation 8 is not necessarily non-negative, unlike in [6]. Because of this it is not clear that the operator

$$u \longrightarrow f(r)^{-2} D_s^2 u, \quad r > 1$$

(and by consequence the operator H_0) is semibounded. We will show that the smallest eigenvalue of Equation 8 vanishes sufficiently rapidly to ensure the semi-boundedness of H_0.

The third complication is that the presence of the boundary in our case will make it harder to prove the relative boundedness of the operators H and H_0, because the boundary terms that arise in integration by parts must be estimated in terms of interior norms.

The final complication is that some of the coefficients of $H - H_0$ are not necessarily of order $O(r^{-2})$, as required in the statement in [6]. However, careful study of the calculations in [6] show that weaker estimates are sufficient for the proof of the Mourre Hypotheses. Since this part of the proof also applies to manifolds without boundary, including some not covered by the hypotheses in [6], this part of the paper (Section 4) might be of independent interest.

The paper is organised as follows. In Section 2, the operators H and H_0 are defined, and H_0 is proven to be self-adjoint and semi-bounded. In Section 3, it is proven that H and H_0 are relatively bounded, and that the difference of their resolvents is compact. In Section 4 the operator A is defined, and the validity in our setting of Mourre Hypotheses 1–4 is proven. In Section 5, the Mourre Hypothesis 5 is proven. This is followed by an appendix, where a number of estimates pertaining to the change of variables $(x, y) \longrightarrow (r, s)$ are proven, followed by estimates for the coefficients of $H - H_0$.

We end this section by remarking that the conclusions of this paper should also hold for domains in higher dimensions with horn-like ends.

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2. **Transformed operator.** The overline symbol ($\overline{}$) shall be used to denote both complex conjugate (when referring to a function) and the topological closure (when referring to a Euclidean domain).

Let $C^\infty(\Omega)$ be the set of infinitely differentiable functions on Ω.

Let

$$\Omega_0 = \{(x, y); x > 0, |y| < f(x)\}.$$

We define coordinates r, s on Ω_0 as in [8]. Let

$$s = \frac{y}{f(x)}.$$

We construct the coordinate r to be orthogonal to s. Note that the slopes of level curves of s are given by

$$\frac{dy}{dx} = sf^{(1)}(x).$$

Hence the slopes of the level curves of an orthogonal coordinate will be given by

$$\frac{dy}{dx} = \frac{-f(x)}{f^{(1)}(x)y}.$$

Solving this equation we obtain

$$\frac{y^2}{2} + \int_{t=0}^{x} \frac{f(t)}{f^{(1)}(t)}\, dt = C,$$

for a constant C. Let

$$F(x) = \int_{t=0}^{x} \frac{f(t)}{f^{(1)}(t)}\, dt.$$

Note that F is decreases monotonically to $-\infty$ because $f^{(1)} < 0$ and $f^{(1)}/f \longrightarrow 0$. The inverse function $F^{-1}(x)$ is well defined on $(-\infty, 0]$, and can be extended infinitely differentiably and monotonically to positive x, so that

$$r = r(x, y) = F^{-1}\Big(\frac{y^2}{2} + F(x)\Big)$$

is well defined for all $(x, y) \in \Omega_0$. The mapping $(x, y) \longrightarrow (r, s)$ is a diffeomorphism from Ω_0 onto a subset of the strip $(-1, 1) \times \mathbf{R}$. It is easy to check, in particular, that the image contains $\{(r, s), r \geq 0, -1 < s < 1\}$.

For simplicity we write

$$\{(r, s), r \leq 0, -1 < s < 1\} = \{r \leq 0\},$$

and so on.

Let r_0 be such that the set $\mathbf{K}$ given by

$$\kappa \cup \{r < r_0\}$$

obeys the segment condition. By translating Ω if necessary, we can assume without loss of generality that $r_0 = 0$. Thus we have the following decomposition of Ω:

$$\Omega = \{r > 0\} \cup \mathbf{K}.$$

Note that for any $a > 0$, $\mathbf{K} \cup \{r < a\}$ will also obey the segment condition.

We establish some notation. Denote $D_r = \partial / \partial r$, $D_s = \partial / \partial s$. Let $g_s = D_s g$, $g_r = D_r g$, $g_{rr} = D_r^2 g$, *etc*.

The metric induced by the change of variables is easily calculated to be $\alpha^2 dr^2 + \beta^2 ds^2$, with

(9)
$$\alpha(r, s) = \left(\frac{f^{(1)}(x)}{f(x)}\right)\left(\frac{f(r)}{f^{(1)}(r)}\right)\left(1 + y^2\left(\frac{f^{(1)}(x)}{f(x)}\right)^2\right)^{-1/2},$$

(10)
$$\beta(r, s) = f(x)\left(1 + y^2\left(\frac{f^{(1)}(x)}{f(x)}\right)^2\right)^{-1/2}.$$

(It is convenient here to use simultaneously the coordinates r, s and x, y). The induced volume element is $\alpha\beta dr ds$, and the associated Laplace-Beltrami operator is

(11)
$$\frac{1}{\alpha\beta}\left(D_r \frac{\beta}{\alpha} D_r + D_s \frac{\alpha}{\beta} D_s\right).$$

It is clear by the choice of r, s that for $r > 0$, the boundary condition $\partial u / \partial \eta|_{\partial\Omega} = 0$ can be written $u_s = 0$ for $s = \pm 1$.

We now define a unitary transformation U on $L^2(\Omega, dx\, dy)$ as follows. Let ω be a positive function in $C^\infty(\Omega)$ such that:

$$\omega^{-1} = \begin{cases} 1 & \text{on } \mathbf{K}, \\ \alpha\beta & \text{for } r > 1. \end{cases}$$

Let $Uv = \omega^{-1/2} v$. Then U is a unitary transformation from $L^2(\Omega, dx\, dy)$ to $L^2(\Omega, \omega dx\, dy)$. Notice that in the coordinates r, s,

$$U: L^2(\{r > 1\}, \alpha\beta\, dr\, ds) \longrightarrow L^2(\{r > 1\}, dr\, ds).$$

Under the transformation U, the Neumann Laplacian transforms to an elliptic differential operator with C^∞ coefficients, H. A direct calculation of using Equation 11 shows that for $r > 1$ the differential operator H can be written as

$$H = -\frac{1}{\alpha^2}D_r^2 - \frac{1}{\beta^2}D_s^2 + 2\frac{\alpha_r}{\alpha^3}D_r + 2\frac{\beta_s}{\beta^3}D_s + V,$$
$$= -D_r\alpha^{-2}D_r - D_s\beta^{-2}D_s + V, \tag{12}$$

with V a C^∞ function which we write for future reference:

$$V = -\frac{1}{2\alpha^4\beta^2}(\beta_r^2\alpha^2/2 + 5\alpha_r^2\beta^2/2 - \alpha\alpha_{rr}\beta^2 - \alpha^2\beta\beta_{rr} + \alpha\alpha_r\beta\beta_r)$$
$$-\frac{1}{2\alpha^2\beta^4}(\alpha_s^2\beta^2/2 + 5\alpha^2\beta_s^2/2 - \alpha\alpha_{ss}\beta^2 - \alpha^2\beta\beta_{ss} + \alpha\alpha_s\beta\beta_s). \tag{13}$$

The boundary conditions for the transformed operator will be:

$$\partial u/\partial\eta = 0 \quad \text{on } \bar{\mathbf{K}} \cap \partial\Omega,$$
$$(\omega_s u/2 + \omega u_s)|_{s=\pm 1} = 0 \quad \text{for } r > 0. \tag{14}$$

In particular, for $r > 1$ this implies $\big(-(\alpha\beta)_s/(2\alpha\beta)u + u_s\big)|_{s=\pm 1} = 0$. It will be convenient to write

$$a(r,s) = -\frac{(\alpha\beta)_s}{2\alpha\beta}(r,s). \tag{15}$$

It will be convenient to give the following definition:

$$\Psi = \{u \in C_0^\infty(\bar{\Omega}) : u \text{ satisfies Equation 14}\}. \tag{16}$$

Here $C_0^\infty(\bar{\Omega})$ denotes functions of bounded support in Ω which extend to C^∞ functions in an open set containing $\bar{\Omega}$.

Since U is unitary, it follows that H is a self-adjoint operator on $L^2(\Omega, \omega dx\, dy)$.

We define a differential operator H_0 on Ω as follows. On $\mathbf{K}$ we set H_0 equal to H as a differential operator. For $r > 0$, the choice of coefficients is motivated by Equation 12. Fix $N \geq 1$ and let

$$H_0 = -D_rA(r,s)^{-2}D_r - D_sB(r,s)^{-2}D_s + V^*; \tag{17}$$

here A, B, V^* are positive, C^∞ functions, satisfying

$$A = \begin{cases} \alpha & \text{for } r < N \\ 1 & \text{for } r \geq N+1, \end{cases}$$
$$B = \begin{cases} \beta & \text{for } r < N \\ f(r) & \text{for } r \geq N+1, \end{cases}$$
$$V^* = \begin{cases} V & \text{for } r < N \\ 0 & \text{for } r \geq N+1. \end{cases}$$

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Here f is viewed as a function on the half-line, so that $f(r)$ is a function on $\{r > N\}$.

We impose the following further conditions on A, B, V^*, and N. First, since the functions α, β are symmetric in s, it follows that we can, without loss of generality, choose A, B to be symmetric in s.

Next, fix $\epsilon > 0$. It follows from Lemma 14 in the appendix that there exists N_1 such that for $r > N_1$, the coefficient functions of H satisfy

$$|\alpha^{-2} - 1| < \epsilon, \quad |\beta^{-2} - f(r)^{-2}| < \epsilon, \quad \left|\frac{\alpha_r}{\alpha^3}\right| < \epsilon, \quad \left|\frac{\beta_s}{\beta^3}\right| < \epsilon, \quad |V| < \epsilon. \tag{18}$$

Note that H and H_0 coincide as differential operators on $K \cup \{r < N\}$, and for $r > N+1$, H_0 coincides with

$$-D_r^2 - f^{-2}(r)D_s^2. \tag{19}$$

It thus follows from Equation 18 that we can choose A, B, V^* above so that the absolute values of the coefficients of $H - H_0$ are globally bounded above by ϵ. The exact choice of ϵ will be made in the proof of Lemma 4.

The next requirement we impose on the coefficients of H_0 is the following. Note that by Equation 18 and the fact that $f(r)^{-2} \longrightarrow \infty$ as $r \longrightarrow \infty$, one can choose N_2 and B such that for $r > N_2$,

$$B(r,s)^{-2} > 2. \tag{20}$$

Finally, we choose $N = \max(N_1, N_2)$.

Before continuing we establish some notation. We denote the norm on $L^2(\Omega, \omega dx\, dy)$ by $\| * \|_{L^2}$. We denote the associated inner product by $\langle *, * \rangle$. The supremum norm is denoted $\| * \|_{L^\infty}$.

The norm for bounded operators on $L^2(\Omega, \omega dx\, dy)$ is given by $\| * \|$.

LEMMA 1. *Consider the operator H_0 acting on Ψ, with Ψ as in Equation 16. Then the operator H_0 is elliptic and essentially self-adjoint in $L^2(\Omega, \omega dx\, dy)$.*

PROOF. Assume for simplicity that $N = 1$. To prove ellipticity note first that for $r < 1$ this follows from the ellipticity of H. For $r > 0$ ellipticity follows from the formula for H_0, and from the fact that the functions α, β, A, B are all strictly positive.

Next we prove that H_0 is symmetric. Let $u, v \in \Psi$. We will view $\langle H_0 u, v\rangle$ as an integral over the components: $\mathbf{K} \cup \{r < 1\}$ and $\{r > 1\}$. It follows from the self-adjointness of H that

$$\int_{\mathbf{K}\cup\{r<1\}} (H_0\bar{u})v = \int_{\mathbf{K}\cup\{r<1\}} \bar{u}H_0 v - \int_{s=-1}^{1} \alpha(1,s)\big(\bar{u}(1,s)v_r(1,s) - \bar{u}_r(1,s)v(1,s)\big)\, ds. \tag{21}$$

Next, applying integration by parts, and applying Equation 14, one obtains

$$\begin{aligned}\int_{\{1<r<\infty\}} \big((-D_r A^{-2} D_r - D_s B^{-2} D_s + V^*)\bar{u}\big) v\, dr\, ds \\ = \int_{\{1<r<\infty\}} \bar{u}(-D_r A^{-2} D_r - D_s B^{-2} D_s + V^*) v\, dr\, ds \\ + \int_{s=-1}^{1} \alpha(1,s)\big(\bar{u}(1,s)v_r(1,s) - \bar{u}_r(1,s)v(1,s)\big)\, ds.\end{aligned} \tag{22}$$

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Adding Equations 21 and 22 one obtains $\langle H_0 u, v\rangle = \langle u, H_0 v\rangle$, and hence H_0 is symmetric. It follows that H_0 admits a symmetric closure, which we again label H_0. Clearly the boundary conditions associated to the closure will be those given by Equation 13.

Let H_0^* be the adjoint of the closed operator H_0. It will be convenient here to distinguish between the closed operator H_0 and the associated differential expression, which we label P. Suppose $v \in \mathrm{Dom}(H_0^*)$. First we show that $H_0^* v = Pv$. For any point $(x, y) \in \Omega$, let $u \in C_0^\infty(\Omega)$ be supported in a neighbourhood of (x, y). Clearly $u \in \mathrm{Dom}(H_0)$ and $H_0 u = Pu$, a.e. Hence by the definition of adjoint,

$$\langle u, H_0^* v\rangle = \langle Pu, v\rangle.$$

Since u is supported away from the boundary it follows from integration by parts that $\langle Pu, v\rangle = \langle u, Pv\rangle$. Thus $\langle u, H_0^* v\rangle = \langle u, Pv\rangle$ when $\langle *, *\rangle$ is viewed as a distributional pairing. It follows that $H_0^* v = Pv$ a.e. in Ω, and $Pv \in L^2(\Omega)$.

Now we prove $\mathrm{Dom}(H_0^*) \subset \mathrm{Dom}(H_0)$. Assume that $v \in \mathrm{Dom}(H_0^*)$ and $u \in \mathrm{Dom}(H_0)$. Since $H_0^* v = Pv$ a.e. and $Pv \in L^2(\Omega)$, the integration by parts shows that $\langle H_0 u, v\rangle = \langle u, H_0^* v\rangle$ only if the following three integrals are zero:

$$\int_{\mathbf{K}\cap\partial\Omega} \bar{u}\partial_\eta v,$$

$$\int_{r=0}^{\infty} \overline{u(r,1)}\big(-\omega_s(r,1)v(r,1)/2 + \omega(r,1)v_s(r,1)\big),$$

$$\int_{r=0}^{\infty} \overline{u(r,-1)}\big(-\omega_s(r,-1)v(r,-1)/2 + \omega(r,-1)v_s(r,-1)\big)\, dr.$$

By a density argument it follows that v satisfies Equation 14, a.e. Hence $v \in \mathrm{Dom}(H_0)$ and thus $\mathrm{Dom}(H_0^*) \subset \mathrm{Dom}(H_0)$. Self-adjointness follows.

LEMMA 2. *The operator H_0 is bounded below.*

PROOF. Assume $u \in \mathrm{Dom}(H_0)$. We write $\langle H_0 u, u\rangle$ as an integral over the regions $\mathbf{K} \cup \{r < N\}$, $\{N > r\}$. Note that $H_0 = H$ in the region $\mathbf{K} \cup \{r < N\}$. Thus the first integral is non-negative because the corresponding integral for H is non-negative.

The second integral equals

$$\int_{\{r>N\}} \big((-D_r A^{-2} D_r - D_s B^{-2} D_s + V^*)\bar{u}\big) u\, dr\, ds.$$

Since $A > 0$, the term involving $-D_r A^{-2} D_r$ can be proven non-negative using integration by parts (the boundary term at $r = N$ will cancel with a corresponding one from the first integral). Also, by construction, the function V^* is bounded below, so

$$\int_{\{r>N\}} V^* |u|^2\, dr\, ds \geq -\|V^*\|_{L^\infty} \|u\|_{L^2}^2.$$

It remains to bound the term involving $D_s B^{-2} D_s$ for $r > N$. We use integration by parts, followed by application of the boundary conditions (Equation 14), the fact that

$B(r,-1) = B(r,1)$, $|a(r,1)| = |a(r,-1)|$, Equation 20, to show that for fixed r, and then the Fundamental Theorem of Calculus to show

$$\begin{aligned}\int_{s=-1}^{1}(-D_sB^{-2}D_s\bar{u})u\,ds &= \int_{s=-1}^{1} B^{-2}|u_s|^2\,ds - (B^{-2}\overline{u_s}u)(r,1) + (B^{-2}\overline{u_s}u)(r,-1)\\ &\geq \int_{-1}^{1} B^{-2}|u_s|^2\,ds - |B^{-2}a|(r,1)|\big(|u(r,1)|^2 - |u(r,-1)|^2\big)\\ &\geq 2\int_{-1}^{1}|u_s|^2\,ds - |B^{-2}a(r,1)|\big(|u(r,1)|^2 - |u(r,-1)|^2\big)\\ &= 2\int_{-1}^{1}|u_s|^2\,ds - |B^{-2}a(r,1)|\int_{s=-1}^{1} D_s(|u|^2)\,ds\\ &\geq 2\int_{-1}^{1}|u_s|^2\,ds - |B^{-2}a(r,1)|\int_{s=-1}^{1}2|u_su|\,ds. \end{aligned} \tag{23}$$

By Lemma 14 in the appendix, $f^{-2}(r)a(r,\pm 1) = O(1)$ as $r \longrightarrow \infty$. Thus by construction of B, $B^{-2}a(r,1)$ is globally bounded. This, and application of the elementary inequality $2cd \leq \epsilon c^2 + \frac{d^2}{\epsilon}$ (for c, d, ϵ positive reals), implies that for sufficiently small ϵ, the right hand side of Equation 23 is bounded below by

$$\int_{-1}^{1}|u_s|^2\,ds - C\int_{s=-1}^{1}|u|^2\,ds,$$

with C some positive constant. It follows that

$$\int_{r=N}^{\infty}\int_{s=-1}^{1}(-D_sB^{-2}D_s\bar{u})u\,ds\,dr \geq -C\int_{r=N}^{\infty}\int_{s=-1}^{1}|u|^2\,ds\,dr.$$

This completes the proof.

3. **Relative boundedness.** We prove a sequence of lemmas regarding the relative boundedness of H_0 and H.

First, we define a cutoff function which localises to a neighbourhood of infinity. Let $\chi(t)$ be a smooth monotone function on $\mathbf{R}$ such that $\chi(t) = 0$ for $t < N+2$ and $\chi(t) = 1$ for $t > N+3$. We define the function χ_R on Ω by

$$\chi_R(r) = \chi(r/R), \quad \text{with } \chi_R = 0 \text{ on } \mathbf{K}. \tag{24}$$

Here $R > 1$ is a constant to be determined later.

Next, note that by the proof of Lemma 2, there exists a positive constant M such that the operators

$$(H_0 + M), \quad \chi_R\big(-f(r)^{-2}D_s^2 + M\big)$$

are positive operators.

Let W^s be the Sobolev spaces associated with H_0, defined as the completion of $\mathrm{Dom}(H_0)$ with respect to the norm

$$\|u\|_s = \|(H_0 + M)^{s/2}u\|_{L^2}.$$

For consistency of notation, we shall write $\|u\|_{L^2} = \|u\|_0$ in the proofs that follow.

Note that since $H_0^s(W^{s/2}) \subset L^2(\Omega)$ and since H_0 is elliptic, it follows by elliptic regularity that $W^6 \subset C^4(\bar{\Omega})$.

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LEMMA 3. *Let W^s be the scale of spaces defined by H_0, and let χ_R be defined as in Equation 24. Then*

i) $\chi_R D_r\colon W^s \longrightarrow W^{s-1}$ *is bounded for* $s \in [-1, 2]$,
ii) $\chi_R D_r^2\colon W^s \longrightarrow W^{s-2}$ *is bounded for* $s \in [0, 2]$,
iii) $\chi_R\big(-f^{-2}(r)D_s^2 + M\big)^{1/2}\colon W^s \longrightarrow W^{s-1}$ *is bounded for* $s \in [-1, 2]$,
iv) $\chi_R\big(-f^{-2}(r)D_s^2 + M\big)\colon W^s \longrightarrow W^{s-2}$ *is bounded for* $s \in [0, 2]$,
v) $\chi_R f^{-1}(r)D_s\colon W^1 \longrightarrow W^0$ *is bounded.*

Furthermore, the bounds are uniform in R for large R.

We remark that the proof of parts i)–iv) runs largely along the lines of the same lemma in [6], although it is complicated in this case by the presence of boundary conditions.

PROOF. Let C denote various positive constants. Denote $D_sD_r(h)$ by h_{rs}, *etc.*

It will be convenient in this proof to denote χ_R simply as χ. We first prove that there exists C such that

$$\|\chi D_r(H_0 + C)^{-1/2}\| \leq 1. \tag{25}$$

Let $\tau = (1-\chi^2)^{1/2}$. Thus τ is a C^∞ function. The IMS localisation formula [C-F-K-S] yields

$$H_0 + M = \chi(H_0 + M)\chi + \tau(H_0 + M)\tau - (\tau_r)^2 - (\chi_r)^2. \tag{26}$$

This, along with the positivity of $H_0 + M$ and $-f(r)^{-2}D_s^2 + M$, implies

$$H_0 + C \geq \chi(-D_r^2)\chi - (\chi_r)^2 - (\tau_r)^2.$$

This inequality should be understood as a form inequality on the set Ψ (see Equation 16 for definition of Ψ). Since $\chi(-D_r^2)\chi = -D_r\chi^2D_r - \chi^{(2)}\chi$, and

$$|-\chi_{rr}\chi - (\chi_r)^2 - (\tau_r)^2| < C,$$

it follows that

$$-D_r\chi^2D_r \leq H_0 + C.$$

Thus, for every $\phi \in \Psi$,

$$\|\chi D_r\phi\|_0 \leq \|(H_0 + C)^{1/2}\phi\|_0,$$

and hence for every $\varphi \in (H_0 + C)^{1/2}(\Psi)$,

$$\|\chi D_r(H_0 + C)^{-1/2}\varphi\|_0 \leq \|\varphi\|_0.$$

The proof of Equation 25 can thus be completed if one can show that the set $(H_0+C)^{1/2}(\Psi)$ is dense in $L^2(\Omega, \omega dx\, dy)$. For this, first note that since Ψ is a core for $H_0 + C$, the set $(H_0 + C)\Psi$ is L^2-dense. A simple topological argument now shows that $(H_0 + C)^{1/2}(\Psi)$ is L^2-dense.

Next, we prove part v). Note first that for $u \in \Psi$, applying integration by parts,

$$\langle(-D_sf^{-2}D_s)\chi u, \chi u\rangle = \|\chi f^{-1}D_su\|_0^2 - \int_{r=0}^{\infty} f^{-2}\chi^2[\overline{u_s}u]_{-1}^1\, dr. \tag{27}$$

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We bound the second term on the right hand side from below.

We recall the boundary conditions associated with W^2, for $r > 1$:

$$(au + u_s)\big|_{s=\pm 1} = 0, \tag{28}$$

with

$$a(r,s) = -\frac{(\alpha\beta)_s}{2\alpha\beta}.$$

Hence

$$\begin{aligned} \big|[\overline{u_s}u]_{-1}^{1}\big| &= \Big|[a|u|^2]_{-1}^{1}\Big| \\ &= \Big|\int_{s=-1}^{1} D_s(a|u|^2)\,ds\Big| \\ &\leq \int_{s=-1}^{1} |a_s|\,|u|^2 + 2|a\overline{u_s}u|\,ds \\ &\leq \int_{s=-1}^{1} |a_s|\,|u|^2 + |a|\Big(\epsilon|u_s|^2 + \frac{1}{\epsilon}|u|^2\Big)\,ds \end{aligned}$$

In the last step, we applied the inequality $2cd \leq \epsilon c^2 + \frac{d^2}{\epsilon}$, for $c, d, \epsilon \geq 0$. By Lemma 14 part vi), the following estimates hold:

$$af^{-2} = O(1), \quad a_s f^{-2} = O(1).$$

Hence, we can choose ϵ so that

$$-f(r)^{-2}[\overline{u_s}u]_{-1}^{1} \geq -\frac{f^{-2}}{2}\int_{s=-1}^{1} |u_s|^2 ds - C\int_{s=-1}^{1} |u|^2\,ds.$$

This and Equation 27 imply

$$\langle -(D_s f^{-2} D_s)\chi u, \chi u\rangle \geq \frac{1}{2}\|f^{-1}D_s\chi u\|_0^2 - C\|u\|_0^2, \tag{29}$$

for $u \in \Psi$. On the other hand, by the I.M.S. localisation formula and the semi-positivity of $-D_r^2$,

$$H_0 + C \geq \chi(-f^{-2}D_s^2)\chi - (\chi_r)^2 - (\tau_r)^2.$$

Combining this and Equation 29, we obtain

$$2\|(H_0 + C)^{1/2}u\|_0^2 \geq \|f^{-1}D_s\chi u\|_0^2,$$

this holding for $u \in \Psi$. Hence

$$\|f^{-1}D_s\chi u\|_0 \leq C\|u\|_1$$

for all $u \in \Psi$. Since Ψ is dense in W^1, part v) now follows.

Next, we prove that

$$\|\chi D_r^2(H_0 + C)^{-1}\| \leq C. \tag{30}$$

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Since we have not established the existence of a subset of $C_0^\infty(\overline{\Omega}) \cap W^4$ which is dense in W^4 (which would play the role of Ψ), some care must be taken in discussing the commutation of derivatives, and also in discussing some of the inner products, in the arguments that follow. Hence, in what follows, we will assume u is an arbitrary element in W^6. Since $W^6 \subset C^4(\Omega)$, derivatives in r and s of order up to 4 commute. It will be convenient to express the inner product $\langle \chi u, \chi v\rangle$ in the following manner:

$$\langle \chi u, \chi v\rangle = \lim_{N\to\infty} \int_{s=-1}^{1} \int_{r=0}^{\infty} \tau_N^2(r)\chi^2 \bar{u} v\, dr\, ds;$$

here $\tau_N^2 \equiv 1 - \chi_N^2$, and $\chi = \chi_R$ as in Equation 24. For notational simplicity we write $\tau_N = \tau$ in what follows.

First, note the following quadratic form inequality which holds for W^6:

$$(H_0 + M)^2 \geq (H_0 + M)\chi^2(H_0 + M). \tag{31}$$

Next, an exercise in commutation shows that the following holds for W^6:

$$\begin{aligned}(H_0 + M)\chi^2(H_0 + M) &= D_r^2\chi^2 D_r^2 - D_r^2\chi^2 Q - \chi^2 Q D_r^2 + Q^2\chi^2 \\ &= D_r^2\chi^2 D_r^2 - 2D_r\chi Q\chi D_r - (\chi^2 Q)^{(2)} + \chi Q^2\chi;\end{aligned} \tag{32}$$

here $Q = f(r)^{-2}D_s^2 + M$ and $(\chi^2 Q)^{(2)} = (\chi^2 f^{-2})_{rr} D_s^2 + M(\chi^2)_{rr}$. We will need to provide lower bounds for the terms on the right hand side.

First, we bound $\chi Q^2 \chi$. We prove that

$$\langle \tau^2 f^{-4} D_s^4 \chi u, \chi u\rangle \geq \|\tau f^{-2} D_s^2 \chi u\|_0^2 - C\|u\|_1^2 - C\|\tau\chi f^{-1} D_s D_r u\|_0 \|u\|_0. \tag{33}$$

It follows from integration by parts that

$$\begin{aligned}&\int_{r=0}^{\infty}\int_{s=-1}^{1} \tau^2 f^{-4} D_s^4 \chi^2 \bar{u} u\, ds\, dr \\ &\qquad = \int_{r=0}^{\infty}\Big(\int_{s=-1}^{1} |\tau f^{-2} D_s^2 \chi u|^2\, ds - \chi^2 f^{-4}\tau^2 [\overline{u_{ss}} u_s - \overline{u_{sss}} u]_{-1}^{1}\Big) dr.\end{aligned} \tag{34}$$

To estimate the second term on the right hand side, we must first discuss the boundary conditions associated to W^6. Since $W^6 \subset W^2$, it follows that Equation 28 holds for $u \in W^6$. Also, since $H_0 u \in W^2$, we have that $H_0 u$ satisfies Equation 28 and hence for $r > 1$

$$\big(a(u_{rr} + f^{-2}u_{ss}) + u_{rrs} + f^{-2}u_{sss}\big)\big|_{s=\pm1} = 0. \tag{35}$$

Since $u \in W^6 \subset C^4(\overline{\Omega})$, we have $u_{rrs} = u_{srr}$. Thus since

$$D_r^2(au + u_s) = 0$$

for $s = \pm 1$, Equation 35 can be rewritten as

$$-a_{rr}u - 2a_r u_r + af^{-2}u_{ss} + f^{-2}u_{sss} = 0 \tag{36}$$

for $s = \pm 1$.

Thus by Equations 28, 36,

$$
\begin{aligned}
f^{-2}[\overline{u_{ss}}u_s - \overline{u_{sss}}u]_{-1}^{1} &= -[a_{rr}|u|^2 + 2a_r\bar{u}_r u]_{-1}^{1} \\
&= -\int_{s=-1}^{1} D_s(a_{rr}|u|^2 + 2a_r\bar{u}_r u)\,ds \\
&= -\int_{s=-1}^{1} a_{rrs}|u|^2 + a_{rr}(\bar{u}_s u + \bar{u}u_s) + 2a_{rs}\bar{u}_r u + 2a_r\bar{u}_r u_s + 2a_r\bar{u}_{rs}u\,ds
\end{aligned}
$$

Hence

$$
\begin{aligned}
&-\int_{r=0}^{\infty}\tau^2\chi^2 f^{-4}[\overline{u_{ss}}u_s - \overline{u_{sss}}u]_{-1}^{1}\,dr \\
&\qquad \geq -C(\|\tau\chi u\|_0^2 + \|\tau\chi f^{-1}u_s u\|_0\|\tau\chi u\|_0 + \|\tau\chi f^{-1}u_s\|_0\|\tau\chi u_r\|_0 \\
&\qquad\qquad +\|\tau\chi u\|_0\|\tau\chi u_r\|_0\|\tau\chi f^{-1}D_sD_r u\|_0\|\tau\chi u\|_0) \\
&\qquad \geq -C\|u\|_1^2 - C\|\tau\chi f^{-1}D_sD_r u\|_0\|u\|_0;
\end{aligned}
$$

the first of the inequalities above follows from the Holders and Schwartz Inequalities and from the bounds

$$
\frac{\partial^k a}{\partial r^i \partial s^j} = O(f(r)^2), \quad i + j = k, \tag{37}
$$

proven in Lemma 14, from the bound $(f^{-2})_r = O(f^{-2})$ (Equation 3), and from parts i) and v) of this lemma. The second of the inequalities follows from the boundedness of χ, τ. Equation 33 now follows from Equation 34.

To prove a lower bound on the quadratic form $-(\chi^2 Q)^{(2)}$, note that χ_r and χ_{rr} are bounded functions. Also, by Equation 3, $(f^{-2})^{(1)} = O(f^{-2})$ and $(f^{-2})^{(2)} = O(f^{-2})$. Hence an argument similar to the proof of part v) shows that

$$
\langle -\tau^2(\chi^2 Q)^{(2)}u, u\rangle \geq -C\|u\|_1^2. \tag{38}
$$

Next, note that by the definition of Q,

$$
\begin{aligned}
&\int_{s=-1}^{1}\int_{r=0}^{\infty}\tau^2(-D_r\chi Q\chi D_r)\bar{u}u\,dr\,ds \\
&\qquad = \int_{s=-1}^{1}\int_{r=0}^{\infty}\tau^2\Big((D_r\chi f^{-2}D_s^2\chi D_r)\bar{u}u + M(-D_r\chi^2 D_r)\bar{u}u\Big)\,dr\,ds.
\end{aligned}
$$

Using integration by parts and part i) of the lemma, the second term on the right hand side is bounded below by $C\|u\|_1^2$. To bound the first term on the right hand side, we first note that by integration by parts:

(39)

$$
\int_{r=0}^{\infty}\tau^2(D_r\chi f^{-2}D_s^2\chi D_r)\bar{u}u\,dr = -\int_{r=0}^{\infty}\tau^2(f^{-2}D_s^2\chi D_r\bar{u})\chi D_r u - (\tau^2)_r f^{-2}(D_s^2\chi D_r)\bar{u}\chi u\,dr.
$$

We bound the last two terms. Note $u_{rs} = u_{sr} = -(au)_r$ at $s = \pm 1$ by Equation 28. Hence

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$$
\begin{aligned}
-\int_{-1}^{1}\int_{r=0}^{\infty}&\big(\tau^2(f^{-2}D_s^2\chi D_r\bar{u})\chi D_r u\big)\,dr\,ds\\
&= \|\tau\chi f^{-1}D_sD_ru\|_0^2-\int_{r=0}^{\infty}\tau^2f^{-2}\chi^2[\bar{u}_{rs}u_r]_{-1}^{1}\,dr\\
&= \|\tau\chi f^{-1}D_sD_ru\|_0^2+\int_{r=0}^{\infty}\tau^2f^{-2}\chi^2[\overline{(au)}_r u_r]_{-1}^{1}\,dr\\
&= \|\tau\chi f^{-1}D_sD_ru\|_0^2+\int_{r=0}^{\infty}\tau^2f^{-2}\chi^2\int_{s=-1}^{1}D_s\big(\overline{(au)}_r u_r\big)\,dr\\
&\geq \|\tau\chi f^{-1}D_sD_ru\|_0^2-C(\|\tau\chi f^{-1}D_sD_ru\|_0\|u\|_1+\|u\|_1^2). \qquad (40)
\end{aligned}
$$

The last inequality follows from Equation 37, the Schwartz Inequality, part i) of the lemma, and the observation $\|\tau\chi u\|_0 \leq \|u\|_0$. To bound the second term on the right hand side of Equation 39, we apply integration by parts in s, followed by the Schwartz inequality and the argument used in the proof of Equation 40 to obtain the estimate:

$$\int_{s=-1}^{1}\int_{r=0}^{\infty}(\tau^2)_r\chi^2f^{-2}(D_s^2D_r\bar{u})u\,dr\,ds \geq -2\|\chi\tau_rf^{-1}D_su\|_0\|\chi\tau f^{-1}D_sD_ru\|_0-C\|u\|_1^2.$$

Noting that $\|\tau_r\|_\infty$ is bounded uniformly in N, we obtain

$$\langle\tau^2(-D_r\chi Q\chi D_r)u,u\rangle \geq \|\tau\chi f^{-1}D_sD_ru\|_0^2-C\|\chi\tau f^{-1}D_sD_ru\|_0\|u\|_1-C\|u\|_1^2. \qquad (41)$$

Using similar arguments, one can prove

$$\langle\tau^2D_r^2\chi^2D_r^2u,u\rangle \geq \|\tau\chi D_r^2u\|_0^2-C\|\tau\chi D_r^2u\|_0\|u\|_1-C\|u\|_1^2. \qquad (42)$$

The details are omitted.

Combining Equations 31, 32, 33, 38, 41, and 42 we obtain

$$
\begin{aligned}
\langle(H_0+M)^2u,u\rangle \geq{}& \|\tau\chi f^{-2}D_s^2u\|_0^2+\|\tau\chi D_r^2u\|_0^2+\|\tau\chi f^{-1}D_sD_ru\|_0^2\\
&-C(\|\tau\chi D_r^2u\|_0\|u\|_1+\|\tau\chi f^{-1}D_sD_ru\|_0\|u\|_1+\|u\|_1^2).
\end{aligned}
$$

By inspecting the proof up to now it is easy to verify that the the constant C is independent of u, N and R. Letting $N\longrightarrow\infty$ we obtain

$$
\begin{aligned}
\langle(H_0+M)^2u,u\rangle \geq{}& \|\chi f^{-2}D_s^2u\|_0^2+\|\chi D_r^2u\|_0^2+\|\chi f^{-1}D_sD_ru\|_0^2\\
&-C(\|\chi D_r^2u\|_0\|u\|_1+\|\chi f^{-1}D_sD_ru\|_0\|u\|_1+\|u\|_1^2). \qquad (43)
\end{aligned}
$$

It follows that

$$\|\chi D_r^2u\|_0^2 \leq C\langle(H_0+M)^2u,u\rangle,$$

and hence

$$\|\chi D_r^2u\|_0^2 \leq C\|u\|_1^2,$$

for $u \in W^6$, with C independent of u and R. This leads to Equation 30 by a density argument.

From Equation 30 it follows that

$$\|\chi D_r^2 (H_0 + C)^z\| \le C, \tag{44}$$

for $\mathrm{Re}(z) = -1$. A similar argument shows that Equation 44 remains true when χD_r^2 is replaced by $D_r^2 \chi$. Taking adjoints it follows that $(H_0 + C)^z \chi_R D_r^2$ extends to a bounded operator with

$$\|(H_0 + C)^z \chi D_r^2\| \le C,$$

for $\mathrm{Re}(z) = -1$. It follows by complex interpolation that

$$\|(H_0 + C)^{-1+z} \chi D_r^2 (H_0 + C)^{-z}\| \le C$$

for $\mathrm{Re}(z) \in [0, 1]$. This proves part ii) of the lemma.

Similarly, the proof of iv) follows from Equation 43 together with an interpolation argument.

To prove part i) of the lemma, it suffices to apply complex interpolation to the inequalities:

$$\|(H_0 + C)^{-1} \chi D_r (H_0 + C)^{1/2}\| \le C,$$
$$\|(H_0 + C)^{1/2} \chi D_r (H_0 + C)^{-1}\| \le C.$$

We prove the first of these inequalities. The second follows by a similar argument. Note that

$$\|(H_0 + C)^{-1} \chi D_r (H_0 + C)^{1/2}\| \le \|\chi D_r (H_0 + C)^{-1/2}\| + \|(H_0 + C)^{-1}[(H_0 + C), \chi D_r](H_0 + C)^{-1/2}\|.$$

The first of the terms on the right hand side is bounded by Equation 25. The second term can be decomposed into two parts; the first, involving $[D_r^2, \chi D_r]$ is bounded Equation 25 and part ii) of the lemma. The second part consists of

$$(\chi f^{-2})_r D_s^2 = \left(-2\chi \frac{f_r}{f^3} + \chi_r \frac{1}{f^2}\right) D_s^2,$$

which is bounded by iv) along with assumption 3 on the function f.

The proof of iii) is similar to the proof of i).

LEMMA 4. *Let E be the differential operator given by $E = H - H_0$. One can choose the coefficient functions of H_0 so that*

i) $E\colon W^s \longrightarrow W^{s-2}$ is bounded for $s \in [0, 2]$.

ii) $\|E(H_0 + M)^{-1}\| < 1$.

iii) The operators H and H_0 are mutually relatively bounded.

iv) The domains of H and H_0 coincide, and the scales of spaces generated by H and H_0 are the same for $s \in [-2, 2]$.

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PROOF. Part i) of the lemma follows immediately from Lemma 3 and the boundedness of the coefficients of E. By the construction of H_0, the coefficients of H_0 can be chosen so that the coefficients of E are arbitrarily small in supremum norm. Part ii) of the lemma thus follows by choosing the coefficients of E to be small enough.

Part iii) follows from the previous lemma along with part ii) and a Neumann series argument. Part iv) follows from part iii) and an interpolation argument.

We will need the following compactness result:

LEMMA 5. *Suppose $h \in C_0^\infty(\mathbf{R})$, $\rho \in C_0^\infty(\bar{\Omega})$, and suppose D is any differential operator with C^∞ coefficients. Then $\rho D h(H)$, $\rho D h(H_0)$ are compact operators on $L^2(\Omega, \omega dx\,dy)$.*

PROOF. First, note that by the Spectral Theorem, $h(H_0)\colon L^2(\Omega, \omega dx\,dy) \longrightarrow W^s(\Omega)$ is a bounded operator of all s. Thus the same holds for $h(H)$ by relative boundedness.

Let U be a bounded open set in Ω. Assume furthermore that U obeys the segment condition and contains the support of ρ. Let s be a positive integer. Define $\tilde{W}^s(U)$ to be the closure of $C^\infty(\bar{U})$ with respect to the norm

$$\left(\int_U \bar{u}(-D_x^2 - D_y^2)^{s/2} u\, dx\, dy\right)^{1/2}.$$

Assume that D is an operator of order d. It then follows from standard arguments involving the ellipticity of H_0 that for all s, ρD is bounded from $W^s(\Omega)$ to $\tilde{W}^{s-d}(U)$.

Now assume $s > d$. Since U obeys the segment condition, it follows ([11]) that the inclusion $\tilde{W}^{s-d}(U) \longrightarrow \tilde{W}^0(U)$ is compact. Also, it is clear that the inclusion $\tilde{W}^0(U) \longrightarrow L^2(\Omega, \omega dx\,dy)$ is bounded. The lemma now follows.

PROPOSITION 2. *i) Assume $z \notin \sigma(H_0) \cup \sigma(H)$. Then*

$$(H + z)^{-1} - (H_0 + z)^{-1}$$

is compact on $L^2(\Omega, \omega dx\,dy)$.

ii) For $h \in C_0^\infty(\mathbf{R})$, the operator $h(H) - h(H_0)$ is compact on $L^2(\Omega, \omega dx\,dy)$.

PROOF. The proof of Proposition 2 runs along the lines of the proof of the corresponding results in [6] (Lemma 1.4, Corollary 1.5) and hence is omitted.

The first of the following corollaries follows from [11]:

COROLLARY 1. *The essential spectrum of H and H_0 coincide.*

COROLLARY 2. *The spectrum of H is $[0, \infty)$.*

PROOF. Fix a constant $\alpha > 0$. Let $v_0(r, s)$ be the eigenfunction corresponding to the smallest eigenvalue of the Sturm-Liouville problem:

$$\begin{aligned} u_{ss}(s) + \lambda u(s) &= 0, \\ au(-1) + u_s(-1) &= 0, \\ au(1) + u_s(1) &= 0, \end{aligned}$$

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with a given by Equation 15. This, of course, is the Sturm-Liouville problem given in Equation 8 and studied in Appendix 2. Approximations to $e^{i\alpha r}v_0(r,s)$ will furnish a Weyl sequence for α^2. Hence the essential spectrum of H_0 contains $[0,\infty)$. By Corollary 1, the same holds for H. But since H is non-negative, Corollary 2 now follows.

4. **Mourre Hypotheses 1–4.** We begin this section by studying the problem:

$$\begin{aligned} u_{ss}(s) + \lambda u(s) &= 0 \\ bu(-1) + u^{(1)}(-1) &= 0 \\ cu(1) + u^{(1)}(1) &= 0. \end{aligned} \tag{45}$$

For the moment we suppose b, c are arbitrary real numbers.

It is well known that this Sturm-Liouville problem has a spectrum consisting of a discrete set of eigenvalues of multiplicity 1.

LEMMA 6. *Let $\lambda_0 < \lambda_1 < \cdots$ be the set of eigenvalues of Sturm-Liouville problem (Equation 45), with corresponding normalised eigenvectors $v_0, v_1, \ldots$. Then the eigenvalues and eigenvectors are jointly analytic in b, c.*

PROOF. The quadratic form associated with the problem above is

$$q(u,u) = \int_{s=-1}^{1} |u_s(s)|^2\, ds - b|u(-1)|^2 + c|u(1)|^2.$$

Thus in each of the variables b and c, the operator associated to the Sturm-Liouville problem is Type B analytic in the sense of Kato; thus the eigenvalues and eigenvectors are analytic in each b and c ([11]). Since analyticity in b, c separately implies joint analyticity in (b,c), the lemma now follows.

We now set

$$b = -\frac{(\alpha\beta)_s}{2\alpha\beta}\Big|_{s=-1}, \quad c = -\frac{(\alpha\beta)_s}{2\alpha\beta}\Big|_{s=1}. \tag{46}$$

Thus $b = c = a(r,\pm 1)$, with a as in Equation 15.

Now λ_j is r dependent.

LEMMA 7. *1. $\lambda_0(r) = O(r^{-1}f(r)^2)$,*
2. $\frac{\partial^n}{\partial r^n}(\lambda_0) = O(r^{-2})$, $n = 1,2,3$.
Furthermore, there exists M such that for $r > M$,

$$\lambda_1(r) > 1.$$

PROOF. In the notation of the previous lemma, $\lambda_0(0,0)$ is the smallest eigenvalue of the Sturm Liouville problem with Neumann boundary conditions. Hence $\lambda_0(0,0) = 0$, and so by analyticity,

$$\lambda_0(b,c) = O(b) + O(c)$$

as $(b,c) \longrightarrow (0,0)$. The estimates for $\lambda_0(r)$ now follow immediately from Lemma 14.

Similarly,

$$\lambda_1(b,c) = \frac{\pi^2}{4} + O(b) + O(c),$$

and thus $\lambda_1(r) > 1$ for large r.

To compute the asymptotics of $\frac{\partial}{\partial r}\lambda_0$, we calculate

$$\frac{\partial}{\partial r}\lambda_0 = (\frac{\partial}{\partial b}\lambda_0)\frac{\partial b}{\partial r} + (\frac{\partial}{\partial c}\lambda_0)\frac{\partial c}{\partial r}.$$

The desired estimates now follow immediately from the estimates for $\big((\alpha\beta)_s/\alpha\beta\big)_r$ given in Lemma 14, together with the continuity of $\frac{\partial}{\partial b}\lambda_0$, $\frac{\partial}{\partial c}\lambda_0$.

Let v_0 be the normalised eigenvector corresponding to λ_0 in the Sturm-Liouville problem (Equations 45, 46). It is straightforward to show that for fixed r and for $\lambda_0 \geq 0$,

$$v_0(s) = \Big(1 + \frac{\sin 2\sqrt{\lambda_0}}{2\sqrt{\lambda_0}}\Big)^{-1/2}\cos(\sqrt{\lambda_0}s). \tag{47}$$

For fixed $r > 1$, let P be the orthogonal projection of $L^2\big((-1,1), ds\big)$ onto v_0. Thus,

$$Pu = v_0\int_{s=-1}^{1} u(s)v_0(s)\,ds. \tag{48}$$

The operator $\chi_R P$ naturally defines a bounded operator $L^2(\Omega, \omega dx\,dy) = W^0$. In what follows we write χ instead of χ_R for notational simplicity.

Since $\chi D_s^2 Pu = \chi\lambda_0 Pu$, it follows that the operator $\chi_R D_s^2 P$ extends to be bounded on $L^2(\Omega, \omega dx\,dy)$, satisfying the estimate

$$\|\chi_R D_s^2 Pu\|_{L^2} \leq C\|f(r)^2 r^{-1}\chi_R u\|_{L^2}, \tag{49}$$

for a positive constant C independent of R.

LEMMA 8. *The following operators extend to bounded operators:*

$$r^2\chi[D_r^j, P]\colon W^{j-1} \longrightarrow W^0.$$

Furthermore, the operator bounds can be chosen independent of R.

ii) P extends to a bounded operator from W^j to W^j for all positive j.

PROOF. Assume first that $u \in \Psi$, and is hence differentiable. Then, for $r > 1$,

$$[D_r, P]u = \frac{\partial v_0}{\partial r}\int_{s=-1}^{1} u(s)v_0(s)\,ds + v_0\int_{s=-1}^{1} u(s)\frac{\partial v_0}{\partial r}(s)\,ds. \tag{50}$$

In the notation of Lemma 7,

$$\frac{\partial v_0}{\partial r} = \frac{\partial v_0}{\partial b}\frac{\partial b}{\partial r} + \frac{\partial v_0}{\partial c}\frac{\partial c}{\partial r}.$$

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It follows from Lemma 14 and the boundedness of $\partial v_0/\partial b$, $\partial v_0/\partial c$ with respect to r that the mapping

$$r \longrightarrow r^2 \frac{\partial v_0}{\partial r}$$

is bounded from $[1,\infty)$ to $L^2\big((-1,1), ds\big)$. In what follows in this lemma, we denote $\partial v_0/\partial r$ by $(v_0)_r$, $\|u\|_{L^2((-1,1),ds)}$ by $\|u\|$, and $\|u\|_{L^2(\Omega,\omega dx\, dy)}$ by $\|u\|_{L^2}$. Let

$$\|(v_0)_r(r,*)\| = r^{-2}M(r);$$

thus M is a bounded function. Then

$$\begin{aligned}
\|r^2\chi(v_0)_r\int_{s=-1}^{1} uv_0\,ds\|_{L^2}^2 &= \int_\Omega r^4\chi^2|(v_0)_r|^2|\int_{s=-1}^{1} uv_0\,ds|^2\,d\tau\,dr\\
&\le \int_{r=0}^{\infty}\int_{\tau=-1}^{1} r^4\chi^2|(v_0)_r|^2\|u(r,*)\|^2\|v_0(r,*)\|^2 d\tau\,dr\\
&= \int_{r,\tau} r^4\chi^2|(v_0)_r|^2\|u(r,*)\|^2\,d\tau\,dr\\
&= 2\int_r \chi^2M(r)^2\|u(r,*)\|^2\,dr\\
&= 2\int_{r,s}\chi^2M(r)^2|u(r,s)|^2\,ds\,dr.
\end{aligned}$$

This gives the desired bound on $\|\chi(v_0)_r\int_{-1}^{1} uv_0\|_{L^2}$. The bound on $\|\chi v_0\int_{-1}^{1} u(v_0)_r\|_{L^2}$ is proven similarly. Thus $\chi r^2[D_r,P]$ is bounded uniformly when applied to elements of Ψ. Uniform boundedness on W^0 now follows by density.

The operator A for the Mourre estimate is defined by

$$A = r\chi_R PD_r + D_rP\chi_R r. \tag{51}$$

Since P is self-adjoint, it is clear that A is skew-adjoint.

LEMMA 9. *Let H and H_0 be as in Section 2, and A is in Equation 51. Then Hypotheses 1 and 3 for the Mourre estimate are satisfied.*

PROOF. It is clear that the set Ψ is contained in $\mathrm{Dom}(A)\cap W^2$, so Hypothesis 1 is satisfied.

To estimate $[H_0,A]$ it suffices to estimate $[D_r^2,A]$ and $[f^{-2}D_s^2,A]$.

We begin with $[f^{-2}D_s^2,A]$. Note that when applied to functions in $W^2=\mathrm{Dom}(H_0)$, the operator P commutes with the operator D_s^2. Also, it is clear from Equation 50 that $[\chi_R P, D_r]$ commutes with multiplication by a function of r. Thus we calculate

$$[f^{-2}D_s^2,A] = -2r\chi_R(f^{-2})^{(1)}PD_s^2. \tag{52}$$

Since f satisfies Equations 3, 4,

$$(f^{-2})^{(1)} = O(f^{-2}r^{-1/2}) \tag{53}$$

This and Equation 49, show that the operator $[f^{-2}D_s^2,A]$ is bounded from W^2 to W^0.

Boundedness for $[D_r^2,A]$ follows by an exercise in differentiation and use of Lemmas 3 and 8. The details are omitted.

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LEMMA 10. *The operator* $\big[[H_0, A], A\big]$ *extends from* $\mathrm{Dom}(H_0A^2)$ *to a bounded operator from* W^2 *to* W^{-2}.

The proof of this lemma involves no new ideas and hence is omitted. We remark in passing that one of the terms that arises in the calculations is $X^2(f^{-2})^{(2)}PD_s^2$, which fails to be bounded when $f(r) = \exp(-x^\alpha)$, $\alpha \in (1/2, 1)$. It is precisely here that Equation 4 is necessary for Theorem 1 to hold, rather than some weaker estimates for $(f^{(1)}/f)^{(j)}$.

To prove Mourre hypotheses 2, 4, and 5, the following estimates are needed:

LEMMA 11. *Let* $E = H - H_0$. *Then:*

i) $[E, A]$ *extends from* Ψ *to a bounded operator from* W^2 *to* W^{-1}.

ii) there exists $\epsilon > 0$ *such that* $r^\epsilon[E, A]$ *extends from* Ψ *to a bounded operator from* W^2 *to* W^{-1}.

iii) $\big[[E, A], A\big]$ *extends from* Ψ *to a bounded operator from* W^2 *to* W^{-2}.

Note that parts i) and iii) of this lemma, along with Lemmas 9 and 10, immediately prove Mourre Hypotheses 2 and 4.

PROOF. Recall that for large r, the differential operator E is given by

$$E = (1 - \alpha^{-2})D_r^2 + \big(f(r)^{-2} - \beta^{-2}\big)D_s^2 + 2\frac{\alpha_r}{\alpha^3}D_r + 2\frac{\beta_s}{\beta^3}D_s + V,$$

with V given by Equation 13.

We cite the bounds obtained in Lemma 14 of the appendix:

- $1 - \alpha^{-2} = O(r^{-2})$,
- $f(r)^{-2} - \beta^{-2} = O(r^{-1})$,
- $\alpha_r/\alpha^3 = O(r^{-2})$,
- $\beta_s/\beta^3 = O\big((f^{(1)}/f)^2\big)$,

(54) • $V = O(r^{-1}), \quad V_r = O(r^{-2}), \quad V_s = O(r^{-2}), \quad V_{rr} = O(r^{-2})$.

As was noted in the introduction, the perturbation coefficients here have weaker decay rates than those hypothesised in [6]. To prove boundness with the weaker decay rates, the following idea is added to the methods in [6]. To fix ideas, consider the term $[V, P]$, which arises in our estimate for $[E, A]$. In [6], the decay rate of the coefficients of this integro-differential operator are obtained simply from the decay rates of each PV and VP. In our proof, we observe that as $r \longrightarrow \infty$, V becomes constant in s in the sense that $V_s = O(r^{-2})$, while P converges to the orthogonal projection onto constants functions on $(-1, 1)$. Consequently, $PV - VP$ decays more rapidly than the individual terms PV and VP.

For notational convenience, let $X = \chi r$. We begin with the proof of part ii). Part i) will obviously follow.

We first prove bounds on $r^\epsilon(1-\alpha^{-2})D_r^2XPD_r$. This term can be bounded by commuting D_r^2 to the right, and then applying Lemmas 3, 6, and the above bound on $\alpha^{-2} - 1$. The other terms in $r^\epsilon[(1 - \alpha^{-2})D_r^2, A]$ can be treated similarly.

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The bound on $r^\epsilon[\alpha_r/\alpha^3 D_r, A]$ is proven similarly.

Next, we prove the bound on $r^\epsilon(f^{-2}-\beta^{-2})D_s^2XPD_r$; the proof of the bound on the other terms in $r^\epsilon[(f^{-2}-\beta^{-2})D_s^2, A]$ is similar. Note that X commutes with D_s^2. Thus

$$r^\epsilon(f^{-2}-\beta^{-2})D_s^2XPD_r = r^\epsilon(f^{-2}-\beta^{-2})X(D_s^2P)D_r.$$

Since

$$f^{-2}-\beta^{-2} = O(r^{-1}),$$

the estimate for D_s^2P (Equation 49), shows that the operator $r^\epsilon(f^{-2}-\beta^{-2})XD_s^2P$ extends to a bounded map on W^0. Thus the operator

$$r^\epsilon(f^{-2}-\beta^{-2})D_s^2XPD_r$$

is bounded from W^1 to W^0, and hence from W^2 to W^{-2}.

The proof of the bound for $r^\epsilon[V, A]$ is slightly more difficult, because individual terms in the expansion of the commutator are not necessarily bounded from W^2 to W^0. We consider the term $r^\epsilon[V, XPD_r]$. The other term in $r^\epsilon[V, A]$ is treated similarly. We have

$$r^\epsilon(VXPD_r - XPD_rV) = r^\epsilon X[V, P]D_r - Pr^\epsilon XV_r.$$

By Equation 54, the second term is bounded from W^0 to W^0, and hence from W^2 to W^{-1}.

To bound the first term, we first note that

$$V(r, s) = V(r, 0) + sV_s(r, z),$$

with z between 0 and s. Note that P commutes with $V(r, 0)$. Thus for fixed r, we have

$$[P, V]u = P(sV_su) - sV_sPu.$$

(Here we have suppressed the dependence on the variables r, s, z for simplicity.) It follows that

$$\int_{s=-1}^{1} |[P, V]u|^2\, ds \le 2\|V_s(r, *)\|^2_{L^\infty(-1,1)} \int_{s=-1}^{1} |u(r, s)|^2\, ds.$$

It now follows from the estimate on V_s that

$$\|r^\epsilon X[V, P]u\|_{L^2} \le C\|\chi_R u\|_{L^2},$$

for some constant C. Thus $r^\epsilon X[V, P]D_r$ is bounded from W^2 to W^{-1} as desired.

The bound on $r^\epsilon[\beta_s/\beta^3 D_s, A]$ is proven as follows. Consider the term

$$r^\epsilon\left(XPD_r\frac{\beta_s}{\beta^3}D_s\right);$$

the other terms will be treated similarly. Commuting $r^\epsilon X$ to the right, we obtain

$$r^\epsilon XPD_r\frac{\beta_s}{\beta^3}D_s = PD_r r^\epsilon X\frac{\beta_s}{\beta^3}D_s - P(r^\epsilon X)_r\frac{\beta_s}{\beta^3}D_s.$$

To bound the first of the two terms on the right hand side, note that

$$PD_r r^\epsilon X\frac{\beta_s}{\beta^3}D_s = PD_r \circ r^\epsilon X\frac{\beta_s}{\beta^3}f\circ f^{-1}D_s.$$

By Lemma 3, $f^{-1}D_s$ is bounded from W^2 to W^0. By the estimate $\frac{\beta_s}{\beta^3} = O(f^{-1}r^{-2})$, the term $r^\epsilon X\frac{\beta_s}{\beta^3}f$ is bounded from W^0 to W^0. Finally, by Lemmas 3 and 8, PD_r is bounded from W^0 to W^{-1}. This gives the desired bounds on the first term. The second term is bounded similarly.

This completes the proof of part i) of the lemma.

The proof of part ii) involves no new ideas and hence is omitted.

5. **Proof of Proposition 1.** In this section we will complete the proof of Proposition 1. The proof of this goes largely along the lines of the corresponding result in [6] (see Lemmas 2.3 and 1.8 of that paper) except at one step. We will sketch the proof except at the one step which we prove in detail.

STEP 1. We show

$$[H_0, A] = 4P\rho H_0\rho P + T.$$

Here $\rho(r) = \sqrt{\chi(r)}$, and T is an operator which, when composed with E_I, is compact. This result is proven using Lemmas 3 and 5, and the hypotheses on f, along with Equation 49.

STEP 2. Using Lemmas 3 and 5 we show that

$$h(H_0)[H_0, A]h(H_0) \geq 4h(H_0)H_0h(H_0) - C\|h(H_0)(P-1)\rho_R\| + K,$$

with K compact and C a positive constant independent of R.

STEP 3. We show that $\|h(H_0)(P-1)\rho_R\|$ can be made arbitrarily small by shrinking the support of h and letting R go to infinity. We prove

A) $\lim_{R\to\infty}\lim_{t\to\infty}\sup\Big\|\rho_R(P-1)h\big(t(H_0-z_0)\big)\Big\| = 0,$

B) $\lim_{R\to\infty}\lim_{t\to\infty}\sup\Big\|\rho_R(P-1)\overline{h}\big(t(H_0-z_0)\big)\Big\| = 0.$

Using a Stone-Weierstrass argument, this reduces to proving A) and B) for $h(x) = (x-z)^{-1}$. It is here that we depart somewhat from the proof in [6]. We have

$$\begin{aligned}\rho_R^2(1-P)^2\big(t(H_0-z_0)-z\big) + \big(t(H_0-z_0)-\overline{z}\big)(1-P)^2\rho_R^2 \\ = 2\rho_R(1-P)\big(t(H_0-z_0)-\mathrm{Re}(z)\big)(1-P)\rho_R \\ +t\rho_R(1-P)[\rho_R(1-P),(H_0-z_0)] \\ +t[(H_0-z_0),(1-P)\rho_R](1-P)\rho_R.\end{aligned}$$

Denote the sum of the last two terms as $t\tilde{T}$.

Let λ_2 be the second smallest eigenvalue of the Sturm-Liouville problem (Equation 8), with a given by Equation 15. By Lemma 6, $\lambda_2 > 1$ for $|a| < 1$. It follows that for r sufficiently large,

$$\rho_R(1-P)(H_0-z_0)(1-P)\rho_R \geq 2\rho_R(1-P)^2\rho_R.$$

This shows that for $|\operatorname{Re}(z/t)| < 1$,

$$2\rho_R(1-P)\big(t(H_0 - z_0) - \operatorname{Re}(z)\big)(1-P)\rho_R \geq 2t\rho_R^2(1-P)^2.$$

Thus we have

$$\rho_R^2(1-P)^2\big(t(H_0 - z_0) - z\big) + \big(t(H_0 - z_0) - \overline{z}\big)(1-P)^2\rho_R^2 \geq 2t\rho_R^2(1-P)^2 + t\tilde{T}. \tag{55}$$

Let $B = (t(H_0 - z_0) - z)^{-1}$. Multiplying Equation 55 by B^* on the left and by B on the right, we obtain

$$B^*\rho_R^2(1-P)^2 + (1-P)^2\rho_R^2 B \geq 2tB^*\rho_R^2(1-P)^2 B + tB^*\tilde{T}B.$$

Applying this quadratic form inequality to a function u of norm one, one obtains

$$\begin{aligned} 2t\|\rho_R(1-P)Bu\|_{L^2}^2 &\leq 2\|\rho_R^2(1-P)^2 Bu\|_{L^2} + |t\langle \tilde{T}Bu, Bu\rangle| \\ &\leq 2\|\rho_R(1-P)Bu\|_{L^2} + t|\langle \tilde{T}Bu, Bu\rangle|. \end{aligned} \tag{56}$$

Applying the quadratic formula to Equation 56, one obtains

$$\|\rho_R(1-P)Bu\|_{L^2} \leq \frac{2 + \sqrt{4 + 8t^2|\langle \tilde{T}Bu, Bu\rangle|}}{4t}.$$

Thus, for any $\epsilon > 0$, one can choose M_0 such that for $t > M_0$,

$$\|\rho_R(1-P)Bu\|_{L^2} \leq \frac{1}{\sqrt{2}}|\langle \tilde{T}Bu, Bu\rangle| + \epsilon.$$

Fix $t > M_0$. We now show that the term $\langle \tilde{T}Bu, Bu\rangle$ vanishes as $R \longrightarrow \infty$. We expand $\tilde{T}$:

$$\begin{aligned} \tilde{T} = &-2(\rho_R')^2(1-P)^2 + 2\rho_R'\rho_R[D_r, P](1-P) - \rho_R^2(1-P)[D_r^2, P] \\ &+\rho_R[D_r^2, P]\rho_R(1-P). \end{aligned} \tag{57}$$

By construction of $\rho_R = \sqrt{\chi_R}$, $\|\rho_R^{(1)}\|_\infty \longrightarrow 0$ as $R \longrightarrow \infty$ (see Equation 24). Thus the first two terms in Equation 57 tend to 0 in norm as $R \longrightarrow \infty$. Lemma 6 shows that the coefficients of $\rho_R[D_r^2, P]$ tend to 0 as $R \longrightarrow \infty$, and thus the last two terms will tend to 0 in norm after being multiplied by B^* on the left and B on the right. In particular, for sufficiently large R,

$$\frac{1}{\sqrt{2}}(\langle \tilde{T}Bu, Bu\rangle) < \epsilon.$$

This proves

$$\|\rho_R(1-P)Bu\|_{L^2} \leq 2\epsilon.$$

Equations A) and B) now follow.

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STEP 4. Now choose a positive number $\epsilon > 0$ satisfying $z_0 - \epsilon > 0$. Choose R sufficiently large, and h with support of h sufficiently close to z_0, so that

$$\|h(H_0)(P-1)\rho_R\|_{L^2} < \epsilon/2, \tag{58}$$

$$h(H_0)H_0h(H_0) - z_0h(H_0)^2 \geq -\frac{\epsilon}{2}h(H_0)^2. \tag{59}$$

Hence one obtains

$$h(H_0)[H_0, A]h(H_0) \geq 4(z_0 - \epsilon/2)h(H_0)^2 - 2\epsilon + K. \tag{60}$$

STEP 5. Recall we defined E_I as the spectral projection of H onto the interval I. We use Lemmas 5 and 11 and Proposition 2 to replace $h(H_0)$ by E_I, and H_0 by H, in Equation 60, absorbing the resulting, compact error terms into K. This completes the proof of Mourre Hypothesis 5, and hence of Proposition 1.

5.1. *Appendix* Recall that r, s are given by

$$s = \frac{y}{f(x)}, \quad r = F^{-1}\Big(\frac{s^2f(x)^2}{2} + F(x)\Big), \tag{61}$$

with $F^{(1)}(x) = f(x)/f^{(1)}(x)$.

LEMMA 12. *A) The function $x = x(r,s)$ satisfies*

$$x - r = O\big(f(r)f^{(1)}(r)\big),$$

the estimate being uniform in s.

B) One has

$$\frac{f^{(1)}(x)}{f(x)}\frac{f(r)}{f^{(1)}(r)} - 1 = O(|f(r)f^{(2)}(r)| + |(f^{(1)}|^2),$$

the estimate uniform in s.

PROOF. To prove part A), we proceed as follows. First, since F, F^{-1} are decreasing, Equation 61 implies the following inequalities:

$$r(x,y) \leq x, \quad r \leq x(r,s).$$

Now since

$$r = F^{-1}\big(y^2/2 + F(x)\big),$$

it follows by Taylor's theorem that

$$\begin{aligned} x - r &= (F^{-1})^{(1)}\big(F(\tilde{x})\big)\frac{y^2}{2}, \quad \tilde{x} \in (r,x) \\ &= \frac{f^{(1)}}{f}(\tilde{x})\frac{s^2f(x)^2}{2}. \end{aligned}$$

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Since f is a decreasing function, it follows that

$$|x-r| \le \frac{f^{(1)}}{f}(\tilde{x})\frac{s^2 f(r)^2}{2}. \tag{62}$$

Next, we estimate $\frac{f^{(1)}}{f}(\tilde{x})$. Note that

$$\frac{f^{(1)}}{f}(\tilde{x}) = \frac{f^{(1)}}{f}(r) + \left(\left(\frac{f^{(1)}}{f}\right)^{(1)}(\tilde{r})\right)(\tilde{x}-r), \quad \tilde{r} \in (r,\tilde{x}).$$

Note that

$$|\tilde{x}-r| \le |x-r| = \left|\frac{f^{(1)}}{f}(\tilde{x})\frac{y^2}{2}\right|.$$

It follows that

$$\frac{f^{(1)}}{f}(\tilde{x}) = \frac{f^{(1)}}{f}(r) + o\left(\frac{f^{(1)}}{f}(\tilde{x})\right),$$

and hence

$$\frac{f^{(1)}}{f}(\tilde{x})\big(1+o(1)\big) = \frac{f^{(1)}}{f}(r).$$

This, along with Equation 62, proves part A).

Part B) easily follows from Taylor's theorem and part A).

REMARK. Part A) of Lemma 12 will allow us to replace x by r in the estimates we make below.

We now estimate the coefficients of $E = H - H_0$. Recall the formulae for H, H_0 for large r:

$$H = -\frac{1}{\alpha^2}D_r^2 - \frac{1}{\beta^2}D_s^2 + 2\frac{\alpha_r}{\alpha^3}D_r + 2\frac{\beta_s}{\beta^3}D_s + V, \tag{63}$$

with

$$V = -\frac{1}{2\alpha^4\beta^2}(\beta_r^2\alpha^2/2 + 5\alpha_r^2\beta^2/2 - \alpha\alpha_{rr}\beta^2 - \alpha^2\beta\beta_{rr} + \alpha\alpha_r\beta\beta_r)$$

$$- - \frac{1}{2\alpha^2\beta^4}(\alpha_s^2\beta^2/2 + 5\alpha^2\beta_s^2/2 - \alpha\alpha_{ss}\beta^2 - \alpha^2\beta\beta_{ss} + \alpha\alpha_s\beta\beta_s); \tag{64}$$

$$H = -D_r^2 - f(r)^{-2}D_s^2. \tag{65}$$

It will be convenient to adopt the following notation:

$$Z = 1 + s^2(f^{(1)})(x)^2.$$

Then we have the following formulae for α, β:

$$\alpha(r,s) = \left(\frac{f^{(1)}(x)}{f(x)}\right)\left(\frac{f(r)}{f^{(1)}(r)}\right)Z^{-1/2}, \tag{66}$$

$$\beta(r,s) = f(x)Z^{-1/2}. \tag{67}$$

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Note that by Equation 3,

$$\lim_{r\longrightarrow\infty} Z = 1. \tag{68}$$

From this and Lemma 12 part B), it is immediate that $\lim_{r\longrightarrow\infty}\alpha = 1$. One calculates

$$1-\frac{1}{\alpha^2} = \left(\frac{f^{(1)}(x)}{f(x)}\frac{f(r)}{f^{(1)}(r)}\right)^{-2}\left(\left(\frac{f^{(1)}(x)}{f(x)}\frac{f(r)}{f^{(1)}(r)}\right)^2 - 1 - s^2 f^{(1)}(x)^2\right).$$

Thus by Lemma 12 part B), followed by Equation 3 and Lemma 12 part A),

$$\begin{aligned} 1-\frac{1}{\alpha^2} &= O(f^{(1)}(x)^2) + O\big(f(r)f^{(2)}(r)\big) \\ &= O(r^{-3}). \end{aligned} \tag{69}$$

Next we estimate $f^{-2}-\beta^{-2}$. Note that Equation 67 along with Taylor's formula give

$$\frac{1}{f(r)^2}-\frac{1}{\beta^2} = \frac{\big(f(x)+f(r)\big)(f^{(1)}(r)\epsilon) - s^2 f(r)^2 f^{(1)}(x)^2}{f(r)^2 f(x)^2},$$

with $|\epsilon| \leq |x-r|$. Applying Lemma 12 part A) followed by Equation 4, one obtains

$$\begin{aligned} 1/f(r)^2 - 1/\beta^2 &= O(f^{(1)}(r)^2/f(r)^2) \\ &= O(r^{-1}). \end{aligned} \tag{70}$$

Before estimating the other coefficients in E, we will need the following.

LEMMA 13. *i)* $\partial x/\partial r = \frac{f^{(1)}(x)}{f(x)}\frac{f(r)}{f^{(1)}(r)}Z^{-1}$
ii) $\partial x/\partial s = -sf(x)f^{(1)}(x)Z^{-1}$.

PROOF. By Equation 61, the derivative J of the transformation $(x,y)\longrightarrow(r,s)$ is given by

$$J = \begin{pmatrix} \frac{f^{(1)}(r)}{f(r)}\frac{f(x)}{f^{(1)}(x)} & sf(x)\frac{f^{(1)}(r)}{f(r)} \\ -\frac{sf^{(1)}(x)}{f(x)} & \frac{1}{f(x)} \end{pmatrix}.$$

Hence

$$J^{-1} = Z^{-1}\begin{pmatrix} \frac{f^{(1)}(x)}{f(x)}\frac{f(r)}{f^{(1)}(r)} & -sf(x)f^{(1)}(x) \\ s\frac{f^{(1)}(x)^2}{f(x)}\frac{f(r)}{f^{(1)}(r)} & f(x) \end{pmatrix}.$$

The lemma follows.

An exercise in differentiation, using Equations 2, 3, 4, 5 for f, and Lemma 11 part A), now shows that

$$\begin{aligned} \partial^2 x/\partial r^2 &= O(|(f^{(1)})^2| + |ff^{(1)}| + |f| + |f^{(1)}f^{(2)}| + |ff^{(2)}|) = O(r^{-1}) \\ \partial^2 x/\partial s^2 &= O(|ff^{(1)}| + |f^2 f^{(1)} f^{(2)}|) = O(r^{-3}) \\ \partial^3 x/\partial s^3 &= O(|f(f^{(1)})^3| + |f^2(f^{(1)})^2| + |f^2 f^{(1)} f^{(2)}| + |f^3 (f^{(1)})^2 f^{(3)}|) = O(r^{-3}) \\ \partial^2 x/\partial r\partial s &= O(|ff^{(2)}| + |(f^{(1)})^2|) = O(r^{-2}) \\ \partial^3 x/\partial r^2\partial s &= O(|f^{(1)}f^{(2)}| + |ff^{(3)}|) = O(r^{-2}). \end{aligned}$$

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The following estimates will also be useful:

$$Z_s = O\big((f^{(1)})^2\big)$$
$$Z_{ss} = O(|f^{(1)}|^2 + |f^3(f^{(1)})^4 f^{(4)}|)$$
$$Z_{rs} = O(f^{(1)}f^{(2)})$$
$$Z_r = O(f^{(1)}f^{(2)})$$
$$Z_{rr} = O\big(|f^2| + |(f^{(1)})^2| + |ff^{(1)}|\big)$$

We now estimate α_r/α^3 as $r \longrightarrow \infty$. It is an exercise to show

$$\frac{\partial}{\partial r}\Big(\frac{f^{(1)}(x)}{f(x)}\frac{f(r)}{f^{(1)}(r)}\Big) = O(|f^{(1)}|^2 + |ff^{(3)}| + |f^{(1)}f^{(2)}| + |ff^{(2)}|).$$

It now follows from this, the estimate above on Z_r, and Equations 2, 3, 4, 5 for f, that

$$\frac{\alpha_r}{\alpha^3} = O(|f^{(1)}|^2 + |ff^{(3)}| + |f^{(1)}f^{(2)}| + |f^{(2)}|).$$

It is a simple calculation to show, using the assumptions on f of Equation 4, that

$$f^{(3)} = O(r^{-1}).$$

From this and the assumptions on f made in Equation 3, it follows that

$$\frac{\alpha_r}{\alpha^3} = O(r^{-2}). \tag{71}$$

To estimate β_s/β^3, note that by Lemma 12 and the estimate above for Z_s,

$$\frac{\partial}{\partial s}(f(x)Z^{-1/2}) = O(ff'^2).$$

It follows immediately that

$$\frac{\beta_s}{\beta^3} = O\big((f^{(1)}/f)^2\big) = O(r^{-1}). \tag{72}$$

LEMMA 14. *The coefficients of $E = H = H_0$ all vanish as $r \longrightarrow \infty$. Furthermore,*

i) $\alpha^{-2} - 1 = O(r^{-2})$,
ii) $\beta^{-2} - f(r)^{-2} = O(r^{-1})$,
iii) $\alpha_r/\alpha^3 = O(r^{-2})$,
iv) $\beta_s/\beta^3 = O(r^{-1})$, $(\beta_s/\beta^3)_r = O(r^{-2})$, $(\beta_s/\beta^3)_{rr} = O(r^{-2})$, $(\beta_s/\beta^3)_s = O(r^{-2})$.
v) $V = O(r^{-1})$, $V_r = O(r^{-3/2})$, $V_s = O(r^{-2})$, $V_{rr} = O(r^{-2})$.
vi) $\frac{\partial^k}{\partial s^i \partial r^j}\big((\alpha\beta)_s/\alpha\beta\big) = O(f^2)$ *for* $i + j = k \leq 3$.

PROOF. Parts i)–iii) of the lemma have been proven (Equations 69, 70, 71), as has the first estimate in part iv) (Equation 72). The reminder of the lemma is a long but straightforward application of differentiation and the estimates above for the derivatives of x and Z. The proof is omitted.

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