Canad. Math. Bull. Vol. **15** (3), 1972.

# ALMOST AUTOMORPHIC INTEGRALS OF ALMOST AUTOMORPHIC FUNCTIONS

BY
M. ZAKI

**Introduction.** Bochner has introduced the idea of almost automorphy in various contexts (see for example [1] and [2]). We shall use the following definition:

A measurable real valued function $f$ of a real variable will be called *almost automorphic* if from every given infinite sequence of real numbers $\{\alpha'_n\}_{n=1}^{\infty}$ we can extract a subsequence $\{\alpha_n\}$ such that

(i) $\lim_{n\to\infty} f(t+\alpha_n)=g(t)$ exists for every real $t$ but no kind of uniformity of convergence is stipulated;

(ii) $\lim_{n\to\infty} g(t-\alpha_n)=h(t)$ exists for every $t$;

(iii) $h(t)=f(t)$ for every $t$.

REMARK. $f(t)$ is bounded. Otherwise we would have a sequence $\{\beta_n\}$ such that $\lim_{n\to\infty} |f(\beta_n)|=\infty$. But from almost automorphy of $f$ there should be a subsequence of $\{\beta_n\}$, say $\{\beta'_n\}$, such that $\lim_{n\to\infty} f(\beta'_n)$ exists which will be a contradiction of $\lim_{n\to\infty} |f(\beta_n)|=\infty$. Hence when $f(t)$ is almost automorphic

$$\sup_{-\infty<t<\infty} |f(t)| = M < \infty. \tag{1}$$

It is easy to verify that

$$\sup_{-\infty<t<\infty} |g(t)| \le \sup_{-\infty<t<\infty} |f(t)| = M < \infty. \tag{2}$$

In this note we prove the following result, which, to the best of our knowledge, is new.

THEOREM 1. *Let $f(t)$ be a real valued almost automorphic function. Then the primitive*

$$F(x) = \int_0^x f(t)\, dt \tag{3}$$

*is almost automorphic if and only if it is bounded on the real line.*

**Proof.** I. Let $\{\alpha'_n\}$ be any given sequence of real numbers. Since $f(t)$ is almost automorphic and

$$|F(x)| \le M < \infty \quad \text{for} \quad -\infty < x < \infty \tag{4}$$

Received by the editors February 26, 1971 and, in revised form, May 12, 1971.

we can choose a subsequence $\{\alpha_n\}$ such that

$$\text{(5)}\qquad \begin{aligned} &\text{(i)}\ \lim_{n\to\infty} f(t+\alpha_n) = g(t) \quad \text{for every } t \text{ in } R;\\ &\text{(ii)}\ \lim_{n\to\infty} g(t-\alpha_n) = f(t) \quad \text{for every } t \text{ in } R;\\ &\text{(iii)}\ \lim_{n\to\infty} F(\alpha_n) = C_1 \quad \text{exists.}\end{aligned}$$

Consider

$$\text{(6)}\qquad I = \int_0^x f(t+\alpha_n)\,dt.$$

Put $t+\alpha_n=\sigma$. Then

$$\begin{aligned} I &= \int_{\alpha_n}^{x+\alpha_n} f(\sigma)\,d\sigma\\ &= F(x+\alpha_n)-F(\alpha_n).\end{aligned}$$

Hence

$$\text{(7)}\qquad F(x+\alpha_n) = F(\alpha_n)+\int_0^x f(t+\alpha_n)\,dt.$$

Since $\sup_{-\infty<t<\infty} |f(t)|<\infty$ and $\lim_{n\to\infty} f(t+\alpha_n)=g(t)$ for every $t$ it follows from Lebesgue's theorem that

$$\lim_{n\to\infty} \int_0^x f(t+\alpha_n)\,dt = \int_0^x g(t)\,dt.$$

Therefore $\lim_{n\to\infty} F(x+\alpha_n)$ exists for every real $x$ and is equal to $C_1+\int_0^x g(t)\,dt$. Let us call it $v(x)$. That is

$$\text{(8)}\qquad \begin{aligned} \lim_{n\to\infty} F(x+\alpha_n) &= v(x)\\ &= C_1+\int_0^x g(t)\,dt.\end{aligned}$$

Therefore, as in (6) and (7)

$$\text{(9)}\qquad v(x-\alpha_n) = v(-\alpha_n)+\int_0^x g(t-\alpha_n)\,dt.$$

Now $v(x)=\lim_{n\to\infty} F(x+\alpha_n)$ for every $x$ and $\sup_{-\infty<x<\infty} |F(x)|=M<\infty$. Therefore

$$\begin{aligned} |v(x)| &\le \lim_{n\to\infty} |F(x+\alpha_n)| \quad \text{for each } x\\ &\le M.\end{aligned}$$

Hence

$$\text{(10)}\qquad \sup_{-\infty<x<\infty} |v(x)| \le M.$$

https://doi.org/10.4153/CMB-1972-078-5 Published online by Cambridge University Press

Therefore, without loss of any kind we may assume that

(11) $$\lim_{n\to\infty} v(-\alpha_n) \quad \text{exists and is equal to } C_2.$$

(If $\{v(-\alpha_n)\}$ is not convergent, then by (10) some subsequence $\{v(-\beta_n)\}$ of $\{v(-\alpha_n)\}$ is convergent. The corresponding statements (5) (i)–(iii) remain valid with $\{\alpha_n\}$ replaced by $\{\beta_n\}$, and so we can repeat the preceding arguments with $\{\beta_n\}$ in place of $\{\alpha_n\}$, and with the *same* $g$, $C_1$, $v$ as before.) Moreover $g(t)$ is bounded over $R$. Hence $\{g(t-\alpha_n)\}_{n=1}^{\infty}$ is a uniformly bounded sequence of functions which converges everywhere to $f(t)$. Applying Lebesque's theorem once again we get

$$\lim_{n\to\infty} \int_0^x g(t-\alpha_n)\,dt = \int_0^x f(t)\,dt \quad \text{for any } x.$$

Therefore, from (9) and (11)

$$\lim_{n\to\infty} v(x-\alpha_n) \quad \text{exists for every } x$$

and

(12) $$\begin{aligned}\lim_{n\to\infty} v(x-\alpha_n) &= C_2+\int_0^x f(t)\,dt\\ &= W(x) \quad \text{say.}\end{aligned}$$

II. If we could show that $C_2$ is zero, it would be established that $F(x)$ is almost automorphic. We have, in fact, shown that

$$\lim_{m\to\infty}\lim_{n\to\infty} F(x+\alpha_n-\alpha_m) = W(x) \quad \text{exists for each } x.$$

Denote this operation of taking double limits by $\mathscr{A}$. That is

(13) $$\mathscr{A}F(x) = W(x).$$

Since $W(x)=\lim_{n\to\infty}v(x-\alpha_n)$, by the same argument it follows from (10)

$$|W(x)| \le M \quad \text{for} \quad -\infty < x < \infty.$$

That is

$$|\mathscr{A}F(x)| \le M \quad \text{for every } x.$$

Now

$$\begin{aligned}\mathscr{A}F(x) &= W(x)\\ &= C_2+\int_0^x f(t)\,dt\\ &= C_2+F(x).\end{aligned}$$

Applying $\mathscr{A}$ again to the two sides above (since it has a meaning) we get

$$\begin{aligned}\mathscr{A}\mathscr{A}F(x) &= \mathscr{A}C_2+\mathscr{A}F(x)\\ &= C_2+C_2+F(x)\\ &= 2C_2+F(x).\end{aligned}$$

In general, for any positive integer $n$

$$\mathscr{A}^n F(x) = nC_2 + F(x). \tag{14}$$

On the other hand starting with $W(x)$ in place of $F(x)$ we find that

$$|\mathscr{A} W(x)| \leq M \quad \text{for every } x.$$
$$|\mathscr{A}^2 F(x)| \leq M, \quad \text{or in general,}$$
$$|\mathscr{A}^n F(x)| \leq M.$$

Hence from (14)

$$\begin{aligned} |nC_2| &\leq |\mathscr{A}^n F(x)| + |F(x)| \\ &\leq 2M. \end{aligned} \tag{15}$$

If $C_2$ is not zero then the left-hand side in (15) becomes larger and larger as $n$ increases which will contradict (15). Hence $C_2$ must be zero and we have $W(x) = F(x)$ which proves that $F(x)$ is almost automorphic when the primitive is bounded. If $F(x) = \int_0^x f(t)\, dt$ is almost automorphic then it is certainly bounded; and this completes the proof of the theorem.

ACKNOWLEDGMENT. My thanks are due to Professor S. Zaidman for suggesting the problem and some useful discussions during the preparation of this note.

## REFERENCES

1. S. Bochner, *Continuous mappings of almost automorphic and almost periodic functions*, Proc. Nat. Acad. Sci. U.S.A., **52**, 1964, 907–910.

2. ——, *Uniform convergence of monotone sequences of functions*, Proc. Nat. Acad. Sci. U.S.A., **47** (1961), 582–585.

SIR GEORGE WILLIAMS UNIVERSITY,
MONTREAL, QUEBEC