

J-NONEXPANSIVE MAPPINGS IN UNIFORM SPACES AND APPLICATIONS

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The purpose of the paper is to introduce a class of “*j*-nonexpansive” mappings and to prove fixed point theorems for such mappings. They naturally arise in the existence theory of functional differential equations. These mappings act in spaces without specific geometric properties as, for instance, uniform convexity. Critical examples are given.

1. INTRODUCTION

The main purpose of the present paper is to introduce a class of “*j*-nonexpansive mappings” and to prove fixed point theorems for such mappings. They naturally arise in the existence theory of functional differential equations.

It is well known that Edelstein [4] has been successful in replacing Banach’s condition $d(Tx, Ty) \leq \kappa d(x, y)$ ($0 < \kappa < 1$) by the weaker one $d(Tx, Ty) < d(x, y)$, $x \neq y$ ((X, d) is a complete metric space and $T: X \rightarrow X$). In the case when $(X, \|\cdot\|_X)$ is a normed space possessing a normal structure, Browder [2], Kirk [7], Gohde [6] have proved the existence of a fixed point under nonexpansive condition $\|Tx - Ty\|_X \leq \|x - y\|_X$. There are many papers dealing with these problems (see, [8, 9]). Other authors have generalised some results to the case of locally convex and uniform spaces [5, 13, 12]. Unfortunately, there are no applications of the mentioned fixed point theorems (see also [3]). That is why we shall consider a class of *j*-nonexpansive mappings in the spaces without specific geometric properties as, for instance, a uniform convexity. Such assumptions restrict the class of functions in which we can find a solution, and hence the spaces L^1 and L^∞ do not have normal structure. With a view to applications it is more useful to introduce supplementary conditions on T instead of requiring that the space X possesses certain geometric properties.

Let (X, \mathcal{A}) be a separated uniform space whose uniformity is generated by a saturated family of pseudometrics $\mathcal{A} = \{d_\alpha(x, y) : \alpha \in A\}$, A being an index set [14]. Let $j: A \rightarrow A$ be a mapping of an index set into itself and $j^\kappa(\alpha) = j(j^{\kappa-1}(\alpha))$ stands for the κ th iterate of j and $j^0(\alpha) = \alpha$, $\alpha \in A$. Let $\{\Phi_\alpha(t) : \alpha \in A\}$ ($\equiv \Phi$) be a family

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of contractive functions possessing properties given in [1]. We shall call a mapping $T: X \rightarrow X$ Φ -contractive if

$$d_\alpha(Tx, Ty) \leq \Phi_\alpha(d_{j(\alpha)}(x, y))$$

for every $x, y \in X$ and $\alpha \in A$. In order to weaken the contractive condition we shall consider j -contractive

$$d_\alpha(Tx, Ty) < d_{j(\alpha)}(x, y)$$

and j -nonexpansive

$$d_\alpha(Tx, Ty) \leq d_{j(\alpha)}(x, y)$$

mappings. The nonexpansive mappings (see [5, 13, 12]) mentioned earlier correspond to the case when j is an identity mapping.

For further motivation we shall consider Myshkis's example in such a "bad" space as $L_{loc}^\infty(\mathbb{R}^1)$:

$$x(t) = x(t-1) + 1, t > 0; \quad x(t) = \varphi(t) = \begin{cases} 0, & -1 \leq t < 0 \\ 1, & t = 0. \end{cases}$$

Step by step we find a solution

$$x(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 2, & 1 \leq t < 2 \\ \vdots & \vdots \\ n, & n-1 \leq t < n \\ \vdots & \vdots \end{cases}$$

Obviously this solution belongs to $L_{loc}^\infty(\mathbb{R}^1)$. Let us form an operator $T: L_{loc}(\mathbb{R}^1) \rightarrow L_{loc}(\mathbb{R}^1)$ by the right-hand side of the above equation. Consider the space $L_{loc}^\infty(\mathbb{R}^1)$ with the topology of uniform convergence on the compact subsets of $\mathbb{R}^1 = (-\infty, \infty)$, that is a saturated family of seminorms in $\mathcal{A} = \{\|f\|_K : K \text{ runs over all compact subsets of } \mathbb{R}^1\}$, where $\|f\|_K = \text{ess sup}\{|f(t)| : t \in K\}$. Then \mathcal{A} consists of all compact subsets of \mathbb{R}^1 . The mapping $j: \mathcal{A} \rightarrow \mathcal{A}$ is defined in the following way: $j(K) = \{t-1 : t \in K\}$. It is easy to verify that T is a j -nonexpansive operator, that is, $\|Tf - T\bar{f}\|_K \leq \|f - \bar{f}\|_{j(K)}$.

But the above constructed solution is obviously a fixed point of the j -nonexpansive operator T and this solution belongs to $L_{loc}^\infty(\mathbb{R}^1)$. It poses the following question: what are the conditions for the nonlinear j -nonexpansive operator to have a fixed point in L_{loc}^∞ ? This fixed point will be a solution of the corresponding nonlinear functional differential equation whose right-hand side generates T .

2. FIXED POINT THEOREMS

Now we shall formulate the main results:

THEOREM 1. *Let $T: X \rightarrow X$ be a *j*-nonexpansive mapping. If there exists an element $x_0 \in X$ such that:*

1. *the sequence $\{T^n x_0\}_{n=0}^\infty$ contains a subsequence which converges to a point of X ;*
2. *for every $\alpha \in A \lim_{n \rightarrow \infty} d_{j^n(\alpha)}(x_0, T x_0) = 0$:*

then T has a fixed point $\xi \in X$.

PROOF: Let $\{T^{n_\kappa} x_0\}_{\kappa=1}^\infty$ be a subsequence of $\{T^n x_0\}_{n=0}^\infty$ whose limit is $\xi \in X$. It is easy to verify that $\lim_{\kappa \rightarrow \infty} T^{n_\kappa+1} x_0 = \xi$.

Assume that $T\xi \neq \xi$. This means there exists $\alpha_0 \in A$ such that $d_{\alpha_0}(T\xi, \xi) > 0$. Let us choose $\alpha \in j^{-1}(\alpha_0) = \{\alpha \in A : j(\alpha) = \alpha_0\}$ and consider the sequence $\{d_{j(\alpha)}(T^{n_\kappa+1} x_0, T^{n_\kappa} x_0)\}$. Two cases are possible:

1. Finitely many members of the above sequence are different from zero. Then $\lim_{k \rightarrow \infty} d_{j(\alpha)}(T^{n_\kappa+1} x_0, T^{n_\kappa} x_0) = d_{\alpha_0}(T\xi, \xi) = 0$ — a contradiction.
2. Infinitely many members of the above sequence are different from zero. Denote by $\{T^{n_{\kappa_s}} x_0\}_{s=1}^\infty$ a subsequence for which $d_{j(\alpha)}(T^{n_{\kappa_s}} x_0, T^{n_{\kappa_s}+1} x_0) > 0$ ($s = 1, 2, \dots$). Then

$$d_{j(\alpha)}(T^{n_{\kappa_s}+1} x_0, T^{n_{\kappa_s}} x_0) \leq d_{j^2(\alpha)}(T^{n_{\kappa_s}} x_0, T^{n_{\kappa_s}-1} x_0) \leq \dots \leq d_{j^{n_{\kappa_s}+1}(\alpha)}(T x_0, x_0).$$

But the subsequence $d_{j^{n_{\kappa_s}+1}(\alpha)}(T x_0, x_0)$ tends to zero as $s \rightarrow \infty$. Therefore $0 \leq d_{\alpha_0}(T\xi, \xi) = d_{j(\alpha)}(T\xi, \xi) \leq 0$. The obtained contradiction implies $T\xi = \xi$, which proves the theorem.

□

THEOREM 2. *If, in addition, we assume that:*

3. *for every $x, y \in X$ and $\alpha \in A, \lim_{n \rightarrow \infty} d_{j^n(\alpha)}(x, y) = 0$
then the fixed point of T is unique and $\xi = \lim_{n \rightarrow \infty} T^n x_0$.*

PROOF: Indeed, let us assume that $x \neq y$ and $Tx = x, Ty = y$. Then for every $\alpha \in A$ we have

$$d_\alpha(x, y) = d_\alpha(Tx, Ty) \leq d_{j(\alpha)}(x, y) \leq \dots \leq d_{j^n(\alpha)}(x, y) \xrightarrow{n \rightarrow \infty} 0$$

which implies uniqueness of the fixed point.

Finally, we have

$$d_\alpha(T^n x_0, \xi) = d_\alpha(T^n x_0, T^n \xi) \leq d_{j^n(\alpha)}(x_0, \xi) \xrightarrow{n \rightarrow \infty} 0$$

which proves Theorem 2. □

Let us note that condition 1 of Theorem 1 can be satisfied in the applications only when T is a compact operator. That is why we shall formulate a fixed point theorem without compactness condition. Therefore condition 2 must be stronger.

THEOREM 3. *Let $T: X \rightarrow X$ be a j -nonexpansive mapping and suppose there exists $x_0 \in X$ such that $\sum_{n=0}^{\infty} d_{j^{n(\alpha)}}(x_0, Tx_0)$ is convergent. Then T has a fixed point $\xi \in X$ and $\lim_{n \rightarrow \infty} T^n x_0 = \xi$.*

PROOF: As usual we form a sequence $\{T^n x_0\}_{n=0}^{\infty}$ and we shall show that it is a Cauchy one. Indeed, for every $\alpha \in A$ we have

$$\begin{aligned} d_{\alpha}(T^n x_0, T^{n+m} x_0) &\leq \sum_{\kappa=0}^{m-1} d_{\alpha}(T^{n+\kappa} x_0, T^{n+\kappa+1} x_0) \\ &\leq \sum_{\kappa=0}^{m-1} d_{j^{n+\kappa}(\alpha)}(x_0, Tx_0) \end{aligned}$$

which completes the proof. □

THEOREM 4. *If, in addition, we assume that for every $x, y \in X$ and $\alpha \in A$, $\lim_{n \rightarrow \infty} d_{j^n(\alpha)}(x, y) = 0$, then the fixed point of T is unique.*

The proof is analogous to the one of Theorem 2.

3. APPLICATIONS

Here we shall present examples of j -nonexpansive operators arising in the theory of some classes of functional equations. At first, we shall consider an integral equation of Abel-Liouville's type with delays. Similar equations without delays have been investigated by Reiner mann and Stallbohm [11]. They obtained the existence of a local solution by means of Edelstein's fixed point theorem in metric spaces. We shall show the existence of a global solution using Theorem 1, since the right-hand side of the mentioned equation generates a j -nonexpansive operator.

Let us consider an initial value problem:

$$\begin{aligned} (1) \quad x(t) &= g(t) + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} F(t, s, x(a_1(s)), \dots, x(a_m(s))) ds, \quad t > 0 \\ x(t) &= \varphi(t), \quad t \leq 0 \end{aligned}$$

where the unknown function $x(t)$ takes values in some Banach space B with a norm $\|\cdot\|_B$, while $g(t)$, $F(t, s, u_1, \dots, u_m)$ and $\varphi(t)$ are prescribed functions, $\Gamma(\mu)$ is Euler's

Gamma-function, $0 < \mu \leq 1$. We shall suppose that $g(t) = 0$ because in an opposite case we can put $\tilde{x}(t) = x(t) - g(t)$. Therefore we shall investigate the following initial value problem:

$$(2) \quad x(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} F(t, s, x(a_1(s)), \dots, x(a_m(s))) ds, t \geq 0.$$

The integral is in Bochner's sense.

Let the following conditions (A) be fulfilled:

- (A1) the functions $a_\ell^{(s)} : \mathbb{R}_+^1 \rightarrow \mathbb{R}_+^1$ ($\ell = 1, \dots, m$) are continuous, $0 \leq a_\ell(s) \leq s$ and $\lim_{n \rightarrow \infty} j^n(K) = 0$ for an arbitrary compact $K \subset \mathbb{R}_+^1$, where $j(K) = \bigcup_{\ell=1}^m j_\ell(K)$, $j_\ell(K) = \{a_\ell(s) : s \in K\}$; $j^n(K) = j(j^{n-1}(K))$, $j^0(K) = K$;
- (A2) the function $F(t, s, u_1, \dots, u_m) : \Delta \times B^m \rightarrow B$ is continuous and bounded on the bounded subsets of $\Delta \times B^m$ and satisfies the condition:

$$\|F(t, s, u_1, \dots, u_m) - F(t, s, \bar{u}_1, \dots, \bar{u}_m)\|_B \leq \frac{\Gamma(\mu + 1)\omega_0(s)}{m} \sum_{\ell=1}^m \|u_\ell - \bar{u}_\ell\|_B$$

where $\Delta = \{(t, s) \in \mathbb{R}_+^1 \times \mathbb{R}_+^1 : 0 \leq s \leq t\}$, $\omega_0(s) = 1/(s^\mu)$, $s > 0$.

THEOREM 5. *If condition (A) is fulfilled, then the initial value problem (2) has a global continuous solution.*

PROOF: As usually, by $C(\mathbb{R}_+^1; B)$ we shall denote the space of all continuous functions $f(t) : \mathbb{R}_+^1 \rightarrow B$. Introduce the set $X = \{f \in C(\mathbb{R}_+^1; B) : f(0) = 0 \text{ and } \|f_1 - f_2\|_B = 0(t^\mu) \text{ as } t \rightarrow 0 \text{ and } t > 0\}$. X will be regarded as a uniform space endowed with a saturated family of pseudometrics

$$d_K(f_1, f_2) = \sup\{\|f_1 - f_2\|_B \omega_0(t) : t \in K\}$$

where K runs over all compact subsets of \mathbb{R}_+^1 . Consequently the index set consists of all these compact sets. In view of the choice of $\omega_0(t)$, every $d_K(f_1, f_2)$ is finite.

Define the operator T by the formula:

$$(Tf)(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} F(t, s, f(a_1(s)), \dots, f(a_m(s))) ds, t \geq 0.$$

It can be verified as in [11] that T maps X into itself. In what follows we shall show that T is j -nonexpansive. Choosing arbitrary $f_1, f_2 \in X$ and $K \subset \mathbb{R}_+^1$ we have

$\omega_0(s)/\omega_0(a_\ell(s)) \leq 1$ and then

$$\begin{aligned} \|(Tf_1)(t) - (Tf_2)(t)\|_B &\leq \frac{\Gamma(\mu + 1)}{m\Gamma(\mu)} \int_0^t (t - s)^{\mu-1} \omega_0(s) \sum_{\ell=1}^m \|f_1(a_\ell(s)) - f_2(a_\ell(s))\|_B ds \\ &\leq \frac{\mu}{m} \int_0^t (t - s)^{\mu-1} \sum_{\ell=1}^m \|f_1(a_\ell(s)) - f_2(a_\ell(s))\|_B \omega_0(a_\ell(s)) \frac{\omega_0(s)}{\omega_0(a_\ell(s))} ds \\ &\leq \frac{\mu}{m} \int_0^t (t - s)^{\mu-1} \sum_{\ell=1}^m d_{j(K)}(f_1, f_2) ds \\ &\leq \mu \int_0^t (t - s)^{\mu-1} ds d_{j(K)}(f_1, f_2) \\ &\leq t^\mu d_{j(K)}(f_1, f_2) \leq \frac{1}{\omega_0(t)} d_{j(K)}(f_1, f_2). \end{aligned}$$

Multiplying by $\omega_0(t)$ we can take a supremum on K which implies $d_K(Tf_1, Tf_2) \leq d_{j(K)}(f_1, f_2)$.

Further on, it is easy to see that the family of functions $\{(Tf)(t)\}$ is equicontinuous on K , when f runs over some bounded subsets of X (the proof is analogous to the one in [11]) and then the Arzela-Ascoli theorem implies that condition 1 of Theorem 1 is satisfied.

In order to verify condition 2 we choose an element $f_0 \in X$ where $f_0(t) \equiv 0$. Then in view of $\lim_{n \rightarrow \infty} j^n(K) = 0$ we obtain $\lim_{n \rightarrow \infty} d_{j^n(K)}(f_0, Tf_0) = 0$.

Thus we prove the existence of a solution of the initial value problem (2) belonging to the set X .

Now we shall formulate theorems which include critical examples from Introduction and from [10]. We shall consider an initial value problem for the functional equation:

$$\begin{aligned} (3) \quad &x(t) = F(t, x(\Delta_1(t)), \dots, x(\Delta_m(t))), t > 0 \\ &x(t) = \varphi(t), t \leq 0. \end{aligned}$$

□

ASSUMPTIONS (B).

(B1) The deviations $\Delta_\ell(t): \mathbb{R}_+^1 \rightarrow \mathbb{R}^1$ are continuous and $\lim_{n \rightarrow \infty} \text{meas } j^n(K) = 0$, where $j(K) = \bigcup_{\ell=1}^m j_\ell(K)$, $j_\ell(K) = \{\Delta_\ell(t) : t \in K\}$, $\Delta_\ell(0) \leq 0$ ($\ell = 1, \dots, m$).

(B2) The function $F: \mathbb{R}_+^1 \times B^m \rightarrow B$ is continuous and

$$\|F(t, u_1, \dots, u_m) - F(t, \bar{u}_1, \dots, \bar{u}_m)\|_B \leq \Omega(t, \|u_1 - \bar{u}_1\|_B, \dots, \|u_m - \bar{u}_m\|_B)$$

where Ω is a comparison function, that is, $\Omega(t, v_1, \dots, v_m) : \mathbb{R}_+^1 \times \mathbb{R}_+^m \rightarrow \mathbb{R}_+^1$ is non-decreasing in v_ℓ and $\Omega(t, v, \dots, v) \leq v$. Besides,

$$\lim_{n \rightarrow \infty} \frac{\sup\{\|F(t, 0, \dots, 0)\| : t \in j^{n+1}(K)\}}{\sup\{\|F(t, 0, \dots, 0)\| : t \in j^n(K)\}} = q(K) < 1.$$

(B3) The initial function $\varphi(t) : \mathbb{R}_-^1 \rightarrow B$ is continuous and $\varphi(0) = F(0, \varphi(\Delta_1(0)), \dots, \varphi(\Delta_m(0))) = 0$.

THEOREM 6. *Let the assumptions (B) hold. Then the problem (3) has a continuous global solution. This solution is unique.*

PROOF: Consider the set X consisting of all continuous functions $f(t) : \mathbb{R}^1 \rightarrow B$ whose restrictions on $(-\infty, 0]$ are equal to $\varphi(t)$. It becomes a uniform space with a saturated family of pseudometrics $d_K(f_1, f_2) = \sup\{\|f_1(t) - f_2(t)\|_B : t \in K\}$. The index set A is $A = \{K : K \text{ runs over all compact subsets of } \mathbb{R}^1\}$.

It is easy to verify that the operator T defined by the formula

$$(Tf)(t) = \begin{cases} F(t, f(\Delta_1(t)), \dots, f(\Delta_m(t))), & t > 0 \\ \varphi(t), & t \leq 0 \end{cases}$$

maps X into itself and is *j*-nonexpansive. Indeed

$$\begin{aligned} \|(Tf)(t) - (T\bar{f})(t)\|_B &\leq \Omega(t, d_{j_1(K)}(f, \bar{f}), \dots, d_{j_m(K)}(f_1, \bar{f}_2)) \\ &\leq d_{j(K)}(f_1, \bar{f}_2) \end{aligned}$$

and then $d_K(Tf, T\bar{f}) \leq d_{j(K)}(f_1, \bar{f}_2)$.

An element $f_0 \in X$ where $f_0(t) = \varphi(t)$ for $t \leq 0$ and $f = 0$ for $t > 0$ has the property $d_{j^{n+1}(K)}(f_0, Tf_0)/d_{j^n(K)}(f_0, Tf_0)_n \rightarrow q(K) < 1$ and therefore $\sum_{n=0}^{\infty} d_{j^n(K)}(f_0, Tf_0) < \infty$; that is, problem (3) has a solution in X in view of Theorem 3. Uniqueness of this solution follows from the condition $\lim_{n \rightarrow \infty} \text{meas } j^n(K) = 0$ (see Theorem 4) which completes the proof. □

ASSUMPTIONS (C).

(C1) Functions $\Delta_\ell(t) : \mathbb{R}_+^1 \rightarrow \mathbb{R}^1$ ($\ell = 1, 2, \dots, m$) are measurable and possess the property: an inverse image by $\Delta_\ell(t)$ of any set with measure equal to zero is measurable. Besides

(C1.1) $\lim_{n \rightarrow \infty} \text{meas } j^n(K) = 0$ or (C1.2) there is n_0 such that for $n \geq n_0$ $j^n(K) \subset \mathbb{R}_-^1$.

(C2) The function $F: \mathbb{R}_+^1 \times B^m \rightarrow B$ satisfies the Caratheodory condition and the inequality

$$\|F(t, u_1, \dots, u_m) - F(t, \bar{u}_1, \dots, \bar{u}_m)\|_B \leq \Omega(t, \|u_1 - \bar{u}_1\|_B, \dots, \|u_m - \bar{u}_m\|_B).$$

Further $F(\cdot, u_1, \dots, u_m)$ is locally essentially bounded for fixed $u_1, u_2, \dots, u_m \in B$;

$$\lim_{n \rightarrow \infty} \frac{\text{ess sup}\{\|F(t, 0, \dots, 0)\|_B : t \in j^{n+1}(K)\}}{\text{ess sup}\{\|F(t, 0, \dots, 0)\|_B : t \in j^n(K)\}} = q(K) < 1$$

(C3) The initial function $\varphi(t): \mathbb{R}_-^1 \rightarrow B$ is locally essentially bounded.

THEOREM 7. *Under the assumptions (C) the problem (3) has a unique locally essentially bounded solution.*

The proof is analogous to the one of Theorem 6.

Finally we shall make a short discussion on some important examples (see [10]):

$$x(t) = \begin{cases} \ell x(\beta t) + h(t), & t > 0 \\ 0, & t = 0, \end{cases} \text{ where } h(0) = 0 \text{ and } 0 \leq \beta \leq 1.$$

The result cited in [10] implies the existence of a solution provided $|\ell\beta| < 1$. When $\ell = 1$, then it must be assumed that $0 < \beta < 1$. Theorem 6 implies the existence of a solution in general, because for every compact $K \subset \mathbb{R}_+^1$ $\lim_{n \rightarrow \infty} \text{meas } j^n(K) = \lim_{n \rightarrow \infty} \beta^n \text{meas } K = 0$.

In the Myshkis example (in the Introduction of the present paper) $\beta = 1$ and the condition $|\ell\beta| < 1$ fails; that is, results from [10] are not applicable. But the retardation $\Delta(t) = t - 1$ has the property: there is an n_0 such that for every $n > n_0$, $j^n(K) \subset \mathbb{R}_-^1$ for arbitrary $K \subset \mathbb{R}_+^1$. Then Theorem 7 guarantees the existence of a L_{loc}^∞ -solution.

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