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ON GENERATORS AND PRESENTATIONS OF SEMIDIRECT PRODUCTS IN INVERSE SEMIGROUPS

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Abstract

In this paper we prove two main results. The first is a necessary and sufficient condition for a semidirect product of a semilattice by a group to be finitely generated. The second result is a necessary and sufficient condition for such a semidirect product to be finitely presented.

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1. Introduction

Investigation of finite generation and finite presentability of various constructions is one of the main areas of research in combinatorial semigroup theory. For example, finite generation and presentability of direct products of semigroups are considered in [1, 7]. In [6] wreath products of monoids are discussed in this respect. Lavers gives a presentation for general products of monoids in [4] and investigates when general products of finitely presented monoids are finitely presented. In [3] finite generation and presentability of Schützenberger products are investigated. These constructions share the property that an action of one of the building blocks is defined on the other component. In the case of direct products, the action is trivial. A common feature of these results is that finite presentability of these constructions requires finite presentability of the component that is acted upon.

In inverse semigroup theory the construction of a semidirect product, which is an important example of general products, plays an important role. For example, every inverse semigroup divides a semidirect product of a semilattice by a group [5, Theorem 7.1.6]. Semilattices and groups are among the best-known examples of inverse semigroups. It is well known that a semilattice is finitely generated if and only if it is finite. The question naturally arises whether there are examples for finitely generated or finitely presented semidirect products of semilattices by groups where the semilattice is infinite. As we shall see such examples do exist. In fact, finite generation or presentability relies on the new notion of finite generation or presentability with respect to the action.

Throughout the paper we are going to work with inverse semigroups. For the definition and basic properties of inverse semigroups see [5]. We briefly summarize basic facts regarding inverse semigroup presentations.

We consider inverse semigroups as algebras of type (2, 1), where the binary operation is multiplication and the unary operation assigns to each element its unique (von Neumann) inverse. First, we recall the description of the free inverse semigroup on a nonempty set X as a factor semigroup of the free semigroup with involution on X; for more details the reader is referred to [5, Ch. 6].

Let X be a nonempty set and $X^{-1} = \{x^{-1} \mid x \in X\}$. Consider the free semigroup $F = (X \cup X^{-1})^+$ and define a unary operation on F in the following way: for each $y \in X \cup X^{-1}$, let

$$y^{-1} = \begin{cases} x^{-1} & \text{if } y = x \in X, \\ x & \text{if } y = x^{-1} \in X^{-1}, \end{cases}$$

and define $(y_1 \ldots y_n)^{-1} = y_n^{-1} \ldots y_1^{-1}$. Then $(F, \cdot, ^{-1})$ is the free semigroup with involution on X, which we shall denote by $FSI(X)$. Define the following binary relation on $FSI(X)$:

$$\mathfrak{R} = \{(uu^{-1}u, u) \mid u \in FSI(X)\} \cup \{(uu^{-1}vv^{-1}, vv^{-1}uu^{-1}) \mid u, v \in FSI(X)\}.$$

The congruence generated by $\mathfrak{R}$ is called the *Wagner congruence* which we denote by ρ. The factor semigroup $FI(X) = FSI(X)/\rho$ is the free inverse semigroup on X. We refer to the elements of $\mathfrak{R}$ as *standard inverse semigroup relations*.

An *inverse semigroup presentation* is an ordered pair $\langle X \mid P \rangle$, where P is a binary relation on $FSI(X)$. Let τ denote the congruence generated by $P \cup \mathfrak{R}$. The semigroup $S = FSI(X)/\tau$ is said to be *presented as an inverse semigroup* by the generators X and relations P and we denote this by $S = \text{Inv}\ \langle X \mid P \rangle$.

Let $S = \text{Inv}\ \langle X \mid P \rangle$ and let w_1, w_2 be words over $X \cup X^{-1}$. We write $w_1 \equiv w_2$, if w_1 and w_2 are identical as words and we write $w_1 = w_2$, if w_1 and w_2 represent the same element of S. If $w_1 = w_2$, then we also say that S satisfies the relation $w_1 = w_2$. We say that w_2 is obtained from w_1 by an *application of a relation of P or of a standard inverse semigroup relation*, if $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$, where $\alpha, \beta \in (X \cup X^{-1})^*$ and $u = v$ or $v = u$ is a relation of P or a standard inverse semigroup relation. We say that $w_1 = w_2$ is a *consequence of relations* in P and of standard inverse semigroup relations, if there exists a sequence of words $w_1 \equiv \alpha_0, \ldots, \alpha_m \equiv w_2$, where α_{j+1} is obtained from α_j by applying a relation of P or a standard inverse semigroup relation. Usual general considerations give the following result.

PROPOSITION 1.1. *Let S be an inverse semigroup generated by a set X. Then $S = \text{Inv}\ \langle X \mid P \rangle$ if and only if the following two conditions hold:*

(i) *S satisfies all relations in $P \cup \mathfrak{R}$;*
(ii) *if $w_1, w_2 \in (X \cup X^{-1})^+$ are such that $w_1 = w_2$ holds in S, then $w_1 = w_2$ is a consequence of relations in $P \cup \mathfrak{R}$.*

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One of the central notions of this paper is the notion of action. Let $(T, *)$ and $(S, \cdot)$ be semigroups. We say that S *acts on* T *by endomorphisms (automorphisms) on the left*, if for every $s \in S$ there exists an endomorphism (automorphism) $\phi_s : T \to T$ so that $\phi_{s_1}\phi_{s_2}t = \phi_{s_1 s_2}t$ holds for all $s_1, s_2 \in S$, $t \in T$. If S is a monoid with identity 1, then we also require $\phi_1 : T \to T$ to be the identity map. For all $s \in S$ and $t \in T$ we denote $\phi_s t$ by ${}^s t$. In this notation, the rule now becomes ${}^{s_1}({}^{s_2}t) = {}^{s_1 s_2}t$.

Let $(T, *)$ and $(S, \cdot)$ be semigroups. Assume that S acts on T on the left by automorphisms. The *semidirect product* $T \rtimes S$ of T by S with respect to this action has as its underlying set $T \times S$ with multiplication defined by $(e, g)(f, h) = (e * {}^g f, gh)$. This easily extends to

$$(e_1, g_1) \ldots (e_n, g_n) = (e_1 * {}^{g_1}e_2 * \cdots * {}^{g_1 \cdots g_{n-1}}e_n, g_1 \ldots g_n).$$

It is well-known that if T is a semilattice and S is a group, then $T \rtimes S$ is an inverse semigroup (see for example [5, Theorem 7.1.1]).

2. Generators

We introduce the concept of finite generation of a semigroup with respect to a semigroup action. This notion will help us to give a necessary and sufficient condition for a semidirect product of a semilattice by a group to be finitely generated. Throughout the paper if we say that a group G acts on a semilattice Y, then we mean that G acts on Y on the left by automorphisms.

Let $(Y, \wedge)$ be a semilattice and let $\le$ denote the natural partial order on Y. Recall that $x \le y$ if and only if $x = x \wedge y$. We say that $y \in Y$ is a *maximal element* of Y if $x \in Y$, $y \le x$ always implies that $x = y$. We say that Y satisfies the *maximum condition* if it has finitely many maximal elements, and for every $x \in Y$ there exists a maximal element $y \in Y$ such that $x \le y$. The following lemma is immediate from the definitions.

LEMMA 2.1. *Let $(Y, \wedge)$ be a semilattice and let φ be an automorphism of $(Y, \wedge)$. Then $x \le y$ if and only if $x\varphi \le y\varphi$. In particular, we have that y is a maximal element of Y if and only if $y\varphi$ is a maximal element of Y.*

The following definition proves to be the key concept for the main result of this section.

DEFINITION 2.2. Let $(S, \cdot)$ be a monoid and $(T, *)$ be a semigroup. Assume that S acts on T on the left by endomorphisms (automorphisms). We say that T is generated by $T_0 \subseteq T$ with respect to the action of S, if $T = \langle {}^S T_0 \rangle$, where ${}^S T_0 = \{{}^s t \mid s \in S, t \in T_0\}$. We say that T is *finitely generated with respect to the action of* S if T_0 can be chosen to be a finite subset of T.

The following is the main result of this section.

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PROPOSITION 2.3. *Let $(Y, \wedge)$ be a semilattice, and $(G, \cdot)$ be a group acting on Y. The semidirect product $S = Y \rtimes G$ is finitely generated if and only if the following conditions hold:*

(i) *G is finitely generated;*
(ii) *Y satisfies the maximum condition;*
(iii) *Y is finitely generated with respect to the action of G.*

PROOF. ($\Longrightarrow$) Assume that $S = Y \rtimes G$ is finitely generated by the elements $A = \{(e_1, g_1), \ldots, (e_n, g_n)\}$, where $A \subseteq S$. Without loss of generality we may assume that $(e, g)^{-1} = ({}^{g^{-1}}e, g^{-1}) \in A$ for all $(e, g) \in A$. Let $X = \{g_1, g_2, \ldots, g_n\}$ and $Y_0 = \{e_1, \ldots, e_n\}$. Let $(e, h) \in S$. Write $(e, h) = (f_1, h_1)(f_2, h_2) \ldots (f_k, h_k)$ where $(f_j, h_j) \in A$ for all $1 \leq j \leq k$. Then, on the one hand $h = h_1 \ldots h_k$, where $h_j \in X$ verifying that G is generated by the finite set X; on the other hand, $e = f_1 \wedge {}^{h_1}f_2 \wedge \cdots \wedge {}^{h_1 \ldots h_{k-1}}f_k$, where $f_1, \ldots, f_k \in Y_0$. It follows that $Y = \langle {}^{G}Y_0 \rangle$. We also obtain that $e \leq f_1$. Hence, the maximal elements of Y are the maximal elements of Y_0 and we deduce that Y satisfies the maximum condition.

($\Longleftarrow$) For the converse assume that G is generated by the finite set X, that Y satisfies the maximum condition and that $\langle {}^{G}Y_0 \rangle = Y$ for some finite set $Y_0 \subseteq Y$. Let Y_m denote the set of maximal elements of Y. Without loss of generality we may assume that $Y_m \subseteq Y_0$. We claim that

$$A = \{(e, 1) \mid e \in Y_0\} \cup \{(e, g) \mid e \in Y_m, g \in X \cup X^{-1}\}$$

generates S.

Let $(e, h) \in Y_0 \times G$. Let $\tilde{e} \in Y_m$ be such that $e \leq \tilde{e}$ and suppose that $h = g_1 g_2 \ldots g_k$ where $g_1, g_2, \ldots, g_k \in X \cup X^{-1}$. By Lemma 2.1, there exist $f_1, f_2, \ldots, f_{k-1} \in Y_m$ such that ${}^{g_1}f_1 = \tilde{e}$ and ${}^{g_j}f_j = f_{j-1}$ for all $2 \leq j \leq k-1$. It follows that ${}^{g_1 \cdots g_j}f_j = \tilde{e}$ for all $1 \leq j \leq k-1$ and we obtain

$$\begin{aligned}(e, 1)(\tilde{e}, g_1)(f_1, g_2) \ldots (f_{k-1}, g_k) &= (e \wedge \tilde{e} \wedge {}^{g_1}f_1 \wedge \cdots \wedge {}^{g_1 g_2 \cdots g_{k-1}}f_{k-1}, h) \\ &= (e \wedge \underbrace{\tilde{e} \wedge \tilde{e} \wedge \cdots \wedge \tilde{e}}_{k}, h) = (e \wedge \tilde{e}, h) = (e, h)\end{aligned}$$

verifying that elements of $Y_0 \times G$ are generated by A.

Let $e \in Y$ and $h \in G$ be arbitrary. Since $\langle {}^{G}Y_0 \rangle = Y$, there exist $h_1, \ldots, h_k \in G$ and $f_1, \ldots, f_k \in Y_0$ such that $e = {}^{h_1}f_1 \wedge \cdots \wedge {}^{h_k}f_k$. Since Y satisfies the maximum condition, there exists $\tilde{e} \in Y_m$ such that $e \leq \tilde{e}$, and so $e = \tilde{e} \wedge e$ holds. Then

$$(e, h) = (\tilde{e}, h_1)(f_1, h_1^{-1}h_2) \ldots (f_{k-1}, h_{k-1}^{-1}h_k)(f_k, h_k^{-1})(\tilde{e}, h).$$

By the above argument each component of the product can be written in terms of elements of A and thus S is finitely generated by A. □

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3. Presentations with respect to a semigroup action

In this section we introduce the notion of finite presentability of an inverse semigroup with respect to a semigroup action. This will be the key concept in giving a necessary and sufficient condition for a semidirect product of a semilattice by a group to be finitely presented.

Let S be a monoid and T be an inverse semigroup on which S acts. Assume that T is generated by T_0 with respect to this action. Let Q be a binary relation on $FSI({}^{S}T_0)$ and let

$$Q_A = \{{}^{s}p = {}^{s}q \mid (p = q) \in Q \cup \Re,\ s \in S\}.$$

If $T = \operatorname{Inv}\langle {}^{S}T_0 | Q_A\rangle$, then we say that T is *presented as an inverse semigroup with respect to the action of* S by the generators T_0 and relations Q and we denote this by $T = \operatorname{InvAct}_S \langle T_0 | Q\rangle$. If T_0 and Q can be chosen to be finite sets, then we say that T is *finitely presented as an inverse semigroup with respect to the action of* S. The following proposition is as expected and can be proved using standard techniques. For further details see [2].

Proposition 3.1. *Let S be a monoid and T be an inverse semigroup. Assume that S acts on T. Assume that the finite sets Y_1 and Y_2 generate T with respect to the action of S and that T can be defined by a finite inverse semigroup presentation with respect to the action of S in terms of Y_1. Then T can be defined by a finite inverse semigroup presentation with respect to the action of S in terms of Y_2 as well.*

The following is the main theorem of the paper.

Theorem 3.2. *Let $(Y, \wedge)$ be a semilattice and let G be a group acting on Y on the left by automorphisms. The semidirect product $S = Y \rtimes G$ is finitely presented as an inverse semigroup if and only if the following conditions hold:*

(i) *G is finitely presented;*
(ii) *Y satisfies the maximum condition;*
(iii) *Y is finitely presented as an inverse semigroup with respect to the action of G.*

Theorem 3.2 will be proved in the following two sections. We first make some preliminary observations and introduce the notation we use.

Let Y be a semilattice and G be a group acting on Y. Assume that the semidirect product $S = Y \rtimes G$ is generated as an inverse semigroup by the finite set A. For a word

$$w \equiv (e_1, g_1) \ldots (e_k, g_k) \in (A \cup A^{-1})^+,$$

we let

$$w^{-1} \equiv ({}^{g_k^{-1}}e_k, g_k^{-1}) \ldots ({}^{g_1^{-1}}e_1, g_1^{-1})$$

and we let

$$\Sigma(w) \equiv e_1 \wedge {}^{g_1}e_2 \wedge \cdots \wedge {}^{g_1 \cdots g_{k-1}}e_k \quad \text{and} \quad \Gamma(w) \equiv g_1 \ldots g_k.$$

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Note that the element of S represented by w is $(\Sigma(w), \Gamma(w))$. Clearly, if $w \equiv uv$, then $\Sigma(w) = \Sigma(u) \wedge {}^{\Gamma(u)}\Sigma(v)$. The following lemma is now immediate.

LEMMA 3.3. *Let $(Y, \wedge)$ be a semilattice and let G be a group acting on Y. Assume that the semidirect product $S = Y \rtimes G$ is generated by A. Then for any $w \in FSI(A)$, $\Sigma(ww^{-1}) = \Sigma(w)$ holds in Y.*

We know by Proposition 2.3 that if $Y \rtimes G$ is finitely generated, then Y satisfies the maximum condition. Let Y_m denote the set of maximal elements of Y. As before, let A denote a finite generating set for S. Let $X = \{g \in G \mid (e, g) \in A$ for some $e \in Y\}$ and $Y_0 = \{e \in Y \mid (e, g) \in A \cup A^{-1}$ for some $g \in X \cup X^{-1}\}$. Without loss of generality we may assume that $(Y_0 \times 1) \cup (Y_m \times X) \subseteq A$. Define the following map

$$\begin{aligned} \mathbf{n} : Y_m \times (X \cup X^{-1})^+ &\to (A \cup A^{-1})^+; \\ (f_1, g_1 g_2 \ldots g_k) &\mapsto (f_1, g_1)(f_2, g_2) \ldots (f_k, g_k), \end{aligned}$$

where ${}^{g_j} f_{j+1} = f_j \, (1 \leq j < k)$. By Lemma 2.1, $f_j \in Y_m$ for all $1 \leq j \leq k$. Note that for all $1 \leq j \leq k-1$, $f_1 = {}^{g_1 \cdots g_j} f_{j+1}$ holds, and thus the word $\mathbf{n}(f_1, g_1 g_2 \ldots g_k)$ represents $(f_1, g_1 g_2 \ldots g_k)$ in S. Throughout the paper, if $u \equiv \mathbf{n}(e, g_1 \ldots g_k)$, then u^{-1} will denote the word $\mathbf{n}(\tilde{e}, g_k^{-1} \ldots g_1^{-1})$ where $e = {}^{g_1 \cdots g_k}\tilde{e}$. It is easy to see that u^{-1} is indeed the inverse of u in S.

Let $\alpha \in (X \cup X^{-1})^+$ and let

$$c(\alpha, l) = \{u \cdot (l, 1) \cdot u^{-1} | u \equiv \mathbf{n}(e, \alpha), e \in Y_m, l \in Y_0, {}^{\alpha}l \leq e\}.$$

The elements of $c(\alpha, l)$ are idempotents, in fact they represent $({}^{\alpha}l, 1)$ in S. Clearly, if Y satisfies the maximum condition, then $c(\alpha, l)$ is a finite set for all $l \in Y_0$ and $\alpha \in (X \cup X^{-1})^+$.

4. Main theorem, Part 1

We prove the direct part of Theorem 3.2. The converse part is proved in the next section. Throughout we are going to work with the notation introduced in the previous section.

Assume that S is given by the finite inverse semigroup presentation $\mathrm{Inv}\langle A | P\rangle$. Without loss of generality we may assume that $(Y_0 \times 1) \cup (Y_m \times X) \subseteq A$. Let $\Gamma(P) = \{\Gamma(r) = \Gamma(p) \mid (r = p) \in P\}$. We show that $G = \mathrm{Grp}\langle X | \Gamma(P)\rangle$. We already know by Proposition 2.3 that G is generated by the finite set X. Clearly, relations in $\Gamma(P)$ hold in G. We verify that any relation that holds in G is a consequence of relations in $\Gamma(P)$. For this, assume that the relation $g_1 \ldots g_m = h_1 \ldots h_k$, $(g_i, h_j \in X \cup X^{-1})$ holds in G. Let $f \in Y_m$. Then $\mathbf{n}(f, g_1 g_2 \ldots g_m) = \mathbf{n}(f, h_1 h_2 \ldots h_k)$ holds in S. Thus, there exists a finite sequence of words

$$\mathbf{n}(f, g_1 \ldots g_m) \equiv q_0, q_1, \ldots, q_t \equiv \mathbf{n}(f, h_1 \ldots h_k)$$

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where q_{j+1} is obtained from q_j by applying a relation in P or a standard inverse semigroup relation. If q_{j+1} is obtained from q_j by applying a relation in P, then $\Gamma(q_{j+1})$ is obtained from $\Gamma(q_j)$ by applying a relation in $\Gamma(P)$. If q_{j+1} is obtained from q_j by applying a standard inverse semigroup relation, then $\Gamma(q_{j+1})$ is obtained from $\Gamma(q_j)$ by applying a sequence of standard group relations. It follows that there exists a finite sequence of words $g_1 \ldots g_m \equiv \alpha_0, \alpha_1, \ldots, \alpha_l \equiv h_1 \ldots h_k$, such that α_{j+1} is obtained from α_j by applying a relation in $\Gamma(P)$ or a standard group relation, verifying that $G = \mathrm{Grp}\langle X | \Gamma(P)\rangle$.

Next, we claim that Y is finitely presented as an inverse semigroup with respect to the action of G. We already know that Y satisfies the maximum condition and that $Y = \langle {}^G Y_0\rangle$. On $FSI({}^G Y_0)$ we define the following set of relations:

$$Q = \{\Sigma(r) = \Sigma(p) \mid (r = p) \in P\} \cup \{e = f \wedge e \mid e, f \in Y_0, \ e \leq f\}.$$

We show that $Y = \mathrm{InvAct}_G \langle Y_0 | Q\rangle$. It is clear that every relation of Q and hence of Q_A holds in Y. Assume that ${}^{g_1}e_1 \wedge {}^{g_2}e_2 \wedge \cdots \wedge {}^{g_m}e_m = {}^{h_1}f_1 \wedge {}^{h_2}f_2 \wedge \cdots \wedge {}^{h_k}f_k$ holds in Y, where $e_i, f_j \in Y_0$ and $g_i, h_j \in G$. For all $1 \leq i \leq m$, $1 \leq j \leq k$, we fix words $\alpha_i, \beta_j \in (X \cup X^{-1})^+$ so that $\alpha_i = g_i$ and $\beta_j = h_j$. Let $c_i \in c(\alpha_i, e_i)$ and $d_j \in c(\beta_j, f_j)$. Let $w_1 \equiv c_1 \ldots c_m$ and $w_2 \equiv d_1 \ldots d_k$. Clearly, the relations

$$\Sigma(w_1) = {}^{g_1}e_1 \wedge {}^{g_2}e_2 \wedge \cdots \wedge {}^{g_m}e_m \quad \text{and} \quad \Sigma(w_2) = {}^{h_1}f_1 \wedge {}^{h_2}f_2 \wedge \cdots \wedge {}^{h_k}f_k$$

are a consequence of relations in $\mathfrak{R}$.

We now verify that $\Sigma(w_1) = \Sigma(w_2)$ is a consequence of relations in Q_A. Since ${}^{g_1}e_1 \wedge \cdots \wedge {}^{g_m}e_m = {}^{h_1}f_1 \wedge \cdots \wedge {}^{h_k}f_k$ holds in Y, the relation $w_1 = w_2$ holds in S. Hence, there exists a finite sequence of words $w_1 \equiv q_0, q_1, \ldots, q_t \equiv w_2$ such that q_{j+1} is obtained from q_j by applying a relation in P or a standard inverse semigroup relation. We consider the following three cases.

(1) If q_{j+1} is obtained from q_j by applying a relation in P, then we may write $q_j \equiv t_1 s t_2$, $q_{j+1} \equiv t_1 z t_2$, where $(s = z) \in P$. It follows that $(\Sigma(s) = \Sigma(z)) \in \Sigma(P)$ and $(\Gamma(s) = \Gamma(z)) \in \Gamma(P)$. In particular, $\Gamma(t_1 s) = \Gamma(t_1 z)$ holds in G and we obtain that

$$\begin{aligned}\Sigma(q_j) &\equiv \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(s) \wedge {}^{\Gamma(t_1 s)}\Sigma(t_2)\\ &= \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(z) \wedge {}^{\Gamma(t_1 s)}\Sigma(t_2)\\ &= \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(z) \wedge {}^{\Gamma(t_1 z)}\Sigma(t_2) \equiv \Sigma(q_{j+1})\end{aligned}$$

verifying that $\Sigma(q_{j+1})$ is obtained from $\Sigma(q_j)$ by applying relations in Q_A.

(2) If q_{j+1} is obtained from q_j by applying a relation of the form $ww^{-1}w = w$, then we may write $q_j \equiv t_1 ww^{-1}wt_2$, $q_{j+1} \equiv t_1 w t_2$. Let $u \equiv ww^{-1}$, and $v \equiv ww^{-1}w$. By Lemma 3.3 we have that $\Sigma(u) = \Sigma(w)$. Clearly $\Gamma(v) = \Gamma(w)$ holds in G and we

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have that

$$\begin{aligned}
\Sigma(q_j) &\equiv \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(u) \wedge {}^{\Gamma(t_1 u)}\Sigma(w) \wedge {}^{\Gamma(t_1 v)}\Sigma(t_2) \\
&= \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(w) \wedge {}^{\Gamma(t_1 u)}\Sigma(w) \wedge {}^{\Gamma(t_1 v)}\Sigma(t_2) && \text{by } Q_A \\
&= \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(w) \wedge {}^{\Gamma(t_1)}\Sigma(w) \wedge {}^{\Gamma(t_1 v)}\Sigma(t_2) && \text{since } \Gamma(t_1 u) = \Gamma(t_1) \\
&= \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(w) \wedge {}^{\Gamma(t_1 v)}\Sigma(t_2) && \text{by } Q_A \\
&= \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(w) \wedge {}^{\Gamma(t_1 w)}\Sigma(t_2) && \text{since } \Gamma(w) = \Gamma(v) \\
&\equiv \Sigma(q_{j+1})
\end{aligned}$$

proving that $\Sigma(q_j) = \Sigma(q_{j+1})$ is a consequence of relations in Q_A.

(3) If q_{j+1} is obtained from q_j by applying a relation of the form $w_1 w_1^{-1} w_2 w_2^{-1} = w_2 w_2^{-1} w_1 w_1^{-1}$, then we may write $q_j \equiv t_1 w_1 w_1^{-1} w_2 w_2^{-1} t_2$ and $q_{j+1} \equiv t_1 w_2 w_2^{-1} w_1 w_1^{-1} t_2$. Let $u \equiv w_1 w_1^{-1}$, $v \equiv w_2 w_2^{-1}$. Then

$$\begin{aligned}
\Sigma(q_j) &\equiv \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(u) \wedge {}^{\Gamma(t_1 u)}\Sigma(v) \wedge {}^{\Gamma(t_1 u v)}\Sigma(t_2) \\
&= \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(u) \wedge {}^{\Gamma(t_1)}\Sigma(v) \wedge {}^{\Gamma(t_1)}\Sigma(t_2) && \text{since } \Gamma(u) = \Gamma(uv) = 1 \\
&= \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(v) \wedge {}^{\Gamma(t_1)}\Sigma(u) \wedge {}^{\Gamma(t_1)}\Sigma(t_2) && \text{by } \Re \\
&= \Sigma(t_1) \wedge {}^{\Gamma(t_1)}\Sigma(v) \wedge {}^{\Gamma(t_1 v)}\Sigma(u) \wedge {}^{\Gamma(t_1 v u)}\Sigma(t_2) && \text{since } \Gamma(v) = \Gamma(vu) = 1 \\
&\equiv \Sigma(q_{j+1})
\end{aligned}$$

proving that $\Sigma(q_j) = \Sigma(q_{j+1})$ is a consequence of relations in Q_A.

It follows that there exists a finite sequence of words $\Sigma(w_1) \equiv u_0, u_1, \ldots, u_l \equiv \Sigma(w_2)$ such that u_{j+1} is obtained from u_j by applying a relation of Q_A and we may deduce that $Y = \mathrm{InvAct}_G \langle Y_0 | Q \rangle$.

5. Main theorem, Part 2

In this section we prove the converse part of our main theorem. Let $G = \mathrm{Grp}\langle X | P \rangle$, where X is a finite set and P is a finite set of relations. Without loss of generality we may assume that for all $g \in X$ the relations $gg^{-1} = g^{-1}g = 1$ and $g = g1 = 1g$ are contained in P. Let Y be a semilattice satisfying the maximum condition and let Y_m denote the set of maximal elements of Y. Assume that $Y = \mathrm{InvAct}_G \langle Y_0 | Q \rangle$, where Y_0 is a finite subset of Y and Q is a finite set of relations on $FSI({}^G Y_0)$. By Proposition 3.1 we may assume that $Y_m \subseteq Y_0$. Consider the semidirect product $S = Y \rtimes G$. We showed in Proposition 2.3 that S is finitely generated as an inverse semigroup by $A = (Y_0 \times \{1\}) \cup (Y_m \times X)$. Clearly $A^{-1} = (Y_0 \times \{1\}) \cup (Y_m \times X^{-1})$. Define the following relations on $(A \cup A^{-1})^+$:

(R1) $(e, 1)(f, h) = (f, 1)(e, h)$, where $e, f \in Y_m$;
(R2) $(e, g)(f, h) = (l, 1)(e, g)(f, h)$, where $e, f, l \in Y_m$ and $l = {}^g f$;
(R3) $(e, 1)(f, h) = (e, 1)(f, h)(l, 1)$, where $e, f, l \in Y_m$ and $e = {}^h l$.

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We let R denote the set of these relations. Let

$$R_G = \{\mathbf{n}(e, p) = \mathbf{n}(e, q) \mid e \in Y_m, (p = q) \in P\}.$$

Since Y_m and P are finite sets, we have that R_G is a finite set of relations. For every relation $p \equiv {}^{g_1}e_1 \wedge \cdots \wedge {}^{g_m}e_m = {}^{h_1}f_1 \wedge \cdots \wedge {}^{h_k}f_k \equiv q$ in Q we fix words α_i, β_j over $X \cup X^{-1}$ such that $\alpha_i = g_i$ and $\beta_j = h_j$. Define the following set of relations

$$R_{p,q} = \{c_{11} \ldots c_{1m} = c_{21} \ldots c_{2k} \mid c_{1j} \in c(\alpha_j, e_j), c_{2j} \in c(\beta_j, f_j)\}$$

and let

$$R_Y = \bigcup_{((p=q)\in Q)} R_{p,q} \cup \{(e, g) = (f, 1)(e, g) \mid e \leq f, e, f \in Y_0\}.$$

Since Y satisfies the maximum condition, the sets $c(\alpha_j, e_j)$ and $c(\beta_j, f_j)$ are finite and hence for all $(p = q) \in Q$ the set $R_{p,q}$ is finite. It follows that R_Y is finite as well.

Let $\mathbf{R} = R \cup R_G \cup R_Y$. Our aim is to show that $S = \operatorname{Inv}\langle A|\mathbf{R}\rangle$. Clearly $\mathbf{R}$ is a finite set of relations and all relations in $\mathbf{R}$ hold in S. According to Proposition 1.1 we need to verify that any relation $w_1 = w_2$ that holds in S is a consequence of relations in $\mathbf{R}$ and of standard inverse semigroup relations. The following key proposition establishes a way of rewriting an arbitrary word over $A \cup A^{-1}$ into a word of special form.

PROPOSITION 5.1. *Let $w \equiv (e_1, g_1) \ldots (e_{k-1}, g_{k-1})(e_k, g_k) \in (A \cup A^{-1})^+$ and let $c_1 = (e_1, 1)$. Then, the relation $w = c_1 c_2 \ldots c_k u$ where $u \equiv \mathbf{n}(f, g_1 \ldots g_k)$ for some $f \in Y_m$ and $c_j \in c(g_1 \ldots g_{j-1}, e_j)$ for all $(2 \leq j \leq k)$ is a consequence of relations in $\mathbf{R}$. Moreover, $c_k(f, 1) = c_k$ also holds in S.*

We need the following lemma to prove Proposition 5.1.

LEMMA 5.2. *Let $w \equiv \mathbf{n}(e, g_1 g_2 \ldots g_k)(f, g_{k+1})$, where $f \in Y_m$. Then there exists $l \in Y_m$ such that the relation $w = (e, 1)\mathbf{n}(l, g_1 g_2 \ldots g_{k+1})$ is a consequence of relations in $\mathbf{R}$. In particular, $l = {}^{g_1 \cdots g_k} f$.*

PROOF. Let $l_k = {}^{g_k} f$. Since $f \in Y_m$, we have that $l_k \in Y_m$. Let $l_1, l_2, \ldots, l_{k-1} \in Y_m$ such that $l_j = {}^{g_j} l_{j+1}$ holds. Such elements exist by Lemma 2.1. Note that ${}^{g_1 \cdots g_k} f = l_1$. Assume that $\mathbf{n}(e, g_1 \ldots g_k) \equiv (e, g_1) \ldots (e_k, g_k)$. Then

$$\begin{aligned}
w &\equiv (e, g_1) \ldots (e_{k-1}, g_{k-1})((e_k, g_k)(f, g_{k+1})) && \\
&= (e, g_1) \ldots (e_{k-1}, g_{k-1})((l_k, 1)(e_k, g_k)(f, g_{k+1})) && \text{by } (R2)\\
&= (e, g_1) \ldots (l_{k-1}, 1)(e_{k-1}, g_{k-1})(l_k, 1)(e_k, g_k)(f, g_{k+1}) && \text{by } (R2)\\
&\ \ \vdots && \vdots\\
&= (l_1, 1)(e, g_1)(l_2, 1) \ldots (l_{k-1}, 1)(e_{k-1}, g_{k-1})(l_k, 1)(e_k, g_k)(f, g_{k+1}) && \text{by } (R2)
\end{aligned}$$

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$$
\begin{aligned}
&= (e,1)(l_1,g_1)(e_2,1)\ldots((e_{k-1},1)(l_{k-1},g_{k-1})(e_k,1))((l_k,g_k)(f,g_{k+1})) && \text{by } (R1)\\
&= (e,1)(l_1,g_1)(e_2,1)\ldots(e_{k-1},1)(l_{k-1},g_{k-1})(l_k,g_k)(f,g_{k+1}) && \text{by } (R3)\\
&\ \ \vdots && \vdots\\
&= (e,1)(l_1,g_1)\ldots(l_{k-1},g_{k-1})(l_k,g_k)(f,g_{k+1}) && \text{by } (R3)\\
&\equiv (e,1)(\mathbf{n}(l_1,g_1\ldots g_{k+1})). && \square
\end{aligned}
$$

Similarly we can prove the following result.

LEMMA 5.3. *Let $f \in Y_m$ and consider $w \equiv (f,1)\mathbf{n}(e, g_1g_2\ldots g_k)$. Then $w = (e,1)\mathbf{n}(f, g_1g_2\ldots g_k)$ is a consequence of relations in R.*

PROOF OF PROPOSITION 5.1. We proceed by induction on the length of w. It is easy to verify that the proposition holds for words of length one. Assume that the proposition is true for all words whose length is less then k.

Let $w \equiv (e_1,g_1)\ldots(e_{k-1},g_{k-1})(e_k,g_k)$. Applying the inductive hypothesis we obtain that $w = c_1c_2\ldots c_{k-1}u(e_k,g_k)$, where $c_1 = (e_1,1)$, $c_j \in c(g_1\ldots g_{j-1},e_j)$, $u \equiv \mathbf{n}(f,g_1\ldots g_{k-1})$ for some $f \in Y_m$ and $c_{k-1}(f,1) = c_{k-1}$. Let $e \in Y_m$ such that $e_k \le e$. By Lemma 5.2, we have that $u(e,1) = (f,1)\mathbf{n}(l,g_1\ldots g_{k-1})$ where $l = {}^{g_1\cdots g_{k-1}}e$. We let $v \equiv \mathbf{n}(l,g_1\ldots g_{k-1})$. Then

$$
\begin{aligned}
w &= c_1c_2\ldots c_{k-1}u(e_k,g_k)\\
&= c_1c_2\ldots c_{k-1}u(e,1)(e_k,g_k) && \text{by } R_Y\\
&= c_1c_2\ldots c_{k-1}(f,1)v(e_k,g_k) && \text{by Lemma 5.2}\\
&= c_1c_2\ldots c_{k-1}v(e_k,g_k) && \text{since } c_{k-1}(f,1) = c_{k-1}\\
&= c_1c_2\ldots c_{k-1}v(e_k,1)(e_k,g_k) && \text{by } R_Y\\
&= c_1c_2\ldots c_{k-1}v(e_k,1)v^{-1}v(e_k,g_k) && \text{by } \Re.
\end{aligned}
$$

Since $l = {}^{g_1\cdots g_{k-1}}e$ and $e_k \le e$ we have that $c_k \equiv v(e_k,1)v^{-1} \in c(g_1\ldots g_{k-1},e_k)$. Moreover, by applying a relation from R_G, we obtain that $c_k(l,1) = c_k$. If $e_k \in Y_m$, that is, if $e = e_k$, then we also obtain that $v(e_k,g_k) \equiv \mathbf{n}(l,g_1\ldots g_k)$. If $e_k \in Y_0 \setminus Y_m$, then by the above we have $w = c_1c_2\ldots c_{k-1}v(e_k,1)$ and so by applying relations in R_Y and standard inverse semigroup relations we obtain that

$$w = c_1c_2\ldots c_{k-1}v(e_k,1)v^{-1}v(e,1).$$

Since $l = {}^{g_1\cdots g_{k-1}}e$ we have that $v(e,1) \equiv \mathbf{n}(l,g_1\ldots g_{k-1}1)$ holds. It now follows that $w = c_1\ldots c_k\mathbf{n}(l,g_1\ldots g_k)$. $\square$

We now complete the proof of Theorem 3.2. Assume that

$$w_1 \equiv (e_1,g_1)\ldots(e_m,g_m) = (f_1,h_1)\ldots(f_k,h_k) \equiv w_2.$$

holds in S. Let $c_1 = (e_1,1)$ and $d_1 = (f_1,1)$. By Proposition 5.1, we have that $w_1 = c_1c_2\ldots c_m u$, where $c_j \in c(g_1\ldots g_{j-1},e_j)$, $u \equiv \mathbf{n}(e,g_1\ldots g_m)$ for some $e \in Y_m$ and $c_m(e,1) = c_m$. Similarly $w_2 = d_1d_2\ldots d_k v$, where $d_j \in c(h_1\ldots h_{j-1},f_j)$, $v \equiv \mathbf{n}(f,h_1\ldots h_k)$ for some $f \in Y_m$ and $d_k(f,1) = d_k$. We write

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$(w_1)_Y \equiv c_1c_2 \ldots c_m$ and $(w_2)_Y \equiv d_1d_2 \ldots d_k$. We prove in three steps that $w_1 = w_2$ is a consequence of relations in $\mathbf{R}$ and of standard inverse semigroup relations.

STEP 1. Since $w_1 = w_2$ in S, we have that $\Gamma(w_1) = \Gamma(w_2)$ holds in G. Hence, there exists a finite sequence of words $\Gamma(w_1) \equiv \alpha_0, \alpha_1, \ldots, \alpha_t \equiv \Gamma(w_2)$ such that α_{j+1} is obtained from α_j by applying a relation in P or a standard group relation. It follows that there exists a finite sequence of words $\mathbf{n}(e, g_1 \ldots g_m) \equiv u_0, \ldots, u_t \equiv \mathbf{n}(e, h_1 \ldots h_k)$ such that β_{j+1} is obtained from β_j by applying a relation in R_G. It follows that $(w_1)_Y u = (w_1)_Y u_t$ holds in S.

STEP 2. Since $w_1 = w_2$ in S, we have that $\Sigma(w_1) = \Sigma(w_2)$ holds in Y, and hence there exists a finite sequence of words $\Sigma(w_1) \equiv \gamma_1, \gamma_2, \ldots, \gamma_q \equiv \Sigma(w_2)$ such that γ_{j+1} is obtained from γ_j by applying a relation in Q_A or a standard inverse semigroup relation. Now we construct a finite sequence of words $(w_1)_Y \equiv \delta_1, \delta_2, \ldots, \delta_q \equiv (w_2)_Y$ in the following way. If γ_{j+1} is obtained from γ_j by applying a relation $p = q$ in Q, then δ_{j+1} is obtained from δ_j by applying a relation in $R_{p,q}$. If γ_{j+1} is obtained from γ_j by applying a relation ${}^g p = {}^g q$ in Q_A, then δ_{j+1} is obtained from δ_j by applying a relation in $R_{p,q}$ and relations in R_G. If γ_{j+1} is obtained from γ_j by applying inverse semigroup relations then δ_{j+1} is obtained from δ_j by applying inverse semigroup relations. Thus, we obtain that $(w_1)_Y u_t = (w_2)_Y u_t$ holds in S.

STEP 3. Finally we show that $(w_2)_Y u_t = (w_2)_Y v$ holds. Since $d_k = d_k(f, 1)$, we have that $(w_2)_Y = (w_2)_Y(f, 1)$. It follows that

$$\begin{aligned}(w_2)_Y u_t &\equiv (w_2)_Y(\mathbf{n}(e, h_1 \ldots h_k))\\ &= (w_2)_Y(f, 1)(\mathbf{n}(e, h_1 \ldots h_k))\\ &= (w_2)_Y(e, 1)(\mathbf{n}(f, h_1 \ldots h_k)) \quad \text{by Lemma 5.3}.\end{aligned}$$

Finally we claim that $(w_2)_Y(e, 1) = (w_2)_Y$. Since $w_1 = c_1 \ldots c_m\mathbf{n}(e, g_1 \ldots g_m)$, we have that $\Sigma(w_1) = \Sigma((w_1)_Y) \wedge e$. On the other hand $\Sigma(w_1) = \Sigma(w_2)$ holds. It follows that $\Sigma(w_2) \wedge e = \Sigma(w_2)$ holds in Y. Thus, there exists a finite sequence of words $(w_2)_Y(e, 1) \equiv \zeta_1, \ldots, \zeta_l = (w_2)_Y$, such that ζ_{j+1} is obtained from ζ_j using relations in $R_Y \cup R_G$ and standard inverse semigroup relations. We may now deduce that $w_1 = w_2$ is indeed a consequence of relations in $\mathbf{R}$ and of inverse semigroup relations.

6. Examples

To illustrate our results, we give two examples. In both examples the infinite antichain $A = \{\ldots e_{-1}, e_0, e_1, \ldots\}$ plays a central role.

EXAMPLE 6.1. Let Y denote the free semilattice generated by infinitely many elements $A = \{\ldots e_{-1}, e_0, e_1, \ldots\}$ with an identity element 1_Y adjoined. Let $G = \langle g\rangle$ be the infinite cyclic group. For each $n \in \mathbb{Z}$ define $\psi_{g^n} : e_i \mapsto e_{i+n}, 1_Y \mapsto 1_Y$ and extend this map to an automorphism of Y. It is clear that an action of G on Y

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is defined. Since Y is the free semilattice with an identity adjoined and since $e_n = {}^{g^n} e_0$, we have that Y is generated by $Y_0 = \{e_0, 1_Y\}$ with respect to this action. It is immediate that Y satisfies the maximum condition. We show that $S = Y \rtimes G$ is finitely presented by showing that Y is finitely presented with respect to the action of G. Let $Q = \{e_0 \wedge e_0 = e_0, 1_Y \wedge 1_Y = 1_Y\}$. We show that $Y = \mathrm{InvAct}_G \langle Y_0 | Q\rangle$. Assume that the relation $p = q$ holds in Y. Since Y is the free semilattice with an identity element adjoined, we have that ${}^{g^n} e_0$ appears in the word p if and only if it appears in q. Thus, we can construct a finite sequence of words $p \equiv u_0, \ldots, u_n \equiv q$ such that u_{j+1} is obtained from u_j by applying relations in Q_A or standard inverse semigroup relations.

EXAMPLE 6.2. Let A be the infinite antichain and adjoin an identity element 1_Y on the top and a zero element 0_Y on the bottom. Denote the semilattice obtained by Y. Let $G = \langle g\rangle$ be the infinite cyclic group. For each $n \in \mathbb{Z}$ define $\psi_{g^n} : e_i \mapsto e_{i+n}$, $1_Y \mapsto 1_Y$, $0_Y \mapsto 0_Y$. It is clear that an action of G on Y is defined. Note that Y is generated by $Y_0 = \{e_0, 1_Y\}$ with respect to this action. Indeed $e_n = {}^{g^n} e_0$ and $e_n \wedge e_m = 0_Y$ for all $n \neq m$. We show that $S = Y \rtimes G$ is not finitely presented by showing that Y is not finitely presented with respect to the action of G.

Assume that Y is finitely presented with respect to the action of G. By Proposition 3.1 we may assume that $Y = \mathrm{InvAct}_G \langle Y_0 | Q\rangle$, where Q is a finite set of relations on $FSI({}^G Y_0)$. Consider an arbitrary word

$$w \equiv {}^{g^{n_1}} e_0 \wedge \cdots \wedge {}^{g^{n_i}} e_0 \wedge \cdots \wedge {}^{g^{n_j}} e_0 \ldots \wedge {}^{g^{n_m}} e_0.$$

Assume that $n_i \leq n_k \leq n_j$ for all $1 \leq k \leq m$ and let $d(w) = n_j - n_i$. We call $d(w)$ the distance in w. Let $d \in \mathbb{N}$ so that $d = d(u)$ where $u = v$ or $v = u$ is a relation in Q and if w is word appearing on one side of a relation in Q then $d(w) \leq d$. Owing to the defined action of G on Y we also have that for all relations $u = v$ in Q_A, $d(u), d(v) \leq d$.

Clearly the relation $p \equiv {}^{g^{d+1}} e_0 \wedge e_0 = {}^{g^{d+2}} e_0 \wedge e_0 \equiv q$ holds in Y and $d < d(p) < d(q)$. If $p = q$ is a consequence of relations in Q_A and of inverse semigroup relations, then there exists a finite sequence of words ${}^{g^{d+1}} e_0 \wedge e_0 \equiv u_0, u_1 \ldots u_t \equiv {}^{g^{d+2}} e_0 \wedge e_0$ so that u_{j+1} is obtained from u_j by applying relations in Q_A and of standard inverse semigroup relations. Since $d < d(p)$ there exists no relation in Q_A that can be applied to p. On the other hand, by applying inverse semigroup relations the distance in p cannot be increased. It follows that $p = q$ is not a consequence of relations in Q_A and of standard inverse semigroup relations and, hence, S is not finitely presented.

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