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A FINITENESS CRITERION FOR ORTHOMODULAR LATTICES

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The main result of this paper is the following:

THEOREM. *Every finitely generated orthomodular lattice L with finitely many maximal Boolean subalgebras (blocks) is finite.*

If L has one block only, our theorem reduces to the well-known fact that every finitely generated Boolean algebra is finite. On the other hand, it is known that a finitely generated orthomodular lattice without any further restrictions can be infinite. In fact, in [**2**] we constructed an orthomodular lattice which is generated by a three-element set with two comparable elements, has infinitely many blocks and contains an infinite chain.

As preliminaries of our main result, we obtain two more theorems, which may be of some independent interest. The first one gives an orthomodular analogue of the well-known direct factorization of finite-dimensional complemented modular lattices, see e.g. [**1**, p. 93, Theorem 10]. The second theorem, of which our main result is an immediate corollary, describes in detail the generating process of an orthomodular lattice with finitely many blocks.

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1. Basic definitions and results. The notions and facts of this section are standard and well known, see for example [**1**, p. 55 f.f.; **3**; **4**].

An ortholattice is an algebra $(L; \vee, \wedge, ', 0, 1)$, where $(L; \vee, \wedge)$ is a lattice with lower bound 0 and upper bound 1 and $x \to x'$ is an orthocomplementation, that is an anti-monotone complementation of period 2. An orthomodular lattice (abbreviated: OML) is an ortholattice satisfying the orthomodular law:

$$\text{if } a \leqq b \text{ then } a \vee (a' \wedge b) = b.$$

If a is an element of an OML L, define $a^0 = a$ and $a^1 = a'$. If A is a finite subset of L the *commutator* $c(A)$ of A is defined by $c(A) = \bigwedge_{\alpha \in 2^A} \bigvee_{a \in A} a^{\alpha(a)}$. If $A = \{x, y\}$ we write $c(x, y)$ instead of $c(\{x, y\})$. If A and B are finite subsets

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of L and $A \subseteq B$ then $c(A) \leqq c(B)$. The elements $a, b \in L$ commute, in symbols: aCb, if and only if $c(a, b) = 0$. This is in particular the case if a and b are comparable. The relation C is symmetric and the set $C(a)$ of all elements of L commuting with the element a is a subalgebra of L. If one of the elements a, b, c commutes with the other two then the distributive law $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and its dual hold. The *center* $C(L)$ of L is the set of all elements of L which commute with every element of L. Clearly $\{0, 1\} \subseteq C(L)$. $C(L)$ is said to be *trivial* if and only if $C(L) = \{0, 1\}$. L is (*directly*) *irreducible* if and only if $C(L)$ is trivial and $0 \neq 1$. A *block* of L is a maximal Boolean subalgebra of L. Every set of pairwise commuting elements is contained in a block and the blocks are exactly the maximal sets of pairwise commuting elements of L. $C(L)$ is the intersection of all blocks of L.

If $a \in L$ then the interval $[0, a] = \{x \in L | 0 \leqq x \leqq a\}$ becomes an OML if if we define the orthocomplement $x^{\#}$ of an element $x \in [0, a]$ by $x^{\#} = a \wedge x'$. The map $x \rightarrow a \wedge x$ is a homomorphism of L onto $[0, a]$ if and only if $a \in C(L)$. In this case the map $x \rightarrow (a \wedge x, a' \wedge x)$ is an isomorphism between L and the direct product $[0, a] \times [0, a']$. If L_1, L_2 are *OML*s then the blocks of the product $L_1 \times L_2$ are exactly the direct products of the blocks of L_1 and the blocks of L_2. If, in particular, L_1 and L_2 are not Boolean, then L_1 and L_2 have fewer blocks than $L_1 \times L_2$.

2. Some preliminary results. Throughout the rest of this paper we assume that L is an *OML* with finitely many blocks, that $\mathfrak{A}$ is the set of all blocks of L and that Ω is the set of all $\mathfrak{B} \subseteq \mathfrak{A}$ satisfying $\bigcap \mathfrak{B} \not\subseteq \bigcup (\mathfrak{A} - \mathfrak{B})$. Here we define the intersection of the empty subset of $\mathfrak{A}$ to be L and the union of the empty subset of $\mathfrak{A}$ to be $\{0, 1\}$, so that $\mathfrak{A} \in \Omega$ if and only if $C(L)$ is not trivial. Note that every $\mathfrak{B} \in \Omega$ is not empty and that Ω is empty if and only if L is a Boolean algebra with at most two elements. We define the *rank* $r(\mathfrak{B})$ of a set $\mathfrak{B} \in \Omega$ to be n $(\geqq 1)$ if and only if for every sequence $\mathfrak{B} = \mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \ldots \subset \mathfrak{B}_k$ in Ω ($\subset$ proper inclusion), $k \leqq n$ holds and there exists such a sequence with $k = n$.

(2.1) *If* $\mathfrak{B} \in \Omega$ *and* $a \in (\bigcap \mathfrak{B}) - (\bigcup (\mathfrak{A} - \mathfrak{B}))$ *then* $C(a) = \bigcup \mathfrak{B}$; *in particular* $\bigcup \mathfrak{B}$ *is a subalgebra of* L. *The blocks of this subalgebra are exactly the elements* $\mathfrak{B}$.

Proof. Clearly $\bigcup \mathfrak{B} \subseteq C(a)$. If $b \in C(a)$ then there exists $B \in \mathfrak{A}$ with $a, b \in B$. Since $a \notin \bigcup (\mathfrak{A} - \mathfrak{B})$ it follows $B \in \mathfrak{B}$ and hence $b \in \bigcup \mathfrak{B}$, proving $C(a) = \bigcup \mathfrak{B}$. Clearly every $B \in \mathfrak{B}$ is a block of $\bigcup \mathfrak{B}$. If there was a block in $\bigcup \mathfrak{B}$ not in $\mathfrak{B}$, such a block would be $B \cap \bigcup \mathfrak{B}$ for some $B \in \mathfrak{A} - \mathfrak{B}$. But if $B \cap \bigcup \mathfrak{B}$ was a block of $\bigcup \mathfrak{B}$, then, since $a \notin B$ there would exist $b \in B \cap \bigcup \mathfrak{B}$ with a $\not{C}\, b$ and it would follow that $b \notin C(a)$ and $b \in \bigcup \mathfrak{B}$, contradicting $C(a) = \bigcup \mathfrak{B}$.

(2.2) $C(L) = \bigcap \{\bigcup \mathfrak{B} | \mathfrak{B} \in \Omega\}$.

Proof. If $x \in C(L)$ then $x \in B$ for every $B \in \mathfrak{A}$; hence, since every $\mathfrak{B} \in \Omega$ is non-empty, $x \in \bigcup \mathfrak{B}$ for every $\mathfrak{B} \in \Omega$ and $x \in \bigcap \{\bigcup \mathfrak{B} | \mathfrak{B} \in \Omega\}$. Assume con-

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versely that $x \notin C(L)$. Then there exists $y \in L$ with $x \not\mathrel{C} y$. Define $\mathfrak{B} = \{B \in \mathfrak{A} | x \notin B, y \in B\}$. Clearly $y \in \bigcap \mathfrak{B}$. But $y \in \bigcup (\mathfrak{A} - \mathfrak{B})$ would imply the existence of a set $B \in \mathfrak{A}$ with $x, y \in B$, contradicting $x \not\mathrel{C} y$. We thus have $\mathfrak{B} \in \Omega$ and $x \notin \bigcup \mathfrak{B}$, i.e. $x \notin \bigcap\{\bigcup \mathfrak{B} | \mathfrak{B} \in \Omega\}$.

(2.3) *Let A be a finite subset of L containing, for every $\mathfrak{B} \in \Omega$, an element of the set $(\bigcap \mathfrak{B}) - (\bigcup (\mathfrak{A} - \mathfrak{B}))$. Then for every $\alpha \in 2^A$, $\bigvee_{x \in A} x^{\alpha(x)} \in C(L)$ and hence $c(A) \in C(L)$.*

Proof. For every $a \in A$, either $a \leqq \bigvee_{x \in A} x^{\alpha(x)}$ or $a' \leqq \bigvee_{x \in A} x^{\alpha(x)}$ and hence $a\, C \bigvee_{x \in A} x^{\alpha(x)}$. It follows from (2.1) and (2.2) that $\bigvee_{x \in A} x^{\alpha(x)} \in \bigcap \{C(a) | a \in A\} = \bigcap \{\bigcup \mathfrak{B} | \mathfrak{B} \in \Omega\} = C(L)$.

(2.4) *If A is a set as in* (2.3), *$A \subseteq X \subseteq L$ and X finite then $c(A) = c(X)$.*

Proof (by induction on the number $|X - A|$ of elements of $X - A$). If $|X - A| = 0$ there is nothing to prove. If $|X - A| \geqq 1$, pick $b \in X - A$ and define $Y = X - \{b\}$. By (2.3), $\bigvee_{y \in Y} y^{\beta(y)} \in C(L)$ holds for every $\beta \in 2^Y$. It follows that

$$c(X) = \bigwedge_{\beta \in 2^Y} \left(\left(b \vee \bigvee_{y \in Y} y^{\beta(y)} \right) \wedge \left(b' \vee \bigvee_{y \in Y} y^{\beta(y)} \right) \right)$$

$$= \bigwedge_{\beta \in 2^Y} \left((b \wedge b') \vee \bigvee_{y \in Y} y^{\beta(y)} \right) = c(Y),$$

which gives the claim by inductive hypothesis.

(2.5) *If A is a set as in* (2.3) *then* $[0, c'(A)]$ is a Boolean algebra.

Proof. Assume $x, y \in [0, c'(A)]$ and let $\bar{c}(x, y)$ be the commutator of $\{x, y\}$ in $[0, c'(A)]$. We have to show that $\bar{c}(x, y) = 0$. But by (2.4): $\bar{c}(x, y) = c(x, y) \wedge c'(A) \leqq c(A \cup \{x, y\}) \wedge c'(A) = c(A \cup \{x, y\}) \wedge c'(A \cup \{x, y\}) = 0$, which implies $\bar{c}(x, y) = 0$.

(2.6) *If A is a set as in* (2.3) *then the OML $[0, c(A)]$ has no non-trivial Boolean factor.*

Proof. If $[0, d]$ is a Boolean factor of $[0, c(A)]$ and if $\bar{c}(A \wedge d)$ is the commutator of $\{a \wedge d | a \in A\}$ in $[0, d]$ then we have by the general finite distributive law in Boolean algebras $0 = \bar{c}(A \wedge d) = c(A) \wedge d = d$, i.e. the factor $[0, d]$ is trivial.

From (2.3), (2.5), (2.6) and the earlier remark that a non-trivial factor of an *OML* without Boolean factor has less blocks than L, the following theorem may be proved easily by induction.

THEOREM 1. *Every OML L with finitely many blocks is isomorphic to a direct product $B_0 \times L_1 \times L_2 \times \ldots \times L_n$ $(n \geqq 0)$ where B_0 is a Boolean algebra and $L_1, L_2, \ldots, L_n$ are irreducible OMLs with at least two blocks each.*

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COROLLARY. *There exists a finite sequence $c_0, \ldots, c_n$ of central elements of L satisfying:*

1. $[0, c_0] \subseteq C(L)$;
2. $c_i \wedge \bigvee_{j \neq i} c_j = 0 \quad (i = 0, 1, \ldots, n)$;
3. *every $a \in C(L)$ is of the form $a = a_0 \vee a_1 \vee \ldots \vee a_n$, where $a_0 \leqq c_0$ and $a_i = 0$ or $a_i = c_i$ for $i = 1, 2, \ldots, n$.*

Proof. If $L \simeq B_0 \times L_1 \times \ldots \times L_n$ is a direct factorization of L as in Theorem 1, then the elements of L corresponding to the elements $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)$ of the product obviously form a sequence with the desired property.

We call a sequence $c_0, \ldots, c_n$ with the properties described in the corollary a *central basis of L*.

3. The fundamental lemma. *Let L be irreducible and let X be a generating set of L containing the elements of a central basis of each subalgebra $\bigcup \mathfrak{B}$ with $\mathfrak{B} \in \Omega$. Define recursively for every $n \geqq 1$ a set X_n, and for every $\mathfrak{B} \in \Omega$ with $r(\mathfrak{B}) = n$, a set $S_{\mathfrak{B}}$ as follows (ΓA is the subalgebra generated by A):*

$$X_1 = X \quad \text{and} \quad S_{\mathfrak{B}} = \Gamma(X \cap \bigcup \mathfrak{B}) \quad \text{if} \quad r(\mathfrak{B}) = 1, \text{and}$$

$$X_n = X \cup \bigcup \{S_{\mathfrak{B}'} | r(\mathfrak{B}') < n\}, \quad S_{\mathfrak{B}} = \Gamma(X_n \cap \bigcup \mathfrak{B}) \quad \text{if } r(\mathfrak{B}) = n \geqq 2.$$

Then $\bigcap \mathfrak{B} \subseteq S_{\mathfrak{B}}$ holds for all $\mathfrak{B} \in \Omega$.

Proof (by induction on $r(\mathfrak{B})$). Assume $n \geqq 1$, that the claim is true for all $k < n$ $(1 \leqq k)$ and let $\mathfrak{B} \in \Omega$ have rank n.

(1) If $\mathfrak{B} \subset \mathfrak{B}' \subseteq \mathfrak{A}$ and $b \in \bigcap \mathfrak{B}'$ then $b \in S_{\mathfrak{B}}$.

Define $\mathfrak{B}'' = \{B \in \mathfrak{A} | b \in B\}$. If $\mathfrak{B}'' = \mathfrak{A}$ then $b \in C(L) = \{0, 1\} \subseteq S_{\mathfrak{B}}$. If $\mathfrak{B}'' \neq \mathfrak{A}$ then $\mathfrak{B}'' \in \Omega$, $r(\mathfrak{B}'') < n$ and it follows by inductive hypothesis that $b \in S_{\mathfrak{B}''} \cap \bigcup \mathfrak{B} \subseteq X_n \cap \bigcup \mathfrak{B} \subseteq S_{\mathfrak{B}}$.

Define now an element $g \in L$ to be *good* if and only if it satisfies the following two conditions:

(G1) If $a \in S_{\mathfrak{B}}$ and $a \vee g \in \bigcap \mathfrak{B}$ then $a \vee g \in S_{\mathfrak{B}}$,
(G2) If $a \in S_{\mathfrak{B}}$ and $a \wedge g \in \bigcap \mathfrak{B}$ then $a \wedge g \in S_{\mathfrak{B}}$.

(2) Every $g \in S_{\mathfrak{B}}$ is good.

This is an immediate consequence of the definition.

(3) Every $g \in L - \bigcup \mathfrak{B}$ is good.

If $a \vee g \in \bigcap \mathfrak{B}$ in this case then $a \vee g \in B$ for some $B \in \mathfrak{A} - \mathfrak{B}$ and the claim of (G1) follows from (1). The condition (G2) follows dually.

(4) Every $g \in X$ is good.

If $g \in \bigcup \mathfrak{B}$ then $g \in S_{\mathfrak{B}}$ and the claim follows from (2). If $g \notin \bigcup \mathfrak{B}$ the claim follows from (3).

(5) If g_1, g_2 are good then $g = g_1 \vee g_2$ satisfies (G1).

If $a \vee g \in \bigcap \mathfrak{B}'$ for some $\mathfrak{B}' \supset \mathfrak{B}$ the claim follows from (1). We may thus assume that $a \vee g \in (\bigcap \mathfrak{B}) - (\bigcup(\mathfrak{A} - \mathfrak{B}))$. Since $a \vee g$ and $a \vee g_i$ $(i = 1, 2)$ are comparable it follows that $a \vee g_i \in \bigcup \mathfrak{B}$ $(i = 1, 2)$. Let $c_0, c_1, \ldots, c_n$ be a central basis of $\bigcup \mathfrak{B}$ belonging to X and let $a \vee g = a_0 \vee a_1 \vee \ldots \vee a_n$ be a representation of $a \vee g$ as in condition 3 of the corollary. Since $a \vee g_1$, $a \vee g_2 \in \bigcup \mathfrak{B}$ it follows from the corollary that $a \vee c_0' \vee g_i \in \bigcap \mathfrak{B}$ $(i = 1, 2)$ and, since g_1, g_2 are good, that $a \vee c_0' \vee g_i \in S_{\mathfrak{B}}$ $(i = 1, 2)$ and hence $a \vee c_0' \vee g \in S_{\mathfrak{B}}$. Since $c_0 \vee a_1 \vee \ldots \vee a_n \in S_{\mathfrak{B}}$ it follows from this that $(a \vee c_0' \vee g) \wedge (c_0 \vee a_1 \vee \ldots \vee a_n) \in S_{\mathfrak{B}}$. But $(a \vee c_0' \vee g) \wedge (c_0 \vee a_1 \vee \ldots \vee a_n) = a \vee g \vee (c_0' \wedge (c_0 \vee a_1 \vee \ldots \vee a_n)) = a \vee g \vee (c_0' \wedge (a_1 \vee \ldots \vee a_n)) = a \vee g$, which proves (5).

(6) If g_1, g_2 are good then $g = g_1 \vee g_2$ satisfies (G2).

By (1) we may again assume that $a \wedge g \notin \bigcup (\mathfrak{A} - \mathfrak{B})$. Let $c_0, c_1, \ldots, c_n$ again be a central basis of $\bigcup \mathfrak{B}$ belonging to X. Since by (3) every $g \in L - \bigcup \mathfrak{B}$ is good we may assume that $g \in \bigcup \mathfrak{B}$ and hence that $c_0' \vee g \in \bigcap \mathfrak{B}$. We claim that $c_0' \vee g \in S_{\mathfrak{B}}$. If $c_0' \vee g \in B$ for some $B \in \mathfrak{A} - \mathfrak{B}$ the claim follows from (1). If $c_0' \vee g \notin \bigcup (\mathfrak{A} - \mathfrak{B})$ we obtain $c_0' \vee g_i \in \bigcap \mathfrak{B}$ and, since the g_i are good, that $c_0' \vee g_i \in S_{\mathfrak{B}}$ and hence $c_0' \vee g \in S_{\mathfrak{B}}$. It follows from this that $c_0 \wedge a \wedge g = c_0 \wedge ((a \wedge g) \vee c_0') = c_0 \wedge ((a \vee c_0') \wedge (c_0' \vee g)) \in S_{\mathfrak{B}}$. Let again $a \wedge g = a_0 \vee a_1 \vee \ldots \vee a_n$ be a representation of $a \wedge g$ as in the corollary. We then obtain $(c_0 \wedge a \wedge g) \vee (a_1 \vee \ldots \vee a_n) \in S_{\mathfrak{B}}$, which, since $(c_0 \wedge a \wedge g) \vee (a_1 \vee \ldots \vee a_n) = a \wedge g \wedge (c_0 \vee a_1 \vee \ldots \vee a_n) = a \wedge g$, implies (6).

By definition the set of all good elements is closed under orthocomplementation. Since it contains X by (4) and is a subalgebra by (5) and (6) it follows that it is equal to L and hence that every element of L is good. To prove the claim, assume now that $g \in \bigcap \mathfrak{B}$. Then $0 \vee g = g \in \bigcap \mathfrak{B}$, hence, since $0 \in S_{\mathfrak{B}}$ and g is good, $g = 0 \vee g \in S_{\mathfrak{B}}$, which proves the lemma.

4. Proof of the main theorem. As an immediate consequence of the fundamental lemma we obtain the following:

THEOREM 2. *Under the assumptions of the fundamental lemma*

$$L = \bigcup \{S_{\mathfrak{B}} | \mathfrak{B} \in \Omega\}.$$

(*Here we define* $\bigcup \{S_{\mathfrak{B}} | \mathfrak{B} \in \Omega\} = \{0, 1\}$ *if* $\Omega = \emptyset$.)

Proof. Clearly the right hand side of the equation is contained in the left. Assume now $a \in L$. If $a \in C(L)$ then $a = 0$ or $a = 1$ and a is trivially contained in the right hand side. If $a \in L - C(L)$ then the set $\mathfrak{B} = \{B \in \mathfrak{A} | a \in B\}$ belongs to Ω and $a \in \bigcap \mathfrak{B}$. It follows from the fundamental lemma that $a \in S_{\mathfrak{B}}$ and hence that a belongs to the right hand side.

Proof of the main theorem. We use induction on the number m of blocks of L. If $m = 1$ the claim reduces to the well-known fact that a finitely generated Boolean algebra is finite. Assume then that $m \geqq 2$ and let $L \simeq B_0 \times L_1 \times \ldots \times L_n$ be a representation of L as in Theorem 1. If $n \geqq 2$ then each of $L_1, \ldots, L_n$ has less blocks than L and the claim follows by inductive hypothesis. Hence we may restrict ourselves to the case that L is irreducible. Let in this case X be a finite generating set of L. We may assume without loss of generality that X contains the elements of a central basis of each of the subalgebras $\bigcup \mathfrak{B}$ with $\mathfrak{B} \in \Omega$. By (2.1) the blocks of $\bigcup \mathfrak{B}$ are exactly the elements of $\mathfrak{B}$. By the remarks preceding (2.1) the irreducibility of L implies that $\bigcup \mathfrak{B}$ has fewer blocks than L. Since $S_{\mathfrak{B}}$ is a subalgebra of $\bigcup \mathfrak{B}$ it has at most as many blocks as $\bigcup \mathfrak{B}$, hence also fewer blocks than L. Since by definition the $S_{\mathfrak{B}}$ are finitely generated they are, by inductive hypothesis, finite. It follows from Theorem 2 that L is finite, which completes the proof.

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