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POLYNOMIALS AND FUNCTIONS WITH FINITE SPECTRA ON LOCALLY COMPACT ABELIAN GROUPS

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In this paper we define polynomials on a locally compact Abelian group G and prove the equivalence of our definition with that of Domar. We explore the properties of polynomials and determine their spectra. We also characterise the primary ideals of certain Beurling algebras $L^1_w(\mathbf{Z})$ on the group of integers $\mathbf{Z}$. This allows us to classify those elements of $L^\infty_w(G)$ that have finite spectrum. If φ is a uniformly continuous function with bounded differences then there is a Beurling algebra naturally associated with φ. We give a condition on the spectrum of φ relative to this algebra which ensures that φ is bounded. Finally we give spectral conditions on a bounded function on $\mathbb{R}$ that ensure that its indefinite integral is bounded.

1. INTRODUCTION, NOTATION AND PRELIMINARIES

Throughout this paper G will denote a locally compact Abelian group equipped with Haar measure. We consider certain unbounded functions $\varphi : G \to \mathbb{C}$ and discuss the notion of Beurling spectrum for φ.

Firstly, the spaces $L^p(G)$ with corresponding norms $\|.\|_p$ are defined as usual using the Haar measure (see [6]). We shall always identify functions on G which agree almost everywhere (see [3, p.36]). In particular, $L^1(G)$ is a Banach algebra under the convolution product and carries an involution defined by

$$f^*(x) = \overline{f(-x)} \quad \text{for} \quad x \in G, \quad f \in L^1(G). \tag{1.1}$$

By $L^\infty_{loc}(G)$ we mean the space of functions $\varphi : G \to \mathbb{C}$ such that $\chi_K \varphi \in L^\infty(G)$ for every compact set $K \subset G$, χ_K denoting the characteristic function of K.

The action of the dual group $\widehat{G}$ on G will be denoted $\gamma(x) = (x, \gamma)$ where $x \in G$, $\gamma \in \widehat{G}$ and the Fourier transform of $f \in L^1(G)$ is defined by $\widehat{f}(\gamma) = \int_G f(x)(-x, \gamma)\, dx$ for $\gamma \in \widehat{G}$. Unless otherwise stated, the group operation on G will be addition and on $\widehat{G}$ multiplication. The unit character in $\widehat{G}$ is denoted by $\widehat{e}$. The translate φ_y and difference $\Delta_y \varphi$ by $y \in G$ of a function $\varphi : G \to \mathbb{C}$ are defined by

$$\varphi_y(x) = \varphi(x + y) \text{ and } \Delta_y \varphi(x) = \varphi(x + y) - \varphi(x) \text{ for } x \in G. \tag{1.2}$$

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Following Reiter [5, p.83] we call a function $w \in L^\infty_{loc}(G)$ a weight function on G if for all $x, y \in G$

$$w(x) \geqslant 1, \tag{1.3}$$

$$w(x+y) \leqslant w(x)w(y). \tag{1.4.}$$

Given a weight on G we define

$$L^1_w(G) = \{f \in L^1(G) : \|f\|_{1,w} = \int_G |f(x)|\, w(x)\, dx < \infty\}. \tag{1.5}$$

Then $L^1_w(G)$ is a subalgebra of $L^1(G)$ which is a Banach algebra under the norm $\|.\|_{1,w}$. The Banach space dual of $L^1_w(G)$ is

$$L^\infty_w(G) = \{\varphi \in L^\infty_{loc}(G) : \|\varphi\|_{\infty,w} = \operatorname{ess\,sup}_{x \in G} \frac{|\varphi(x)|}{w(x)} < \infty\}. \tag{1.6}$$

As noted by Reiter [5, p.84], we may assume that w is upper semicontinuous, and so each continuous function $\varphi \in L^\infty_w(G)$ satisfies

$$|\varphi(x)| \leqslant \|\varphi\|_{\infty,w}\, w(x) \quad \text{for all } x \in G \tag{1.7}$$

It is known that $L^1_w(G)$ is a Wiener algebra if and only if w satisfies the following condition of Beurling-Domar [5, p.132], [1, Theorem 2.11]:

$$\sum_{m=1}^{\infty} \frac{\log w(mx)}{m^2} < \infty \text{ for all } x \in G. \tag{1.8}$$

These algebras were introduced by Beurling for the case $G = \mathbb{R}$ and studied in the general case by Domar (see [1]).

If $\varphi \in L^\infty_w(G)$, $f^* * \varphi$ is well-defined for all $f \in L^1_w(G)$ and $I_w(\varphi) = \{f \in L^1_w(G) : f^* * \varphi = 0\}$ is an ideal of the algebra $L^1_w(G)$. We define the Beurling spectrum of φ relative to $L^1_w(G)$ to be the set

$$sp_w(\varphi) = \text{hull } I_w(\varphi) = \{\gamma \in \widehat{G} : \widehat{f}(\gamma) = 0 \text{ for all } f \in I_w(\varphi)\}. \tag{1.9}$$

As each such function $\widehat{f}$ is continuous on $\widehat{G}$, $sp_w(\varphi)$ is a closed subset of $\widehat{G}$.

If in addition w is bounded, so that $L^1_w(G) = L^1(G)$, we shall write $I(\varphi) = I_w(\varphi)$ and $sp(\varphi) = sp_w(\varphi)$.

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PROPOSITION 1.1. *Let w be a weight function on G satisfying the condition (1.8) and let $\varphi \in L^\infty_w(G)$.*

(a) $sp_w(f^* * \varphi) \subset \operatorname{supp} \widehat{f} \cap sp_w(\varphi)$ *for all* $f \in L^1_w(G)$.

(b) *Given a neighbourhood V of a compact set W in $\widehat{G}$, there exists $f \in L^1_w(G)$ such that $\widehat{f} = 1$ on W and* $\operatorname{supp} \widehat{f} \subset V$.

(c) *If $f \in L^1_w(G)$ and $\widehat{f} = 1$ on a neighbourhood of $sp_w(\varphi)$ then $f^* * \varphi = \varphi$.*

(d) $sp_w(\varphi) = \emptyset$ *if and only if* $\varphi = 0$ *and* $sp_w(\varphi) = \{\widehat{e}\}$ *if φ is a non-zero constant.*

(e) $sp_w(\gamma\varphi) = \gamma sp_w(\varphi)$ *for all* $\gamma \in \widehat{G}$.

(f) $sp_w(\Delta_y\varphi) \subset sp_w(\varphi) \subset \{\widehat{e}\} \bigcup_{z \in G} sp_w(\Delta_z\varphi)$ *for all* $y \in G$.

PROOF: Since $L^1_w(G)$ is a Wiener algebra the proof of (a)-(e) can be carried out in the same manner as for $L^1(G)$. See [**2**, p.988] or [**5**, pp.140–141].

Let $y \in G$. Since $L^1_w(G)$ is translation invariant, $I_w(\varphi) \subset I_w(\Delta_y\varphi)$. Hence $sp_w(\Delta_y\varphi) \subset sp_w(\varphi)$. Conversely, suppose $\gamma \notin sp_w(\Delta_z\varphi)$ for all $z \in G$, and $\gamma \neq \widehat{e}$. Then there exists $z \in G$ such that $\gamma(z) \neq 1$ and $f \in I_w(\Delta_z\varphi)$ such that $\widehat{f}(\gamma) \neq 0$. If $g = \Delta_{-z}f$ then $\widehat{g}(\gamma) = (\gamma(-z) - 1)\widehat{f}(\gamma) \neq 0$ whereas $g^* * \varphi = f^* * \Delta_z\varphi = 0$. So $g \in I_w(\varphi)$ which implies $\gamma \notin sp_w(\varphi)$. This proves (f). □

REMARK 1.2. Let w_1 and w_2 be two weight functions on G satisfying the condition (1.8). If $\varphi \in L^\infty_{w_1}(G) \cap L^\infty_{w_2}(G)$ then $sp_{w_1}(\varphi) = sp_{w_2}(\varphi)$. Indeed, $w = w_1w_2$ is also a weight function on G satisfying (1.8). Since $w_1 \leqslant w$ and $w_2 \leqslant w$ it follows from [**5**, p.138] that $sp_{w_1}(\varphi) = sp_w(\varphi) = sp_{w_2}(\varphi)$. If $\varphi \in L^\infty(G)$ and w is a weight function satisfying (1.8), then $\varphi \in L^\infty_w(G)$ and so $sp_w(\varphi) = sp(\varphi)$.

2. POLYNOMIALS AND THEIR SPECTRA

In this section we define polynomials on G and prove the equivalence of our definition with that of Domar [**1**, Definition 2.3.2]. We explore the properties of polynomials, and prove in particular that $sp_w(p) = \{\widehat{e}\}$ for each non-zero polynomial $p \in L^\infty_w(G)$.

A continuous function $\varphi : G \to \mathbb{C}$ will be called a polynomial of degree n if

$$\Delta_{y_1} \dots \Delta_{y_{n+1}} \varphi = 0 \text{ for all } y_1, \dots, y_{n+1} \in G \tag{2.1}$$

$$\Delta_{y_1} \dots \Delta_{y_n} \varphi \neq 0 \text{ for some } y_1, \dots, y_n \in G. \tag{2.2}$$

Denote by $\mathbb{Z}_+$ the semigroup of non-negative integers with the discrete topology. Set $\mathbb{Z}_+^k = (\mathbb{Z}_+)^k$, $k \geqslant 1$. For arbitrary topological semigroups H denote by $C(H)$ the set of continuous complex valued functions on H. The following two propositions will be found useful. The first is proved by induction and the second, a simple chain rule for difference operators, is obvious.

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PROPOSITION 2.1. *If $\varphi \in C(G)$ then $\varphi(x+my) = \sum_{j=0}^{m} \binom{m}{j} \Delta_y^j \varphi(x)$ for all $m \in \mathbb{Z}_+$ and $x, y \in G$.*

PROPOSITION 2.2. *Given $\Phi \in C(G^n)$, define $\Psi \in C(\mathbb{Z}_+^n \times G^n)$ by $\Psi(t_1, \ldots, t_n, y_1, \ldots, y_n) = \Phi(t_1 y_1, \ldots, t_n y_n)$, where $t_j \in \mathbb{Z}_+$ and $y_j \in G$. If $\Delta_{y,j}$ denotes difference by y in position j only, then $\Delta_{y_j,j}\Phi(t_1 y_1, \ldots, t_n y_n) = \Delta_{1,j}\Psi(t_1, \ldots, t_n, y_1, \ldots, y_n)$.*

Recall that a function $\varphi \in C(G)$ is a polynomial of degree n in the sense of Domar if $\varphi(x+ty)$ is a polynomial in $t \in \mathbb{Z}_+$ of degree at most n for each $x, y \in G$ and of degree n for some $x, y \in G$.

PROPOSITION 2.3. *For a function $\varphi \in C(G)$ the following are equivalent:*

(a) $\Delta_{y_1} \ldots \Delta_{y_{n+1}} \varphi = 0$ *for all* $y_1, \ldots y_{n+1} \in G$.

(b) $\Delta_y^{n+1} \varphi = 0$ *for all* $y \in G$.

(c) $\varphi(x+ty)$ *is a polynomial in* $t \in \mathbb{Z}_+$ *of degree at most n for all* $x, y \in G$.

(d) $\varphi(x + t_1 y_1 + \ldots + t_k y_k)$ *is a polynomial in* $(t_1, \ldots, t_k) \in \mathbb{Z}_+^k$ *of degree at most n for all* $x, y_1, \ldots, y_k \in G$, *and* $k > 0$.

PROOF: That (a) implies (b) is obvious. If (b) holds then by Proposition 2.1 we have $\varphi(x+ty) = \sum_{j=0}^{n} \binom{t}{j} \Delta_y^j \varphi(x)$, showing that (b) implies (c). Given (c) and $x, y_1, \ldots, y_k \in G$, define $\psi(t) = \varphi(x + t_1 y_1 + \ldots + t_k y_k)$ for $(t_1, \ldots, t_k) \in \mathbb{Z}_+^k$. Then ψ is a polynomial in each t_j of degree at most n. So ψ is a polynomial in t of degree at most kn. Hence

$$\psi(t) = \sum_{|\alpha| \leqslant kn} a_\alpha t^\alpha, \text{ where } a_\alpha \in \mathbb{C},\ \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_+^k \text{ and } |\alpha| = \alpha_1 + \ldots + \alpha_k.$$

Moreover, for $\lambda \in \mathbb{Z}_+$, the function

$$\psi(\lambda t) = \sum_{|\alpha| \leqslant kn} a_\alpha \lambda^{|\alpha|} t^\alpha = \varphi(x + \lambda t_1 y_1 + \ldots + \lambda t_k y_k)$$

is again a polynomial in λ of degree at most n. Hence $a_\alpha = 0$ for $|\alpha| > n$, showing that (c) implies (d). Finally, given (d) and $x, y_1, \ldots, y_{n+1} \in G$, define $\psi(t) = \varphi(x + t_1 y_1 + \cdots + t_{n+1} y_{n+1})$ for $(t_1, \ldots, t_{n+1}) \in \mathbb{Z}_+^{n+1}$. Then ψ is a polynomial in t of degree at most n. So all (n+1)th differences of ψ are zero. Hence, by Proposition 2.2, $\Delta_{y_1} \ldots \Delta_{y_{n+1}} \varphi(x) = \Delta_{1,1} \ldots \Delta_{1,n+1} \psi(0) = 0$, showing that (d) implies (a). □

THEOREM 2.4. *For a function $\varphi \in C(G)$ the following are equivalent:*

(a) φ *is a polynomial of degree n.*

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(b) $\Delta_y^{n+1}\varphi = 0$ *for all* $y \in G$ *and* $\Delta_y^n\varphi \neq 0$ *for some* $y \in G$.

(c) φ *is a polynomial of degree n in the sense of Domar.*

(d) $\varphi(x + t_1y_1 + \ldots + t_ky_k)$ *is a polynomial in* $(t_1, \ldots, t_k) \in \mathbb{Z}_+^k$ *of degree at most n for all* $x, y_1, \ldots, y_k \in G$, $k > 0$ *and of degree n for some* $x, y_1, \ldots, y_k \in G$, $k > 0$.

PROOF: Given (d), we have Proposition 2.3(c). Moreover we can choose $x, y_1, \ldots, y_k$ such that $\varphi(x + t_1y_1 + \ldots + t_ky_k)$ is a polynomial in $(t_1, \ldots, t_k) \in \mathbb{Z}_+^k$ of degree n. So $\varphi(x + t_1y_1 + \ldots + t_ky_k) = \sum_{|\alpha| \leqslant n} a_\alpha t^\alpha$, where $a_\alpha \in \mathbb{C}$, and $\sum_{|\alpha|=n} a_\alpha t^\alpha$ is not identically zero. Choose t so that $\sum_{|\alpha|=n} a_\alpha t^\alpha \neq 0$ and set $y = t_1y_1 + \ldots + t_ky_k$. Then $\varphi(x + \lambda y) = \sum_{|\alpha| \leqslant n} a_\alpha t^\alpha \lambda^{|\alpha|}$ is a polynomial in $\lambda \in \mathbb{Z}_+$ of degree n, showing that (d) implies (c). Given(c), we have Proposition 2.3(b). Moreover, there exist $x, y \in G$ such that $\varphi(x + ty)$ is of degree n in $t \in Z_+$. So $\varphi(x + ty) = \sum_{j=0}^{n} a_j t^j$ where $a_j \in \mathbb{C}, a_n \neq 0$. By Proposition 2.2, $\Delta_y^n\varphi(x) = \sum_{j=0}^{n} a_j \Delta_1^n t^j = a_n n! \neq 0$, showing that (c) implies (b). That (b) implies (a) is obvious. Finally, if (a) holds then Proposition 2.3(d) holds. Moreover, there exist $x, y_1, \ldots, y_n \in G$ such that $\Delta_{y_1} \ldots \Delta_{y_n}\varphi(x) \neq 0$. So $\varphi(x + t_1y_1 + \ldots + t_ny_n) = \sum_{|\alpha| \leqslant n} a_\alpha t^\alpha$ and by Proposition 2.2, $\Delta_{y_1} \ldots \Delta_{y_n}\varphi(x) = a_\beta$ where $\beta = (1, 1, \ldots, 1) \in \mathbb{Z}_+^n$. So $a_\beta \neq 0$ and $\varphi(x + t_1y_1 + \ldots + t_ny_n)$ is a polynomial in $(t_1, \ldots, t_n)$ of degree n, showing that (a) implies (d). □

REMARKS 2.5. (a) On any Abelian group G the polynomials of degree 0 are precisely the constant functions. Polynomials of degree larger than 0 are unbounded. So the only polynomials on a compact group are the constant functions. Any polynomial φ on G of degree 1 is of the form $\varphi(x) = \varphi(0) + \alpha(x)$, where $\alpha(x + y) = \alpha(x) + \alpha(y)$ for $x, y \in G$.

(b) The definition of polynomials and the results proved so far in this section require only that G be an Abelian topological semigroup.

We turn now to the investigation of spectra of polynomials.

PROPOSITION 2.6. *Let p be a non-zero polynomial on G. Suppose that w is a weight function satisfying (1.8) and* $p \in L_w^\infty(G)$. *Then* $sp_w(\gamma p) = \{\gamma\}$ *for all* $\gamma \in \widehat{G}$. *In particular* $sp_w(p) = \{\widehat{e}\}$

PROOF: Let p have degree n. By Theorem 2.4, $\Delta_{y_0}^n p \neq 0$ for some $y_0 \in G$. Applying Proposition 1.1(f) n times we obtain $sp_w(\Delta_{y_0}^n p) \subset sp_w(p) \subset \{\widehat{e}\} \bigcup_{y \in G} sp_w(\Delta_y^n p)$. But by Proposition 1.1 (d), $sp_w(\Delta_{y_0}^n p) = \{\widehat{e}\}$ and $sp_w(\Delta_y^n p) \subset \{\widehat{e}\}$ for each $y \in G$.

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Hence $sp_w(p) = \{\widehat{e}\}$ and by Proposition 1.1(e), $sp_w(\gamma p) = \{\gamma\}$. □

3. Primary ideals and functions with finite spectrum

In this section we study Beurling algebras $L^1_w(G)$ when the weight function w satisfies the following two conditions for some $N \in \mathbb{Z}_+$:

$$\sup_{m\in\mathbb{Z}} \frac{w(mx)}{1+|m|^{N+\alpha(x)}} < \infty \text{ for all } x \in G \text{ and some } \alpha(x) < 1; \tag{3.1}$$

$$\inf_{m\in\mathbb{Z}} \frac{w(mx)}{|m|^N} > 0 \text{ for some } x \in G. \tag{3.2}$$

It should be noted that condition (3.1) implies the Beurling-Domar condition (1.8). Using conditions (3.1) and (3.2) we characterise those functions $\varphi \in L^\infty_w(G)$ for which $sp_w(\varphi)$ is finite.

We begin by studying the primary ideals of $L^1_w(\mathbb{Z})$ when w satisfies (3.1) and (3.2) with $G = \mathbb{Z}$. Recall that a primary ideal is one that is contained in exactly one maximal ideal, and the cospectrum of an ideal J of $L^1_w(G)$ is the set $\{\gamma \in \widehat{G} : \widehat{f}(\gamma) = 0 \text{ for all } f \in J\}$.

When $G = \mathbb{Z}$, conditions (3.1) and (3.2) are equivalent to the existence of $c_1 > 0, c_2 > 0$ and $\alpha \in (0,1)$ such that

$$c_1\left(1+|m|^N\right) \leqslant w(m) \leqslant c_2\left(1+|m|^{N+\alpha}\right) \text{ for all } m \in \mathbb{Z}. \tag{3.3}$$

In this case, $\{\widehat{f} : f \in L^1_w(\mathbb{Z})\}$ is a subalgebra of $C^N(T)$, the algebra of continuous functions on $T = \mathbb{R}/2\pi\mathbb{Z}$ with derivatives up to order N also continuous on T. We then define

$$I_k = \{f \in L^1_w(\mathbb{Z}) : \widehat{f}^{(j)}(\widehat{e}) = 0 \text{ for } 0 \leqslant j \leqslant k\}, \text{ where } 0 \leqslant k \leqslant N. \tag{3.4}$$

So I_k is a closed primary ideal of $L^1_w(\mathbb{Z})$ with cospectrum $\{\widehat{e}\}$.

Theorem 3.1. *Let w be a weight function on $\mathbb{Z}$ satisfying (3.3). Then $I_0, I_1, \ldots, I_N$ are the only closed primary ideals of $L^1_w(\mathbb{Z})$ with cospectrum $\{\widehat{e}\}$.*

Proof: Let $e_m(n) = \delta_{m,n}$ denote the Kronecker function, where $m, n \in \mathbb{Z}$. Then e_1 and e_{-1} generate the algebra $L^1_w(\mathbb{Z})$, and $\widehat{e_1}(t) = e^{it}$ and $\widehat{e_{-1}}(t) = e^{-it}$ for $t \in T$. Moreover, $e_1^m = e_m$ so $\alpha_m = \|e_1^m\|_{1,w} = w(m)$ for each $m \in \mathbb{Z}$. It follows from (3.3) that $\lim_{r\to 1-}(1-r)^{N+2}\sum_{m=0}^{\infty}\alpha_m r^m = \lim_{r\to 1-}(1-r)^{N+2}\sum_{m=0}^{\infty}\alpha_{-m}r^{-m} = 0$. By a theorem of Gelfand [4, p.209], $L^1_w(\mathbb{Z})$ has exactly N+1 ideals with cospectrum $\{\widehat{e}\}$, namely $I_0, I_1, \ldots, I_N$. □

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PROPOSITION 3.2. *Let w be a weight function on $\mathbb{Z}$ satisfying (3.3). If $0 \leqslant k \leqslant N$ then $\Delta_m^{k+1} f \in I_k$ for all $m \in \mathbb{Z}$ and all $f \in L_w^1(\mathbb{Z})$.*

PROOF: If $g = \Delta_m^{k+1} f$ then $\widehat{g}(t) = \left(e^{itm} - 1\right)^{k+1} \widehat{f}(t)$. Hence $g \in I_k$ as claimed. □

COROLLARY 3.3. *Let w be a weight function on $\mathbb{Z}$ satisfying (3.3). If $\varphi \in L_w^1(\mathbb{Z})$ and $sp_w(\varphi) = \{\widehat{e}\}$, then φ is a polynomial of degree at most N.*

PROOF: Since $I_w(\varphi)$ is a closed ideal with cospectrum $\{\widehat{e}\}$, we conclude from Theorem 3.1 that $I_w(\varphi) = I_k$ for some k, $0 \leqslant k \leqslant N$. Now for arbitrary $f \in L_w^1(\mathbb{Z})$ and $m \in \mathbb{Z}$, we have $\Delta_{-m}^{k+1} f \in I_k$ and so $f^* * \Delta_m^{k+1} \varphi = \left(\Delta_{-m}^{k+1} f\right)^* * \varphi = 0$. So $I_w\left(\Delta_m^{k+1} \varphi\right) = L_w^1(\mathbb{Z})$ and $sp_w\left(\Delta_m^{k+1} \varphi\right) = \emptyset$. By Proposition 1.1 (d), $\Delta_m^{k+1} \varphi = 0$, and by Proposition 2.3, φ is a polynomial of degree at most k. □

We now consider functions on more general groups G.

THEOREM 3.4. *Let w be a weight function satisfying (3.1) and (3.2) and let $\varphi \in L_w^\infty(G)$. Let $sp_w(\varphi) = \{\gamma_1, \ldots, \gamma_n\}$. Then there exist polynomials $p_1, \ldots, p_n$ in $L_w^\infty(G)$ of degree at most N such that $\varphi = \sum_{k=1}^{n} \gamma_k p_k$. In particular if $sp_w(\varphi) = \{\widehat{e}\}$ then $\varphi = p$ for some polynomial p of degree at most N.*

PROOF: As $sp_w(\varphi)$ is compact, φ coincides almost everywhere with a continuous function. (See [5, p.142].) We may therefore assume $\varphi \in C(G)$. Now suppose $sp_w(\varphi) = \{\widehat{e}\}$. For $y \in G$, let H_y be the subgroup generated by y and let K_y be its closure. Let w_y be the restriction of w to K_y and $\varphi_y(h) = \varphi(h + y)$ for $h \in K_y$. Then $\varphi_y \in L_{w_y}^\infty(K_y)$ and by a theorem of Reiter (see [5, p.144]) $sp_{w_y}(\varphi_y) \subset \pi(sp_w(\varphi)) = \{\widehat{e_y}\}$, where $\pi : \widehat{G} \to \widehat{K}_y$ is the canonical projection and $\widehat{e}_y$ is the unit in $\widehat{K}_y$. By a theorem of Weil (see [5, p.88]) either K_y is compact or K_y is isomorphic to $\mathbb{Z}$ ($H_y = \mathbb{Z}$). If K is compact then φ is bounded with $sp(\varphi_y) \subset \{\widehat{e}_y\}$. Therefore φ_y is a constant. (See Rudin [6, 6.8.3].) On the other hand, by Corollary 3.3, if $H_y = \mathbb{Z}$ then φ is a polynomial of degree at most N. Applying these results to ψ, defined by $\psi(z) = \varphi(x + z)$ for $x, z \in G$, and noting that $sp_w(\psi) = sp_w(\varphi)$ we conclude that $\varphi(x + my)$ is a polynomial in $m \in \mathbb{Z}_+$ of degree at most N for each $x, y \in G$. So φ is a polynomial of degree at most N. For the general case, $sp_w(\varphi) = \{\gamma_1, \ldots, \gamma_n\}$, choose a neighbourhood V of $\{\widehat{e}\}$ in $\widehat{G}$ and functions $f_1, \ldots, f_n \in L_w^1(G)$ such that $\widehat{f}_k = 1$ on $\gamma_k V$ and $\gamma_j V \cap \gamma_k V = \operatorname{supp} \widehat{f}_j \cap \operatorname{supp} \widehat{f}_k = \emptyset$ for $j, k = 1, \ldots, n$ and $j \neq k$. (See Proposition 1.1(b).) Then $sp_w(f_k^* * \varphi) \subset \operatorname{supp} \widehat{f}_k \cap sp_w(\varphi) = \{\gamma_k\}$ and by the first part of this proof, $\gamma_k^{-1}(f_k^* * \varphi) = p_k$, a polynomial of degree at most N. So $f^* * \varphi = \sum_{k=1}^{n} \gamma_k p_k$, where $f = \sum_{k=1}^{n} f_k$. But $\widehat{f} = 1$ on a neighbourhood of $sp_w(\varphi)$ and so, again by Proposition 1.1, $f^* * \varphi = \varphi$. □

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4. Uniformly continuous functions with bounded differences

Let $C_u(G)$ be the space of uniformly continuous complex-valued functions on G. Let $C_{ub}(G) = C_u(G) \cap L^\infty(G)$ and denote by $C_{ud}(G)$ the linear subspace of $C_u(G)$ consisting of functions $\varphi \in C_u(G)$ satisfying

$$\Delta_h \varphi \in C_{ub}(G) \text{ for all } h \in G. \tag{4.1}$$

We notice that if G is connected then by a theorem of Pontrjagin (see [5, p.97]) $G = K \times \mathbb{R}^n$, where K is compact and connected and $n \geqslant 0$. It follows that $C_{ud}(G) = C_u(G)$.

In this section we study the spectra of functions $\varphi \in C_{ud}(G)$. To do this we construct associated Beurling algebras. Indeed, define $w_\varphi : G \to \mathbb{C}$ by

$$w_\varphi(y) = 1 + \frac{1}{2}\left(\|\Delta_y \varphi\|_\infty + \|\Delta_{-y}\varphi\|_\infty\right),\ y \in G. \tag{4.2}$$

Proposition 4.1. *If $\varphi \in C_{ud}(G)$ then w_φ is a uniformly continuous weight function on G satisfying the condition*

$$\lim_{|m|\to\infty} \frac{w_\varphi(mx)}{m^2} = 0 \text{ for all } x \in G. \tag{4.3}$$

Moreover $\varphi \in L^\infty_{w_\varphi}(G)$.

Proof: Let $\alpha(y) = w_\varphi(y) - 1$. Then $\alpha(y_1 + y_2) \leqslant \alpha(y_1) + \alpha(y_2)$, $\alpha(y) = \alpha(-y)$, and $|\alpha(y_1 + y_2) - \alpha(y_2)| \leqslant \alpha(y_1)$ for $y, y_1, y_2 \in G$. It follows that α and w_φ are uniformly continuous on G, and that w_φ is a weight function. Moreover, for $n \in \mathbb{Z}$ and $y \in G$, $w_\varphi(my) = 1 + \alpha(my) \leqslant 1 + |m|\,\alpha(y)$, from which (4.3) follows. Finally, $|\varphi(x)| \leqslant |\varphi(x) - \varphi(0)| + |\varphi(0)| \leqslant 2\alpha(x) + |\varphi(0)| \leqslant (2 + |\varphi(0)|)w_\varphi(x)$, for $x \in G$, showing $\varphi \in L^\infty_{w_\varphi}(G)$. □

We remark that condition (4.3) implies condition (3.1) with N=1. This in turn implies the Beurling-Domar condition (1.8).

Theorem 4.2. *If $\varphi \in C_{ud}(G)$ and $\widehat{e} \notin sp_{w_\varphi}(\varphi)$ then φ is bounded.*

Proof: Let V be a compact neighbourhood of $\widehat{e}$ such that $V \cap sp_{w_\varphi}(\varphi) = \emptyset$. By Proposition 1.1 there exists $f \in L^1_{w_\varphi}(G)$ such that $\operatorname{supp} \widehat{f} \subset V$ and $\widehat{f}(\widehat{e}) = 1$. So $sp_{w_\varphi}(f^* * \varphi) = \emptyset$ and hence $f^* * \varphi = 0$. Hence $\varphi(t) = \varphi(t) - f^* * \varphi(t) = \int_G [\varphi(t) - \varphi(t-s)] f^*(s)\,ds$ and $|\varphi(t)| \leqslant 2\int_G w_\varphi(s)\,|f^*(s)|\,ds = 2\|f\|_{1,w_\varphi}$, for $t \in G$. So φ is bounded. □

As is customary, we identify $\widehat{\mathbb{R}^n}$ with $\mathbb{R}^n$ so that $\widehat{\mathbb{R}^n}$ becomes an additive group with unit 0. We now consider the indefinite integrals $P\varphi(x_1, x_2, .., x_n) = \int_0^{x_1} \varphi(t, x_2, .., x_n)\,dt$ of functions $\varphi \in L^\infty(\mathbb{R}^n)$.

THEOREM 4.3. *If $\varphi \in L^\infty(\mathbb{R}^n)$, $0 \notin sp(\varphi)$ and $P\varphi \in C_u(\mathbb{R}^n)$ then $P\varphi$ is bounded.*

PROOF: If $w(x_1, \dots, x_n) = 1 + |x_1|$ then w is a weight on $\mathbb{R}^n$ satisfying (1.8) and $\varphi, P\varphi \in L^\infty_w(\mathbb{R}^n)$. By remark 1.2, $sp(\varphi) = sp_w(\varphi)$ and similar to Proposition 1.1(f), $sp_w(P\varphi) \subset sp(\varphi) \cup \{0\}$. Let V be a neighbourhood of 0 with $sp(\varphi) \cap V = \emptyset$. By Proposition 1.1(b), there is a function $h \in L^1_w(\mathbb{R}^n)$ such that $\widehat{h} = 1$ on a neighbourhood W of 0 and $\operatorname{supp} \widehat{h} \subset V$. By Proposition 1.1 (a), $sp_w(h^* * P\varphi) \subset \{0\}$ and $h^* * \varphi = 0$. Since w also satisfies the conditions (3.1) and (3.2) with N=1, we have by Theorem 3.4, $h^* * P\varphi = p$, a polynomial of degree at most 1 in $L^\infty_w(\mathbb{R}^n)$. Since, by (1.7) $|p(x)| \leqslant c(1 + |x_1|)$ for some constant $c > 0$ and all $x \in \mathbb{R}^n$, we conclude that $p(x) = h^* * P\varphi = c_1 + c_2 x_1$ for some $c_1, c_2 \in \mathbb{C}$. But $(\partial(h^* * P\varphi))/(\partial x_1) = h^* * \varphi = 0$. Hence $c_2 = 0$ and $h^* * P\varphi = c_1$. If $\eta = P\varphi - c_1$ then $0 \notin sp_w(\eta)$. Indeed, $h^* * \eta = h^* * P\varphi - h^* * c_1 = c_1 - c_1 = 0$. Thus $h \in I_w(\eta)$ and $\widehat{h}(0) = 1$, showing $0 \notin sp_w(\eta)$. Finally, $\eta \in C_u(\mathbb{R}^n) = C_{ud}(\mathbb{R}^n)$ so by Theorem 4.2, η is bounded. Hence $P\varphi = \eta + c_1$ is bounded. □

COROLLARY 4.4. *If $\varphi \in L^\infty(\mathbb{R})$ and $0 \notin sp(\varphi)$ then $P\varphi$ is bounded.*

PROOF: Since $|P\varphi(x) - P\varphi(y)| \leqslant \|\varphi\|_\infty |x - y|$, $P\varphi \in C_u(\mathbb{R})$ and the corollary follows from the theorem. □

REMARK 4.5. Let $\varphi(x) = xe^{ix}$ and $w(x) = 1 + |x|$ for $x \in \mathbb{R}$. By Proposition 2.6 $sp_w(\varphi) = \{1\}$. So $0 \notin sp_w(\varphi)$ yet φ is unbounded. Hence the condition $\varphi \in C_{ud}(G)$ is necessary in Theorem 4.2. Let $\varphi(x, y) = e^{iy}$ for $(x, y) \in \mathbb{R}^2$. Then $sp(\varphi) = \{(0, 1)\}$. So $(0, 0) \notin sp(\varphi)$ yet $P\varphi(x, y) = xe^{iy}$ is unbounded. Hence the condition $P\varphi \in C_u(\mathbb{R})$ is necessary in Theorem 4.3.

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