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UNIFORM LOCAL CONSTANCY OF ÉTALE COHOMOLOGY OF RIGID ANALYTIC VARIETIES

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Abstract We prove some ℓ-independence results on local constancy of étale cohomology of rigid analytic varieties. As a result, we show that a closed subscheme of a proper scheme over an algebraically closed complete non-archimedean field has a small open neighbourhood in the analytic topology such that, for every prime number ℓ different from the residue characteristic, the closed subscheme and the open neighbourhood have the same étale cohomology with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients. The existence of such an open neighbourhood for each ℓ was proved by Huber. A key ingredient in the proof is a uniform refinement of a theorem of Orgogozo on the compatibility of the nearby cycles over general bases with base change.

Keywords and phrases: Rigid analytic varieties; Étale cohomology; Tubular neighbourhoods; Nearby cycles over general bases

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Contents

1. Introduction

Let K be an algebraically closed complete non-archimedean field whose topology is given by a valuation $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ of rank 1. Let $\mathcal{O} = K^{\circ}$ be the ring of integers of K. In this paper, we study local constancy of étale cohomology of rigid analytic varieties over K, or more precisely, of adic spaces of finite type over $\mathrm{Spa}(K,\mathcal{O})$.

1.1. A main result

The theory of étale cohomology for adic spaces was developed by Huber (see [14]). Huber obtained several finiteness results on étale cohomology of adic spaces in a series of papers [15, 16, 18]. Let us recall one of the main results of [16] (see [16, Theorem 3.6] for a more precise statement).

Theorem 1.1 (Huber [16, Theorem 3.6]). *We assume that K is of characteristic zero. Let X be a separated adic space of finite type over $\mathrm{Spa}(K,\mathcal{O})$ and Z a closed adic subspace of X. Let n be a positive integer invertible in $\mathcal{O}$. Then there exists an open subset V of X containing Z such that the restriction map*

$$H^i(V,\mathbb{Z}/n\mathbb{Z}) \to H^i(Z,\mathbb{Z}/n\mathbb{Z})$$

on étale cohomology groups is an isomorphism for every integer i. Moreover, we can assume that V is quasicompact.

It is a natural question to ask whether we can take an open subset V as in Theorem 1.1 independent of n. In the present paper, we answer this question in the affirmative for adic spaces which arise from schemes of finite type over $\mathcal{O}$.

More precisely, we will prove the following theorem. For a scheme $\mathcal{X}$ of finite type over $\mathcal{O}$, let $\widehat{\mathcal{X}}$ denote the ϖ-adic formal completion of $\mathcal{X}$, where $\varpi \in K^{\times}$ is an element with $|\varpi| < 1$. The Raynaud generic fibre of $\widehat{\mathcal{X}}$ is denoted by $(\widehat{\mathcal{X}})^{\mathrm{rig}}$ in this section, which is an adic space of finite type over $\mathrm{Spa}(K,\mathcal{O})$ (it is denoted by $d(\widehat{\mathcal{X}})$ in [14] and in the main body of this paper).

Theorem 1.2 (Theorem 4.9). *Let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed immersion of separated schemes of finite type over $\mathcal{O}$. We have a closed embedding $(\widehat{\mathcal{Z}})^{\mathrm{rig}} \hookrightarrow (\widehat{\mathcal{X}})^{\mathrm{rig}}$. Then there exists an open subset V of $(\widehat{\mathcal{X}})^{\mathrm{rig}}$ containing $(\widehat{\mathcal{Z}})^{\mathrm{rig}}$ such that, for every positive integer n invertible in $\mathcal{O}$, the restriction map*

$$H^i(V,\mathbb{Z}/n\mathbb{Z}) \to H^i((\widehat{\mathcal{Z}})^{\mathrm{rig}},\mathbb{Z}/n\mathbb{Z})$$

on étale cohomology groups is an isomorphism for every integer i. Moreover, we can assume that V is quasicompact.

A more precise statement is given in Theorem 4.9. In this paper, we will use de Jong's alterations in several ways. This is the main reason why we restrict ourselves to the case where adic spaces arise from schemes of finite type over $\mathcal{O}$. We remark that, in our case, we need not impose any conditions on the characteristic of K. We will also prove an analogous statement for étale cohomology with compact support (see Theorem 4.8).

Remark 1.3. In [29], Scholze proved the weight-monodromy conjecture for a projective smooth variety X over a non-archimedean local field L of mixed characteristic $(0,p)$ which is a set-theoretic complete intersection in a projective smooth toric variety, by reduction to the function field case proved by Deligne. In the proof, Scholze used Theorem 1.1 to construct, for a fixed prime number $\ell \neq p$, a projective smooth variety Y over a function field of characteristic p and an appropriate mapping from étale cohomology with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients of X to that of Y. The initial motivation for the present study is, following the method of Scholze, to prove that an analogue of the weight-monodromy conjecture holds for étale cohomology with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients of such a variety X for all but finitely many $\ell \neq p$ by reduction to an ultraproduct variant of Weil II established by Cadoret [5]. For this, we shall use Theorem 1.2 instead of Theorem 1.1. The details are given in [21].

1.2. Local constancy of higher direct images with proper support

For the proof of Theorem 1.2, we need to investigate local constancy of higher direct images with proper support for an algebraizable morphism of adic spaces whose target is the unit disc. To see this, let us give an outline of the proof of Theorem 1.1.

Sketch of the proof of Theorem 1.1. We assume that K is of characteristic zero. For simplicity, we assume that the closed embedding $Z \hookrightarrow X$ is of the form $(\widehat{\mathcal{Z}})^{\mathrm{rig}} \hookrightarrow (\widehat{\mathcal{X}})^{\mathrm{rig}}$ for a closed immersion of finite presentation $\mathcal{Z} \hookrightarrow \mathcal{X}$ of separated schemes of finite type over $\mathcal{O}$. By considering the blow-up of $\mathcal{X}$ along $\mathcal{Z}$, we may assume further that the closed subscheme $\mathcal{Z}$ is defined by one global function $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. Let

$$f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}[T]$$

be the morphism defined by $T \mapsto f$. The Raynaud generic fibre of the ϖ-adic formal completion of $\operatorname{Spec} \mathcal{O}[T]$ is the unit disc $\mathbb{B}(1) := \operatorname{Spa}(K\langle T\rangle, \mathcal{O}\langle T\rangle)$. The set of K-rational points of $\mathbb{B}(1)$ is identified with the set

$$\mathbb{B}(1)(K) = \{x \in K \,|\, |x| \leq 1\}.$$

The morphism f induces the following morphism of adic spaces:

$$f^{\mathrm{rig}}\colon (\widehat{\mathcal{X}})^{\mathrm{rig}} \to \mathbb{B}(1).$$

The inverse image $(f^{\mathrm{rig}})^{-1}(0)$ of the origin $0 \in \mathbb{B}(1)$ is the closed subspace $(\widehat{\mathcal{Z}})^{\mathrm{rig}}$.

We fix a positive integer n invertible in $\mathcal{O}$. We want to take an open subset V in Theorem 1.1 as the inverse image

$$V = (f^{\mathrm{rig}})^{-1}(\mathbb{B}(\epsilon))$$

of the disc $\mathbb{B}(\epsilon) \subset \mathbb{B}(1)$ of radius ϵ centred at 0 for a small $\epsilon \in |K^{\times}|$. Such a subset is called a *tubular neighbourhood* of $(\widehat{\mathcal{Z}})^{\mathrm{rig}}$. For this, we have to compute étale cohomology with $\mathbb{Z}/n\mathbb{Z}$-coefficients of $(f^{\mathrm{rig}})^{-1}(\mathbb{B}(\epsilon))$ for a small $\epsilon \in |K^{\times}|$. By the Leray spectral sequence for f^{rig}, it suffices to compute the cohomology group

$$H^i(\mathbb{B}(\epsilon), R^j f^{\mathrm{rig}}_{*} \mathbb{Z}/n\mathbb{Z})$$

for all i,j. The key steps are as follows.

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- By [16, Theorem 2.1], the étale sheaf $R^j f_*^{\mathrm{rig}}\mathbb{Z}/n\mathbb{Z}$ is an oc-quasiconstructible étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules in the sense of [16, Definition 1.4]. It follows that there exists an element $\epsilon_1 \in |K^\times|$ such that the restriction $(R^j f_*^{\mathrm{rig}}\mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(\epsilon_1)\backslash\{0\}}$ is a locally constant $\mathbb{Z}/n\mathbb{Z}$-sheaf of finite type.
- By the p-adic Riemann existence theorem of Lütkebohmert [22, Theorem 2.2], there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \leq \epsilon_1$ such that $(R^j f_*^{\mathrm{rig}}\mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(\epsilon_0)\backslash\{0\}}$ is trivialized by a Kummer covering $\varphi_m \colon \mathbb{B}(\epsilon_0^{1/m})\backslash\{0\} \to \mathbb{B}(\epsilon_0)\backslash\{0\}$ defined by $T \mapsto T^m$.

Then the desired result can be obtained by explicit calculations. □

In our case, the problem is to show that ϵ_0 and ϵ_1 in the above argument can be taken independent of n. To overcome this problem, by de Jong's alterations and by cohomological descent, we reduce to the case where there exists an element $\epsilon \in |K^\times|$ with $\epsilon \leq 1$ such that the restriction

$$(f^{\mathrm{rig}})^{-1}(\mathbb{B}(\epsilon)\backslash\{0\}) \to \mathbb{B}(\epsilon)\backslash\{0\}$$

of f^{rig} is *smooth*. In this case, we will analyse the higher direct image sheaf with proper support

$$R^j f_!^{\mathrm{rig}}\mathbb{Z}/n\mathbb{Z}$$

on $\mathbb{B}(1)$, which is defined in [14, Definition 5.4.4]. An important fact is that, since f^{rig} is smooth over $\mathbb{B}(\epsilon)\backslash\{0\}$, the restriction $(R^j f_!^{\mathrm{rig}}\mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(\epsilon)\backslash\{0\}}$ is a constructible étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules (in the sense of [14, Definition 2.7.2]) for every positive integer n invertible in $\mathcal{O}$ by [14, Theorem 6.2.2].

The following theorem is the most fundamental result in this paper. We do not suppose that K is of characteristic zero. For elements $a,b \in |K^\times|$ with $a < b \leq 1$, let $\mathbb{B}(a,b) \subset \mathbb{B}(1)$ be the annulus with inner radius a and outer radius b centred at 0.

Theorem 1.4 (Proposition 6.6 and Theorem 6.10). *Let $f \colon \mathcal{X} \to \operatorname{Spec}\mathcal{O}[T]$ be a separated morphism of finite presentation. We assume that there exists an element $\epsilon \in |K^\times|$ with $\epsilon \leq 1$ such that the induced morphism*

$$f^{\mathrm{rig}} \colon (\widehat{\mathcal{X}})^{\mathrm{rig}} \to \mathbb{B}(1)$$

is smooth over $\mathbb{B}(\epsilon)\backslash\{0\}$. Then there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \leq \epsilon$ such that, for every positive integer n invertible in $\mathcal{O}$, the following two assertions hold:

(1) *The restriction $(R^i f_!^{\mathrm{rig}}\mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(\epsilon_0)\backslash\{0\}}$ is a locally constant $\mathbb{Z}/n\mathbb{Z}$-sheaf of finite type for every i.*

(2) *For elements $a,b \in |K^\times|$ with $a < b \leq \epsilon_0$, there exists a composition*

$$h \colon \mathbb{B}(c^{1/m}, d^{1/m}) \xrightarrow{\varphi_m} \mathbb{B}(c,d) \xrightarrow{g} \mathbb{B}(a,b)$$

of a Kummer covering φ_m of degree m, where m is invertible in $\mathcal{O}$, with a finite Galois étale morphism g, such that $(R^i f_!^{\mathrm{rig}}\mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(a,b)}$ is trivialized by h for every i. If K is of characteristic zero, then we can take g as a Kummer covering (the morphism g can be taken independent of n although the integer m possibly depends on n).

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Remark 1.5. For the proof of Theorem 1.4 (1), we need Huber's result [14, Theorem 6.2.2]. However, our methods are different from the ones used in [15, 16, 18].

Remark 1.6. We will prove Theorem 1.4 in a slightly more general setting involving certain sheaves on $\mathcal{X}$ which are not necessary constant (see Section 6 for details).

Under the assumptions of Theorem 1.4, the same results hold for the higher direct image sheaf $R^i f_*^{\mathrm{rig}}\mathbb{Z}/n\mathbb{Z}$ by Poincaré duality [14, Corollary 7.5.5], which will imply Theorem 1.2.

1.3. Nearby cycles over general bases

A key ingredient in the proof of Theorem 1.4 is the following uniform refinement of a theorem of Orgogozo [25, Théorème 2.1] on the compatibility of *the sliced nearby cycles functors* with base change. We also obtain a result on uniform unipotency of the sliced nearby cycles functors (see Section 2.1 for the definition of the sliced nearby cycles functors, and see Definition 2.3 for the terminology used in the following theorem).

Theorem 1.7 (Corollary 2.9). *Let S be an excellent Noetherian scheme and $g\colon Y \to S$ a separated morphism of finite type. There exists an alteration $S' \to S$ such that, for every positive integer n invertible on S, the following assertions hold:*

(1) *The sliced nearby cycles complexes for the base change $g_{S'}\colon Y_{S'} \to S'$ of g and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ are compatible with any base change.*

(2) *The sliced nearby cycles complexes for $g_{S'}\colon Y_{S'} \to S'$ and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ are unipotent.*

Theorem 1.7 is a corollary of a more general result (Theorem 2.8), which may be of independent interest. For the proof, we use a combination of the methods of [25] and of [26] and we need de Jong's alteration.

By using a comparison theorem of Huber [14, Theorem 5.7.8], we will study the relation between higher direct images with proper support for morphisms of adic spaces and the sliced nearby cycles functors. Then we will deduce Theorem 1.4 from Theorem 1.7. Roughly speaking, Theorem 1.4 (1) can be deduced from Theorem 1.7 (1) by considering a specialization map from an adic space of finite type over $\mathrm{Spa}(K,\mathcal{O})$ to its reduction (see Section 5.3 and Section 6.2 for details). Theorem 1.4 (2) can be deduced from Theorem 1.7 (2) and some properties of the *discriminant function* $\delta_h\colon [0,\infty) \to \mathbb{R}_{\geq 0}$ associated with a finite Galois étale covering $h\colon Y \to \mathbb{B}(1)\backslash\{0\}$ defined in [22, 23, 27] (see Section 6.1 and Appendix A for details). In the proofs of both parts of Theorem 1.4, points of rank 2 of (finite Galois étale coverings of) $\mathbb{B}(1)$ play important roles.

1.4. The organisation of this paper

This paper is organised as follows. In Section 2, we first recall the definition of the sliced nearby cycles functors. Then we formulate our main result (Theorem 2.8) on the sliced nearby cycles functors. In Section 3, we prove Theorem 2.8.

In Section 4, we recall the definition of tubular neighbourhoods, and then we state our main results (Theorem 4.8 and Theorem 4.9) on étale cohomology of tubular neighbourhoods. In Section 5, we recall a comparison theorem of Huber and use it to study the relation between higher direct images with proper support for morphisms of adic spaces and the sliced nearby cycles functors. In Section 6, we prove Theorem 1.4 in a slightly more general setting. In Section 7, we prove Theorem 4.8 and Theorem 4.9 (and, hence, Theorem 1.2) by using Theorem 1.4.

Finally, in Appendix A, we prove two theorems (Theorem 6.2 and Theorem 6.3) on finite étale coverings of annuli, which basically follow from the results in [22, 23, 27].

2. Nearby cycles over general bases

In this section, we formulate our main results on nearby cycles over general bases. We will use the following notation throughout this paper. Let $f\colon X \to S$ be a morphism of schemes. For a morphism $T \to S$ of schemes, the base change $X \times_S T$ of X is denoted by X_T and the base change of f is denoted by $f_T\colon X_T \to T$. For a commutative ring Λ, let $D^+(X,\Lambda)$ be the derived category of bounded below complexes of étale sheaves of Λ-modules on X. For a complex $\mathcal{K} \in D^+(X,\Lambda)$, the pullback of $\mathcal{K}$ to X_T is denoted by $\mathcal{K}_T$. We often call an étale sheaf on X simply a sheaf on X.

We will use the following terminology (see also [26, 1.2.4]).

Definition 2.1. Let G be a group and X a scheme. We say that a sheaf $\mathcal{F}$ of Λ-modules on X with a G-action is *G-unipotent* if $\mathcal{F}$ has a finite filtration which is stable by the action of G such that the action of G on each successive quotient is trivial. We say that a complex $\mathcal{K} \in D^+(X,\Lambda)$ with a G-action is *G-unipotent* if its cohomology sheaves are G-unipotent.

Remark 2.2. Assume that $\mathcal{F}$ as above is G-unipotent. Then every subquotient of $\mathcal{F}$ (as a sheaf of Λ-modules with a G-action) is G-unipotent. We also note that higher direct images of $\mathcal{F}$ (and, hence, of G-unipotent complexes) with induced G-actions are G-unipotent, and the same statements hold for higher direct images with proper support, pullbacks, etc.

2.1. Sliced nearby cycles functor

In this paper, a scheme is called a *strictly local scheme* if it is isomorphic to an affine scheme $\operatorname{Spec} R$ where R is a strictly Henselian local ring. Let $f\colon X \to S$ be a morphism of schemes. Let $q\colon U \to S$ be a morphism from a strictly local scheme U. The closed point of U is denoted by u. Let $\eta \in U$ be a point. Let $\overline{\eta} \to U$ be an algebraic geometric point lying above η, that is, it is a geometric point lying above η such that the residue field $\kappa(\overline{\eta})$ is a separable closure of the residue field $\kappa(\eta)$ of η. The strict localization of U at $\overline{\eta} \to U$ is denoted by $U_{(\overline{\eta})}$. We have the following commutative diagram:

$$\begin{array}{ccccc} X_{U_{(\overline{\eta})}} & \xrightarrow{\ j\ } & X_U & \xleftarrow{\ i\ } & X_u \\ \downarrow & & \downarrow{\scriptstyle f_U} & & \downarrow \\ U_{(\overline{\eta})} & \longrightarrow & U & \longleftarrow & u. \end{array}$$

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Let Λ be a commutative ring. We have the following functor:

$$R\Psi_{f_U,\overline{\eta}} := i^*Rj_*j^* \colon D^+(X_U,\Lambda) \to D^+(X_u,\Lambda).$$

This functor is called the *sliced nearby cycles functor* in [19]. For a complex $\mathcal{K} \in D^+(X_U,\Lambda)$, we have an action of the absolute Galois group $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$ on $R\Psi_{f_U,\overline{\eta}}(\mathcal{K})$ via the canonical isomorphism

$$\mathrm{Aut}(U_{(\overline{\eta})}/\mathrm{Spec}(\mathcal{O}_{U,\eta})) \cong \mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta)).$$

Let $q\colon V \to U$ be a local morphism of strictly local schemes over S, that is, a morphism over S which sends the closed point v of V to the closed point u of U. Let $\xi \in V$ be a point with image $\eta = q(\xi) \in U$. For an algebraic geometric point $\overline{\xi} \to V$ lying above ξ, we have an algebraic geometric point $\overline{\eta} \to U$ lying above η by taking the separable closure of $\kappa(\eta)$ in $\kappa(\overline{\xi})$. We call $\overline{\eta} \to U$ the image of $\overline{\xi} \to V$ under the morphism q. We have the following commutative diagram:

$$\begin{array}{ccccc} X_{V_{(\overline{\xi})}} & \longrightarrow & X_V & \longleftarrow & X_v \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ X_{U_{(\overline{\eta})}} & \longrightarrow & X_U & \longleftarrow & X_u, \end{array}$$

where the vertical morphisms are induced by q. For a complex $\mathcal{K} \in D^+(X_U,\Lambda)$, we have the following base change map:

$$q^*R\Psi_{f_U,\overline{\eta}}(\mathcal{K}) \to R\Psi_{f_V,\overline{\xi}}(\mathcal{K}_V).$$

Definition 2.3. Let $f\colon X \to S$ be a morphism of schemes. Let Λ be a commutative ring and $\mathcal{K} \in D^+(X,\Lambda)$ a complex.

(1) We say that *the sliced nearby cycles complexes for f and $\mathcal{K}$ are compatible with any base change* (or simply that *the nearby cycles for f and $\mathcal{K}$ are compatible with any base change*) if for every local morphism $q\colon V \to U$ of strictly local schemes over S and every algebraic geometric point $\overline{\xi} \to V$ with image $\overline{\eta} \to U$, the base change map

$$q^*R\Psi_{f_U,\overline{\eta}}(\mathcal{K}_U) \to R\Psi_{f_V,\overline{\xi}}(\mathcal{K}_V)$$

is an isomorphism.

(2) We say that *the sliced nearby cycles complexes for f and $\mathcal{K}$ are unipotent* (or simply that *the nearby cycles for f and $\mathcal{K}$ are unipotent*) if, for every morphism $q\colon U \to S$ from a strictly local scheme U, a point $\eta \in U$ and an algebraic geometric point $\overline{\eta} \to U$ lying above η, the complex $R\Psi_{f_U,\overline{\eta}}(\mathcal{K}_U)$ is $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$-unipotent in the sense of Definition 2.1.

Remark 2.4. We can restate Definition 2.3 (1) in terms of vanishing topoi as follows. Let $f\colon X \to S$ be a morphism of schemes. Let

$$X \overleftarrow{\times}_S S$$

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be the vanishing topos, where the étale topos of a scheme X is also denoted by X by abuse of notation. See [19] for the definition and basic properties of the vanishing topos $X \overleftarrow{\times}_S S$. Let Λ be a commutative ring. We have a morphism of topoi $\Psi_f\colon X \to X \overleftarrow{\times}_S S$. The direct image functor

$$R\Psi_f\colon D^+(X,\Lambda) \to D^+(X \overleftarrow{\times}_S S,\Lambda)$$

defined by Ψ_f is called the *nearby cycles functor*. For a morphism $q\colon T \to S$ of schemes, we have a morphism of topoi $\overleftarrow{q}\colon X_T \overleftarrow{\times}_T T \to X \overleftarrow{\times}_S S$ and a 2-commutative diagram

$$\begin{array}{ccc} X_T & \longrightarrow & X \\ \downarrow{\scriptstyle R\Psi_{f_T}} & & \downarrow{\scriptstyle R\Psi_f} \\ X_T \overleftarrow{\times}_T T & \xrightarrow{\overleftarrow{q}} & X \overleftarrow{\times}_S S, \end{array}$$

where $X_T \to X$ is the projection. For a complex $\mathcal{K} \in D^+(X,\Lambda)$, we have the base change map

$$c_{f,q}(\mathcal{K})\colon (\overleftarrow{q})^* R\Psi_f(\mathcal{K}) \to R\Psi_{f_T}(\mathcal{K}_T).$$

For a morphism $f\colon X \to S$ of schemes and a complex $\mathcal{K} \in D^+(X,\Lambda)$, the sliced nearby cycles complexes for f and $\mathcal{K}$ are compatible with any base change in the sense of Definition 2.3 (1) if and only if, for every morphism $q\colon T \to S$ of schemes, the base change map $c_{f,q}(\mathcal{K})$ is an isomorphism (see also the proof of [25, Lemme 4.1]). This follows from the following descriptions of the stalks of the nearby cycles functor and the sliced nearby cycles functors.

Let $x \to X$ be a geometric point of X, and let $s \to S$ denote the composition $x \to X \to S$. Let $t \to S$ be a geometric point with a specialization map $\alpha\colon t \to s$, that is, an S-morphism $\alpha\colon S_{(t)} \to S_{(s)}$, where $S_{(s)}$ (resp. $S_{(t)}$) is the strict localization of S at $s \to S$ (resp. $t \to S$). The triple (x,t,α) defines a point of the vanishing topos $X \overleftarrow{\times}_S S$, and every point of $X \overleftarrow{\times}_S S$ is of this form (up to equivalence). The topos $X \overleftarrow{\times}_S S$ has enough points. For the stalk $R\Psi_f(\mathcal{K})_{(x,t,\alpha)}$ of $R\Psi_f(\mathcal{K})$ at (x,t,α), we have an isomorphism

$$R\Psi_f(\mathcal{K})_{(x,t,\alpha)} \cong R\Gamma(X_{(x)} \times_{S_{(s)}} S_{(t)},\mathcal{K})$$

(see [19, (1.3.2)]). Here, the pullback of $\mathcal{K}$ to $X_{(x)} \times_{S_{(s)}} S_{(t)}$ is also denoted by $\mathcal{K}$ (we will use this notation in this paper when there is no possibility of confusion).

We have a similar description of the stalks of the sliced nearby cycles functors. More precisely, let $q\colon U \to S$ be a morphism from a strictly local scheme U and $\overline{\eta} \to U$ an algebraic geometric point. Let $x \to X_u$ be a geometric point of the special fibre X_u of X_U. Then, since the morphism $X_{U_{(\overline{\eta})}} \to X_U$ is quasicompact and quasiseparated, we have

$$R\Psi_{f_U,\overline{\eta}}(\mathcal{K}_U)_x \cong R\Gamma((X_U)_{(x)} \times_U U_{(\overline{\eta})},\mathcal{K}_U). \tag{2.1}$$

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2.2. Main results on nearby cycles over general bases

A proper surjective morphism $f\colon X \to Y$ of Noetherian schemes is called an *alteration* if it sends every generic point of X to a generic point of Y and it is generically finite, that is, there exists a dense open subset $U \subset Y$ such that the restriction $f^{-1}(U) \to U$ is a finite morphism. If, furthermore, X and Y are integral schemes, then f is called an integral alteration. An alteration $f\colon X \to Y$ is called a *modification* if there exists a dense open subset $U \subset Y$ such that the restriction $f^{-1}(U) \to U$ is an isomorphism.

Let $f\colon X \to S$ be a morphism of finite type of excellent Noetherian schemes. In [25], Orgogozo proved the following result:

Theorem 2.5 (Orgogozo [25, Théorème 2.1]). *For a positive integer n invertible on S and for a constructible sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on X, there exists a modification $S' \to S$ such that the sliced nearby cycles complexes for $f_{S'}$ and $\mathcal{F}_{S'}$ are compatible with any base change in the sense of Definition 2.3 (1).*

Proof. See [25, Théorème 2.1] for the proof and for a more general result (actually, Orgogozo formulated his results in terms of vanishing topoi, see Remark 2.4). □

To prove Theorem 1.2, we need a uniform refinement of Theorem 2.5. More precisely, we need a modification (or an alteration) $S' \to S$ such that, for every positive integer n invertible on S, the sliced nearby cycles complexes for $f_{S'}$ and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ are compatible with any base change.

In order to prove the existence of such a modification, we will use the methods developed in a recent paper [26] of Orgogozo. In fact, by the same methods, we can also prove that there exists an *alteration* $S' \to S$ such that, for every positive integer n invertible on S, the sliced nearby cycles complexes for $f_{S'}$ and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ are unipotent in the sense of Definition 2.3 (2). Such an alteration is also needed in the proof of Theorem 1.2.

We need to recall the definition of a *locally unipotent sheaf* on a Noetherian scheme from [26]. Let X be a Noetherian scheme. In this paper, we call a finite set $\mathfrak{X} = \{X_\alpha\}_\alpha$ of locally closed subsets of X a *stratification* if we have $X = \coprod_\alpha X_\alpha$ (set theoretically).

Definition 2.6 (Orgogozo [26, 1.2.1]). Let X be a Noetherian scheme and $\mathfrak{X}$ a stratification of X. We say that an abelian sheaf $\mathcal{F}$ on X is *locally unipotent along* $\mathfrak{X}$ if, for every morphism $U \to X$ from a strictly local scheme U and every $X_\alpha \in \mathfrak{X}$, the pullback of $\mathcal{F}$ to $U \times_X X_\alpha$ has a finite filtration whose successive quotients are constant sheaves.

Remark 2.7. If a constructible abelian sheaf $\mathcal{F}$ on a Noetherian scheme X is locally unipotent along a stratification $\mathfrak{X}$, then it is constructible along $\mathfrak{X}$, that is, for every $X_\alpha \in \mathfrak{X}$, the pullback of $\mathcal{F}$ to X_α is locally constant (see [26, 1.2.2]).

Our main result on nearby cycles over general bases is as follows.

Theorem 2.8. *Let S be an excellent Noetherian scheme. Let $f\colon X \to S$ be a proper morphism. Let $\mathfrak{X}$ be a stratification of X. Then there exists an alteration $S' \to S$ such that, for every positive integer n invertible on S and every complex $\mathcal{K} \in D^+(X, \mathbb{Z}/n\mathbb{Z})$ whose*

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cohomology sheaves are constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules and are locally unipotent along $\mathfrak{X}$, the following two assertions hold.

(1) *The sliced nearby cycles complexes for $f_{S'}\colon X_{S'} \to S'$ and $\mathcal{K}_{S'}$ are compatible with any base change.*

(2) *The sliced nearby cycles complexes for $f_{S'}\colon X_{S'} \to S'$ and $\mathcal{K}_{S'}$ are unipotent.*

For future reference, we state the following corollary.

Corollary 2.9. *Let S be an excellent Noetherian scheme and $f\colon X \to S$ a separated morphism of finite type. There exists an alteration $S' \to S$ such that, for every positive integer n invertible on S, the sliced nearby cycles complexes for $f_{S'}\colon X_{S'} \to S'$ and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ are compatible with any base change and are unipotent.*

Proof. This follows immediately from Theorem 2.8. □

In fact, as in [25], we can show a more precise result for the compatibility of the sliced nearby cycles functors with base change as a corollary of Theorem 2.8:

Corollary 2.10. *Under the assumptions of Theorem 2.8, there exists a modification $S' \to S$ such that, for every positive integer n invertible on S and every complex $\mathcal{K} \in D^+(X,\mathbb{Z}/n\mathbb{Z})$ whose cohomology sheaves are constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules and are locally unipotent along $\mathfrak{X}$, the sliced nearby cycles complexes for $f_{S'}\colon X_{S'} \to S'$ and $\mathcal{K}_{S'}$ are compatible with any base change.*

Proof. Theorem 2.8, together with [25, Lemme 3.2 and Lemme 3.3], implies the result. □

3. Proof of Theorem 2.8

3.1. Nodal curves

In this subsection, we recall some results on nodal curves from [7, 26]. Let $f\colon X \to S$ be a morphism of Noetherian schemes. We say that f is a *nodal curve* if it is a flat projective morphism such that every geometric fibre of f is a connected reduced curve having at most ordinary double points as singularities. We say that f is a *nodal curve adapted to a pair* (X°,S°) of dense open subsets X° and S° of X and S, respectively, if the following conditions are satisfied:

- f is a nodal curve which is smooth over S°.
- There is a closed subscheme D of X which is étale over S and is contained in the smooth locus of f. Moreover, we have $f^{-1}(S^\circ) \cap (X \setminus D) = X^\circ$.

The following proposition will be used in the proof of Theorem 2.8, which is one of the main reasons why we introduce the notion of locally unipotent sheaves.

Proposition 3.1 (Orgogozo [26, Proposition 2.3.1]). *Let S be a Noetherian scheme and $f\colon X \to S$ a nodal curve adapted to a pair (X°,S°) of dense open subsets X° and S° of X and S, respectively. Let $u\colon X^\circ \hookrightarrow X$ denote the open immersion. Assume that S° is normal. Then, for every positive integer n invertible on S and every locally constant*

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constructible sheaf $\mathcal{L}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on X° such that $u_!\mathcal{L}$ is locally unipotent along the stratification $\mathfrak{X} = \{X^\circ, X \backslash X^\circ\}$ of X, the sheaf

$$R^i f_*(u_!\mathcal{L})$$

is locally unipotent along the stratification $\mathfrak{S} = \{S^\circ, S \backslash S^\circ\}$ of S for every i.

Proof. See [26, Proposition 2.3.1]. □

Remark 3.2. The proof of Theorem 2.8 is inspired by that of Proposition 3.1. In fact, we can show that, with the notation of Proposition 3.1, the nearby cycles for f and $u_!\mathcal{L}$ are compatible with any base change and unipotent. Since we will not use this fact in the proof of Theorem 2.8, we omit the proof of it.

We say that a morphism $f\colon X \to S$ of Noetherian integral schemes is a *pluri nodal curve adapted to a dense open subset* $X^\circ \subset X$ if there are an integer $d \geq 0$, a sequence

$$(X = X_d \underset{f_d}{\to} X_{d-1} \to \cdots \to X_1 \underset{f_1}{\to} X_0 = S)$$

of morphisms of Noetherian integral schemes and dense open subsets $X_i^\circ \subset X_i$ for every $0 \leq i \leq d$ with $X_d^\circ = X^\circ$ such that $f_i\colon X_i \to X_{i-1}$ is a nodal curve adapted to the pair $(X_i^\circ, X_{i-1}^\circ)$ for every $1 \leq i \leq d$. If $d = 0$, by convention, it means that $X = S$ and f is the identity map.

The following theorem of de Jong plays an important role in the proof of Theorem 2.8.

Theorem 3.3 (de Jong [7, Theorem 5.9]). *Let $f\colon X \to S$ be a proper surjective morphism of excellent Noetherian integral schemes. Let $X^\circ \subset X$ be a dense open subset. We assume that the geometric generic fibre of f is irreducible. Then there is the following commutative diagram:*

$$\begin{array}{ccc} X_0 & \xrightarrow{f'} & S' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S, \end{array}$$

where the vertical maps are integral alterations and f' is a pluri nodal curve adapted to a dense open subset $X_0^{\circ\circ} \subset X_0$ which is contained in the inverse image of $X^\circ \subset X$.

Proof. See [7, Theorem 5.9] and the proof of [7, Theorem 5.10]. We note that if the dimension of the generic fibre of f is zero, then f is an integral alteration. Hence, we can take S' as X and take f' as the identity map on X in this case. □

3.2. Preliminary lemmas

We shall give two lemmas, which will be used in the proof of Theorem 2.8.

We will need the following terminology.

Definition 3.4. Let $f\colon X \to S$ be a morphism of schemes. Let Λ be a commutative ring and $\mathcal{K} \in D^+(X, \Lambda)$ a complex. Let ρ be an integer.

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(1) We say that *the sliced nearby cycles complexes for f and $\mathcal{K}$ are ρ-compatible with any base change* (or simply that *the nearby cycles for f and $\mathcal{K}$ are ρ-compatible with any base change*) if for every local morphism $q\colon V \to U$ of strictly local schemes over S and every algebraic geometric point $\overline{\xi} \to V$ with image $\overline{\eta} \to U$, we have $\tau_{\leq \rho}\Delta = 0$ for the cone Δ of the base change map:

$$q^* R\Psi_{f_U,\overline{\eta}}(\mathcal{K}_U) \to R\Psi_{f_V,\overline{\xi}}(\mathcal{K}_V) \to \Delta \to .$$

(2) We say that *the sliced nearby cycles complexes for f and $\mathcal{K}$ are ρ-unipotent* (or simply that *the nearby cycles for f and $\mathcal{K}$ are ρ-unipotent*) if, for every morphism $q\colon U \to S$ from a strictly local scheme U, a point $\eta \in U$ and an algebraic geometric point $\overline{\eta} \to U$ lying above η, the complex

$$\tau_{\leq \rho} R\Psi_{f_U,\overline{\eta}}(\mathcal{K}_U)$$

is $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$-unipotent (in the sense of Definition 2.1).

As in [25], we need some results on cohomological descent (see [1, Exposé Vbis] and [8, Section 5] for the terminology used here). Let $f\colon Y \to X$ be a morphism of schemes. Let

$$\beta\colon Y_\bullet := \mathrm{cosq}_0(Y/X) \to X$$

be the augmented simplicial object in the category of schemes defined as in [8, (5.1.4)], so Y_m is the $(m+1)$-times fibre product $Y \times_X \cdots \times_X Y$ for $m \geq 0$. We can associate to the étale topoi of Y_m $(m \geq 0)$ a topos $(Y_\bullet)^\sim$ (see [8, (5.1.6)–(5.1.8)]). Moreover, as in [8, (5.1.11)], we have a morphism of topoi

$$(\beta_*, \beta^*)\colon (Y_\bullet)^\sim \to X^\sim_{\text{ét}}$$

from $(Y_\bullet)^\sim$ to the étale topos $X^\sim_{\text{ét}}$ of X.

Lemma 3.5. *Let $f\colon X \to S$ be a morphism of schemes. Let $\beta_0\colon Y \to X$ be a proper surjective morphism. We put $\beta\colon Y_\bullet := \mathrm{cosq}_0(Y/X) \to X$. Let $\mathcal{F}$ be a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X and $\mathcal{F}_m := \beta_m^*\mathcal{F}$ the pullback of $\mathcal{F}$ by $\beta_m\colon Y_m \to X$. The composition $f \circ \beta_m$ is denoted by f_m. Let $\rho \geq -1$ be an integer.*

(1) *If the nearby cycles for f_m and $\mathcal{F}_m$ are $(\rho - m)$-compatible with any base change for every $0 \leq m \leq \rho + 1$, then the nearby cycles for f and $\mathcal{F}$ are ρ-compatible with any base change.*

(2) *If the nearby cycles for f_m and $\mathcal{F}_m$ are $(\rho - m)$-unipotent for every $0 \leq m \leq \rho$, then the nearby cycles for f and $\mathcal{F}$ are ρ-unipotent.*

Proof. The assertion (1) is [25, Lemme 4.1] (see also Remark 2.4). Although it is stated for constant sheaves, the same proof works for sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules (or more generally, for torsion abelian sheaves).

The assertion (2) can be proved by the same arguments as in the proof of [25, Lemme 4.1]. We shall give a sketch here. Let $q\colon U \to S$ be a morphism from a strictly local scheme U and $\overline{\eta} \to U$ an algebraic geometric point with image $\eta \in U$. Let $u \in U$ be the closed

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point. We have the following diagram:

$$\begin{array}{ccccc}
(Y_\bullet)_{U(\overline{\eta})} & \xrightarrow{j_\bullet} & (Y_\bullet)_U & \xleftarrow{i_\bullet} & (Y_\bullet)_u \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\
X_{U(\overline{\eta})} & \xrightarrow{j} & X_U & \xleftarrow{i} & X_u,
\end{array}$$

where $\beta\colon (Y_\bullet)_U \to X_U$ is the base change of β, etc. By [1, Exposé Vbis, Proposition 4.3.2], the morphism $\beta_0\colon Y \to X$ is universally of cohomological descent, and, hence, we have $\mathcal{F}_U \cong R\beta_*\beta^*\mathcal{F}_U$. Using this isomorphism and the proper base change theorem, we obtain

$$R\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U) \cong R\beta_*(i_\bullet)^*R(j_\bullet)_*(j_\bullet)^*\beta^*\mathcal{F}_U.$$

The pullback of the complex

$$(i_\bullet)^*R(j_\bullet)_*(j_\bullet)^*\beta^*\mathcal{F}_U$$

to $(Y_m)_u$ is isomorphic to $R\Psi_{(f_m)_U,\overline{\eta}}((\mathcal{F}_m)_U)$ for every $m \geq 0$. Thus, we have the following spectral sequence:

$$E_1^{k,l} = R^l(\beta_k)_*R\Psi_{(f_k)_U,\overline{\eta}}((\mathcal{F}_k)_U) \Rightarrow R^{k+l}\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U)$$

(see [8, (5.2.7.1)]). The assertion follows from this spectral sequence since our assumption implies that the sheaf

$$R^l(\beta_k)_*R\Psi_{(f_k)_U,\overline{\eta}}((\mathcal{F}_k)_U) \cong R^l(\beta_k)_*\tau_{\leq l}R\Psi_{(f_k)_U,\overline{\eta}}((\mathcal{F}_k)_U)$$

is $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$-unipotent if $k+l \leq \rho$. □

3.3. Proof of Theorem 2.8

In this subsection, we prove Theorem 2.8. Our main technique is a combination of the methods of [25] and [26].

In this section, we use the following terminology.

Definition 3.6. Let S be a Noetherian scheme and $f\colon X \to S$ a morphism of finite type. Let ρ be an integer.

(1) Let $\mathfrak{X}$ be a stratification of X. We say that an alteration $S' \to S$ is *ρ-adapted to the pair* $(f,\mathfrak{X})$ if, for every positive integer n invertible on S and every constructible sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on X which is locally unipotent along $\mathfrak{X}$, the nearby cycles for $f_{S'}\colon X_{S'} \to S'$ and $\mathcal{F}_{S'}$ are ρ-compatible with any base change and ρ-unipotent.

(2) Let $u\colon U \hookrightarrow X$ be an open immersion. We say that an alteration $S' \to S$ is *ρ-adapted to the pair* (f,U) if, for every positive integer n invertible on S and every locally constant constructible sheaf $\mathcal{L}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on U such that $u_!\mathcal{L}$ is locally unipotent along the stratification $\{U, X\backslash U\}$, the nearby cycles for $f_{S'}\colon X_{S'} \to S'$ and $(u_!\mathcal{L})_{S'}$ are ρ-compatible with any base change and ρ-unipotent.

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Let S be an excellent Noetherian integral scheme. Let ρ and d be two integers. We shall consider the following statement $\mathbf{P}(S,\rho,d)$:

$\mathbf{P}(S,\rho,d)$: For every integral alteration $T \to S$, for every proper morphism $f\colon Y \to T$ such that the dimension of the generic fibre of f is less than or equal to d and for any stratification $\mathfrak{Y}$ of Y, there exists an alteration $T' \to T$ which is ρ-adapted to $(f,\mathfrak{Y})$ in the sense of Definition 3.6 (1).

Remark 3.7.

(1) $\mathbf{P}(S,-2,d)$ holds trivially for every excellent Noetherian integral scheme S and every integer d.

(2) For an integral scheme T and a proper morphism $f\colon Y \to T$, the condition that the dimension of the generic fibre is less than or equal to -1 means that f is not surjective. The statement $\mathbf{P}(S,\rho,-1)$ is not trivial.

Lemma 3.8. *To prove Theorem 2.8, it is enough to show that the statement* $\boldsymbol{P}(S,\rho,d)$ *holds for every triple* (S,ρ,d).

Proof. The assertion can be proved by standard arguments, using the following fact proved in [25, Proposition 3.1] (see also Remark 2.4): Let S be a Noetherian scheme and $f\colon X \to S$ a morphism of finite type. Let N be the supremum of dimensions of fibres of f. Let $q\colon U \to S$ be a morphism from a strictly local scheme U and $\overline{\eta} \to U$ an algebraic geometric point. Then, for every sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on X, where n is a positive integer, we have $R^i\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U) = 0$ for $i > 2N$. □

We will prove $\mathbf{P}(S,\rho,d)$ by induction on the triples (S,ρ,d). For two excellent Noetherian integral schemes S and S', we denote

$$S' \prec S$$

if S' is isomorphic to a proper closed subscheme of an integral alteration of S. For an excellent Noetherian integral scheme S and an integer ρ, we also consider the following statements.

- $\mathbf{P}(S,\rho,*)$: The statement $\mathbf{P}(S,\rho,d')$ holds for every integer d'.
- $\mathbf{P}(*\prec S,\rho,*)$: The statement $\mathbf{P}(S',\rho,d')$ holds for every excellent Noetherian integral scheme S' with $S' \prec S$ and every integer d'.

We begin with the following lemma.

Lemma 3.9. *Let S be an excellent Noetherian integral scheme and ρ an integer. If* $\boldsymbol{P}(*\prec S,\rho,*)$ *holds, then* $\boldsymbol{P}(S,\rho,-1)$ *holds.*

Proof. This lemma can be proved by the same arguments as in [25, Section 4.2] by using [26, Proposition 1.6.2] instead of [25, Lemme 4.3]. □

Our next task is to show the following proposition, which is the most difficult part.

Proposition 3.10. *Let (S,ρ,d) be a triple of an excellent Noetherian integral scheme S and two integers ρ and d. Assume that $d \geq 0$. If* $\boldsymbol{P}(S,\rho,d-1)$, $\boldsymbol{P}(S,\rho-1,*)$ *and* $\boldsymbol{P}(*\prec S,\rho,*)$ *hold, then* $\boldsymbol{P}(S,\rho,d)$ *holds.*

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The proof of Proposition 3.10 is divided into two steps. The first step is to reduce to the case of pluri nodal curves:

Lemma 3.11. *We assume that* $\boldsymbol{P}(S,\rho,d-1)$*,* $\boldsymbol{P}(S,\rho-1,*)$ *and* $\boldsymbol{P}(*\prec S,\rho,*)$ *hold. Under this assumption, to prove* $\boldsymbol{P}(S,\rho,d)$*, it suffices to prove the following statement* $\boldsymbol{P}_{nd}(S,\rho,d)$*:*

> $\boldsymbol{P}_{nd}(S,\rho,d)$*: For every integral alteration* $T\to S$ *and for every pluri nodal curve* $f\colon Y\to T$ *adapted to a dense open subset* $Y^\circ\subset Y$ *such that the dimension of the generic fibre of f is less than or equal to d, there is an alteration* $T'\to T$ *which is* ρ*-adapted to* (f,Y°) *in the sense of Definition 3.6 (2).*

Proof. This lemma can be proved by an argument similar to that in the proof of [26, Théorème 3.1.1] together with Theorem 3.3, Lemma 3.5, and Lemma 3.9 (see especially [26, 3.5.2]). □

Next, we prove $\mathbf{P}_{\mathrm{nd}}(S,\rho,d)$ in Lemma 3.11 under the assumptions:

Lemma 3.12. *We assume that* $\boldsymbol{P}(S,\rho,d-1)$*,* $\boldsymbol{P}(S,\rho-1,*)$ *and* $\boldsymbol{P}(*\prec S,\rho,*)$ *hold. Then the statement* $\boldsymbol{P}_{nd}(S,\rho,d)$ *in Lemma 3.11 is true.*

Proof. Let $T\to S$ be an integral alteration and $f\colon Y\to T$ a pluri nodal curve adapted to a dense open subset $Y^\circ\subset Y$ such that the dimension of the generic fibre of f is less than or equal to d. Let $u\colon Y^\circ\hookrightarrow Y$ denote the open immersion. If f is an isomorphism, then there is nothing to prove. Hence, we may assume that f is not an isomorphism, and, hence, there are a factorisation

$$Y\xrightarrow{h}X\xrightarrow{g}T,\qquad f=g\circ h$$

and a dense open subset $X^\circ\subset X$ such that $h\colon Y\to X$ is a nodal curve adapted to the pair (Y°,X°) and $g\colon X\to T$ is a pluri nodal curve adapted to X°. Since $\mathbf{P}(S,\rho,d-1)$ holds, we may assume that the identity map $T\to T$ is ρ-adapted to the following two pairs

$$(Y\backslash Y^\circ\to T,\{Y\backslash Y^\circ\})\quad\text{and}\quad(g,\{X^\circ,X\backslash X^\circ\}).$$

By replacing T with its normalization, we may assume that T is normal.

We claim that the identity map $T\to T$ is ρ-adapted to (f,Y°). The proof is divided into two parts. First, we prove the assertion after restricting to the smooth locus $Y'\subset Y$ of h. Then, we prove our claim by using the results on the smooth locus Y'.

Claim 3.13. *Let* $a\colon Y'\to T$ *denote the restriction of f to* Y'*. Let n be a positive integer invertible on T and* $\mathcal{L}$ *a locally constant constructible sheaf of* $\mathbb{Z}/n\mathbb{Z}$*-modules on* Y° *such that* $u_!\mathcal{L}$ *is locally unipotent along the stratification* $\mathfrak{Y}:=\{Y^\circ,Y\backslash Y^\circ\}$*. Let* $\mathcal{F}$ *be the pullback of* $u_!\mathcal{L}$ *to* Y'*. Then the following assertions hold:*

(1) *The nearby cycles for* $a\colon Y'\to T$ *and* $\mathcal{F}$ *are* ρ*-compatible with any base change.*

(2) *The nearby cycles for* $a\colon Y'\to T$ *and* $\mathcal{F}$ *are* ρ*-unipotent.*

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Proof. (1) We fix a local morphism $q\colon V\to U$ of strictly local schemes over T and an algebraic geometric point $\overline{\xi}\to V$ with image $\overline{\eta}\to U$. In the following, for a morphism $\phi\colon Z\to T$ and a complex $\mathcal{K}\in D^+(Z,\mathbb{Z}/n\mathbb{Z})$, the cone of the base change map

$$q^*R\Psi_{\phi_U,\overline{\eta}}(\mathcal{K}_U)\to R\Psi_{\phi_V,\overline{\xi}}(\mathcal{K}_V)$$

is denoted by $\Delta(\phi,\mathcal{K})$. For a morphism $\psi\colon Z\to W$ of T-schemes and a T-scheme T', the base change $Z_{T'}\to W_{T'}$ is often denoted by the same letter ψ when there is no possibility of confusion.

We want to show $\tau_{\leq\rho}\Delta(a,\mathcal{F})=0$. It suffices to prove that $\tau_{\leq\rho}\Delta(a,\mathcal{F})_x=0$ at every geometric point $x\to Y'_s$, where $s\in V$ is the closed point. The morphism

$$(q^*R\Psi_{a_U,\overline{\eta}}(\mathcal{F}_U))_x\to R\Psi_{a_V,\overline{\xi}}(\mathcal{F}_V)_x$$

on the stalks can be identified with the pullback map

$$R\Gamma((Y'_U)_{(x)}\times_U U_{(\overline{\eta})},u_!\mathcal{L})\to R\Gamma((Y'_V)_{(x)}\times_V V_{(\overline{\xi})},u_!\mathcal{L})$$

(see also (2.1) in Remark 2.4).

In order to show $\tau_{\leq\rho}\Delta(a,\mathcal{F})_x=0$, we can assume that $\mathcal{L}$ is a constant sheaf on Y°. Indeed, let $\mathcal{L}'$ denote the pullback of $\mathcal{L}$ to $(Y'_U)_{(x)}\times_Y Y^\circ$. It has a finite filtration

$$0=\mathcal{L}'_0\subset\cdots\subset\mathcal{L}'_i\subset\mathcal{L}'_{i+1}\subset\cdots\subset\mathcal{L}'$$

such that each successive quotient $\mathcal{L}'_{i+1}/\mathcal{L}'_i$ is a constant sheaf since $u_!\mathcal{L}$ is locally unipotent along $\mathfrak{Y}=\{Y^\circ,Y\backslash Y^\circ\}$. The pullback of $u_!\mathcal{L}$ to $(Y'_U)_{(x)}$ is isomorphic to $u'_!\mathcal{L}'$, where $u'\colon (Y'_U)_{(x)}\times_Y Y^\circ\hookrightarrow (Y'_U)_{(x)}$ denotes the open immersion. Let Δ_i (resp. $\Delta_{i+1,i}$) denote the cone of the map

$$R\Gamma((Y'_U)_{(x)}\times_U U_{(\overline{\eta})},u'_!\mathcal{L}'_i)\to R\Gamma((Y'_V)_{(x)}\times_V V_{(\overline{\xi})},u'_!\mathcal{L}'_i)$$

(resp.

$$R\Gamma((Y'_U)_{(x)}\times_U U_{(\overline{\eta})},u'_!(\mathcal{L}'_{i+1}/\mathcal{L}'_i))\to R\Gamma((Y'_V)_{(x)}\times_V V_{(\overline{\xi})},u'_!(\mathcal{L}'_{i+1}/\mathcal{L}'_i))).$$

We obtain a distinguished triangle $\Delta_i\to\Delta_{i+1}\to\Delta_{i+1,i}\to$ for each i. Thus, if we have shown $\tau_{\leq\rho}\Delta_{i+1,i}=0$ for every i, then we can prove $\tau_{\leq\rho}\Delta_i=0$ for every i inductively, which, in turn, implies $\tau_{\leq\rho}\Delta(a,\mathcal{F})_x=0$. Since $\mathcal{L}'_{i+1}/\mathcal{L}'_i$ is a constant sheaf (and, hence, is the pullback of a constant sheaf on Y°), this means that it suffices to treat the case where $\mathcal{L}$ is a constant sheaf.

Now we assume that $\mathcal{L}=\Lambda$ is a constant sheaf on Y°. Note that Y° is contained in Y'. Since we have the following exact sequence of sheaves on Y'

$$0\to u_!\Lambda\to\Lambda\to v_*\Lambda\to 0,$$

where $v\colon Y'\backslash Y^\circ\hookrightarrow Y'$ is the closed immersion and we denote the open immersion $Y^\circ\hookrightarrow Y'$ by the same letter u, it suffices to prove that $\tau_{\leq\rho}\Delta(a,\Lambda)=0$ and $\tau_{\leq\rho}\Delta(a,v_*\Lambda)=0$.

It follows from the assumption on T that the nearby cycles for $a\circ v$ and the constant sheaf Λ are ρ-compatible with any base change. Hence, we have $\tau_{\leq\rho}\Delta(a,v_*\Lambda)=0$. By the assumption on T again, the nearby cycles for g and the constant sheaf Λ are ρ-compatible with any base change. Since the composition $b\colon Y'\hookrightarrow Y\to X$ is smooth, we

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have $\Delta(a,\Lambda) \cong b^*\Delta(g,\Lambda)$ by the smooth base change theorem. Hence, we obtain that

$$\tau_{\leq\rho}\Delta(a,\Lambda) \cong \tau_{\leq\rho}b^*\Delta(g,\Lambda) \cong b^*\tau_{\leq\rho}\Delta(g,\Lambda) = 0.$$

(2) Let $q\colon U \to T$ be a morphism from a strictly local scheme U, a point $\eta \in U$ and an algebraic geometric point $\overline{\eta} \to U$ lying above η. Let $s \in U$ be the closed point. We want to show that the complex

$$\tau_{\leq\rho}R\Psi_{a_U,\overline{\eta}}(\mathcal{F}_U)$$

is $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$-unipotent.

We first claim that for every $i \leq \rho$, the sheaf $R^i\Psi_{a_U,\overline{\eta}}(\mathcal{F}_U)$ is constructible. Since we have already shown that the nearby cycles for a and $\mathcal{F}$ are ρ-compatible with any base change, we may assume that U is the strict localization of T at $s \to T$, in particular, we may assume that U is Noetherian. Then, by using [11, Proposition 7.1.9], we may assume that U is the spectrum of strictly Henselian discrete valuation ring, and in this case, the claim follows from [9, Th. finitude, Théorème 3.2] (see also [25, Section 8]). Now, it suffices to prove that, for every geometric point $x \to Y'_s$, the complex

$$\tau_{\leq\rho}R\Psi_{a_U,\overline{\eta}}(\mathcal{F}_U)_x \cong \tau_{\leq\rho}R\Gamma((Y'_U)_{(x)} \times_U U_{(\overline{\eta})}, u_!\mathcal{L})$$

is $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$-unipotent (see [26, Lemme 1.2.5]). Since the sheaf $u_!\mathcal{L}$ is locally unipotent along $\mathfrak{Y} = \{Y^\circ, Y\backslash Y^\circ\}$, we reduce to the case where $\mathcal{L} = \Lambda$ is a constant sheaf on Y° as in the proof of (1).

By the exact sequence $0 \to u_!\Lambda \to \Lambda \to v_*\Lambda \to 0$, it suffices to prove that the nearby cycles for a and $v_*\Lambda$ and the nearby cycles for a and Λ are ρ-unipotent. By using the assumption on T, we conclude by the same argument as in the proof of (1). □

Claim 3.14. *Let n be a positive integer invertible on T and $\mathcal{L}$ a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on Y° such that $\mathcal{F} := u_!\mathcal{L}$ is locally unipotent along the stratification $\mathfrak{Y} = \{Y^\circ, Y\backslash Y^\circ\}$. Then the following assertions hold:*

(1) *The nearby cycles for f and $\mathcal{F}$ are ρ-compatible with any base change.*

(2) *The nearby cycles for f and $\mathcal{F}$ are ρ-unipotent.*

Proof. (1) We fix a local morphism $q\colon V \to U$ of strictly local schemes over T and an algebraic geometric point $\overline{\xi} \to V$ with image $\overline{\eta} \to U$. We retain the notation of the proof of Claim 3.13 (1). We write $\Delta := \Delta(f,\mathcal{F})$. We want to show $\tau_{\leq\rho}\Delta = 0$. Let $c\colon Z \hookrightarrow Y$ be a closed immersion whose complement is the smooth locus Y' of h. By Claim 3.13 (1), we have

$$\tau_{\leq\rho}\Delta \cong c_*c^*\tau_{\leq\rho}\Delta,$$

and, hence, it suffices to show that $c^*\tau_{\leq\rho}\Delta = 0$. Since the composition $d\colon Z \to Y \to X$ is a finite morphism, it is enough to prove that

$$d_*c^*\tau_{\leq\rho}\Delta = 0.$$

By using $\tau_{\leq\rho}\Delta \cong c_*c^*\tau_{\leq\rho}\Delta$, we obtain an isomorphism $d_*c^*\tau_{\leq\rho}\Delta \cong \tau_{\leq\rho}Rh_*\Delta$. By the proper base change theorem, we have $Rh_*\Delta \cong \Delta(g,Rh_*\mathcal{F})$. Note that X° is normal since T is normal. Hence, the cohomology sheaves of $Rh_*\mathcal{F}$ are locally unipotent along the stratification $\{X^\circ, X\backslash X^\circ\}$ by Proposition 3.1. By the assumption on T, we have $\tau_{\leq\rho}\Delta(g, R^ih_*\mathcal{F}) = 0$ for every i. It follows that $\tau_{\leq\rho}\Delta(g, Rh_*\mathcal{F}) = 0$. This completes the proof of (1).

(2) Let $q\colon U \to T$ be a morphism from a strictly local scheme U, a point $\eta \in U$ and an algebraic geometric point $\overline{\eta} \to U$ lying above η. We write $\mathcal{K} := R\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U)$. Let $e\colon Y' \to Y$ denote the open immersion. We have the following distinguished triangle:

$$e_!e^*\tau_{\leq\rho}\mathcal{K} \to \tau_{\leq\rho}\mathcal{K} \to c_*c^*\tau_{\leq\rho}\mathcal{K} \to .$$

By Claim 3.13 (2), it suffices to prove that $c^*\tau_{\leq\rho}\mathcal{K}$ is $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$-unipotent. Since d is a finite morphism, it suffices to prove that

$$d_*c^*\tau_{\leq\rho}\mathcal{K}$$

is $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$-unipotent. We have the following distinguished triangle:

$$Rh_*e_!e^*\tau_{\leq\rho}\mathcal{K} \to Rh_*\tau_{\leq\rho}\mathcal{K} \to d_*c^*\tau_{\leq\rho}\mathcal{K} \to .$$

Since the complex $Rh_*e_!e^*\tau_{\leq\rho}\mathcal{K}$ is $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$-unipotent by Claim 3.13 (2), it is enough to show that $\tau_{\leq\rho}Rh_*\tau_{\leq\rho}\mathcal{K} \cong \tau_{\leq\rho}Rh_*\mathcal{K}$ is $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$-unipotent. By the proper base change theorem, we have

$$Rh_*\mathcal{K} \cong R\Psi_{g_U,\overline{\eta}}((Rh_*\mathcal{F})_U).$$

As above, by Proposition 3.1 and the assumption on T, it follows that the complex $\tau_{\leq\rho}R\Psi_{g_U,\overline{\eta}}((Rh_*\mathcal{F})_U)$ is $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$-unipotent, whence, (2). □

The proof of Lemma 3.12 is complete. □

Now Proposition 3.10 follows from Lemma 3.11 and Lemma 3.12. Finally, we prove the following proposition which completes the proof Theorem 2.8.

Proposition 3.15. *For every triple (S,ρ,d) of an excellent Noetherian integral scheme S and two integers ρ and d, the statement $\boldsymbol{P}(S,\rho,d)$ holds.*

Proof. We assume that $\mathbf{P}(S,\rho,d)$ does not hold. Then, by Lemma 3.9 and Proposition 3.10, we can find infinitely many triples $\{(S_n,\rho_n,d_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ with the following properties:

(1) $\mathbf{P}(S_n,\rho_n,d_n)$ does not hold for every $n \in \mathbb{Z}_{\geq 0}$.

(2) $(S_0,\rho_0,d_0) = (S,\rho,d)$.

(3) For every $n \in \mathbb{Z}_{\geq 0}$, we either have
  - (a) $S_{n+1} \prec S_n$,
  - (b) $S_{n+1} = S_n$, $\rho_{n+1} = \rho_n - 1$, and $d_n \geq 0$ or
  - (c) $S_{n+1} = S_n$, $\rho_{n+1} = \rho_n$, and $d_{n+1} = d_n - 1 \geq -1$.

By [26, Lemme in 3.4.4], there is an integer $N \geq 0$ such that $S_{n+1} = S_n$ for every $n \geq N$. Since $\mathbf{P}(S', -2, d')$ holds trivially for every excellent Noetherian integral scheme S' and every integer d', there is an integer $N' \geq N$ such that $d_{n+1} = d_n - 1 \geq -1$ for every $n \geq N'$. This leads to a contradiction. □

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4. Tubular neighbourhoods and main results

In this section, we will state our main results on étale cohomology of tubular neighbourhoods.

4.1. Adic spaces and pseudo-adic spaces

In this paper, we will freely use the theory of adic spaces and pseudo-adic spaces developed by Huber. Our basis references are [12, 13, 14]. We shall recall the definitions very roughly. We will use the terminology in [14, Section 1.1], such as a valuation of a ring, an affinoid ring, a Tate ring or a strongly Noetherian Tate ring.

An *adic space* is by definition a triple

$$X = (X, \mathcal{O}_X, \{v_x\}_{x \in X}),$$

where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings on the topological space X and v_x is an equivalence class of valuations of the stalk $\mathcal{O}_{X,x}$ at $x \in X$ which is locally isomorphic to the *affinoid adic space* $\mathrm{Spa}(A, A^+)$ associated with an affinoid ring (A, A^+) (see [14, Section 1.1] for details). In this paper, unless stated otherwise, we assume that every adic space is locally isomorphic to the affinoid adic space $\mathrm{Spa}(A, A^+)$ associated with an affinoid ring (A, A^+) such that A is a strongly Noetherian Tate ring. So we can use the results in [14] (see [14, (1.1.1)]). In particular, we only treat analytic adic spaces (see [14, Section 1.1] for the definition of an analytic adic space).

A *pseudo-adic space* is a pair

$$(X, S),$$

where X is an adic space and S is a subset of X satisfying certain conditions (see [14, Definition 1.10.3]). If X is an adic space and $S \subset X$ is a locally closed subset, then (X, S) is a pseudo-adic space. Almost all pseudo-adic spaces which appear in this paper are of this form. A morphism $f\colon (X, S) \to (X', S')$ of pseudo-adic spaces is by definition a morphism $f\colon X \to X'$ of adic spaces with $f(S) \subset S'$.

We have a functor $X \mapsto (X, X)$ from the category of adic spaces to the category of pseudo-adic spaces. We often consider an adic space as a pseudo-adic space via this functor.

A typical example of an adic space is the following. Let K be a non-archimedean field, that is, it is a topological field whose topology is induced by a valuation $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ of rank 1. We assume that K is complete. Let $\mathcal{O} = K^\circ$ be the valuation ring of $|\cdot|$. We call $\mathcal{O}$ the ring of integers of K. Let $\varpi \in K^\times$ be an element with $|\varpi| < 1$. Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{O}$. The ϖ-adic formal completion of $\mathcal{X}$ is denoted by $\widehat{\mathcal{X}}$ or $\mathcal{X}^\wedge$. Following [14, Section 1.9], the Raynaud generic fibre of $\widehat{\mathcal{X}}$ is denoted by $d(\widehat{\mathcal{X}})$, which is an adic space of finite type over $\mathrm{Spa}(K, \mathcal{O})$. In particular, $d(\widehat{\mathcal{X}})$ is quasicompact. For example, we have

$$d((\mathrm{Spec}\, \mathcal{O}[T])^\wedge) = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}\langle T\rangle) =: \mathbb{B}(1).$$

We often identify $d((\mathrm{Spec}\, \mathcal{O}[T])^\wedge)$ with $\mathbb{B}(1)$. For a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of schemes of finite type over $\mathcal{O}$, the induced morphism $d(\widehat{\mathcal{Y}}) \to d(\widehat{\mathcal{X}})$ is denoted by $d(f)$ (rather than $d(\widehat{f})$).

Important examples of pseudo-adic spaces for us are tubular neighbourhoods of adic spaces. In the next subsection, we will define them in the case where adic spaces arise from schemes of finite type over $\mathcal{O}$.

4.2. Tubular neighbourhoods

Let $X = (X, \mathcal{O}_X, \{v_x\}_{x\in X})$ be an adic space. Let $U \subset X$ be an open subset and $g \in \mathcal{O}_X(U)$ an element. Following [14], for a point $x \in U$, we write $|g(x)| := v_x(g)$ (strictly speaking, we implicitly choose a valuation from the equivalence class v_x).

As in the previous subsection, let K be a complete non-archimedean field with ring of integers $\mathcal{O}$.

Proposition 4.1. *Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. Let $\epsilon \in |K^\times|$ be an element.*

(1) *There exist subsets*

$$S(\mathcal{Z},\epsilon) \subset d(\widehat{\mathcal{X}}) \quad \text{and} \quad T(\mathcal{Z},\epsilon) \subset d(\widehat{\mathcal{X}})$$

satisfying the following properties; for any affine open subset $\mathcal{U} \subset \mathcal{X}$ and any set $\{g_1,\dots,g_q\} \subset \mathcal{O}_{\mathcal{U}}(\mathcal{U})$ of elements defining the closed subscheme $\mathcal{Z}\cap\mathcal{U}$ of $\mathcal{U}$, we have

$$\begin{aligned} S(\mathcal{Z},\epsilon) \cap d(\widehat{\mathcal{U}}) &= \{x \in d(\widehat{\mathcal{U}}) \mid |g_i(x)| < \epsilon \text{ for every } 1 \le i \le q\} \\ &:= \{x \in d(\widehat{\mathcal{U}}) \mid |g_i(x)| < |\varpi(x)| \text{ for every } 1 \le i \le q\} \end{aligned}$$

and

$$\begin{aligned} T(\mathcal{Z},\epsilon) \cap d(\widehat{\mathcal{U}}) &= \{x \in d(\widehat{\mathcal{U}}) \mid |g_i(x)| \le \epsilon \text{ for every } 1 \le i \le q\} \\ &:= \{x \in d(\widehat{\mathcal{U}}) \mid |g_i(x)| \le |\varpi(x)| \text{ for every } 1 \le i \le q\}, \end{aligned}$$

where $\varpi \in K^\times$ is an element with $\epsilon = |\varpi|$ and the element of $\mathcal{O}_{d(\widehat{\mathcal{U}})}(d(\widehat{\mathcal{U}}))$ arising from g_i is denoted by the same letter. Moreover, they are characterised by the above properties.

(2) *The subset $T(\mathcal{Z},\epsilon)$ is a quasicompact open subset of $d(\widehat{\mathcal{X}})$. The subset $S(\mathcal{Z},\epsilon)$ is closed and constructible in $d(\widehat{\mathcal{X}})$ (see [14, (1.1.13)] for the definition of a constructible subset).*

(3) *For a morphism $f : \mathcal{Y} \to \mathcal{X}$ of finite type, we have*

$$S(\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y},\epsilon) = d(f)^{-1}(S(\mathcal{Z},\epsilon)) \quad \text{and} \quad T(\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y},\epsilon) = d(f)^{-1}(T(\mathcal{Z},\epsilon)).$$

Proof. (1) Let $\varpi \in K^\times$ be an element with $\epsilon = |\varpi|$. Let $\mathcal{U} \subset \mathcal{X}$ be an affine open subset. It suffices to show that the subsets

$$\{x \in d(\widehat{\mathcal{U}}) \mid |g_i(x)| < \epsilon \text{ for every } 1 \le i \le q\}$$

and

$$\{x \in d(\widehat{\mathcal{U}}) \mid |g_i(x)| \le \epsilon \text{ for every } 1 \le i \le q\}$$

are independent of the choice of a set $\{g_1,\ldots,g_q\} \subset \mathcal{O}_{\mathcal{U}}(\mathcal{U})$ of elements defining the closed subscheme $\mathcal{Z} \cap \mathcal{U}$ of $\mathcal{U}$. Let $\{h_1,\ldots,h_r\} \subset \mathcal{O}_{\mathcal{U}}(\mathcal{U})$ be another set of such elements. Then, for every i, we have

$$g_i = \Sigma_{1 \leq j \leq r} s_{ij} h_j$$

for some elements $\{s_{ij}\} \subset \mathcal{O}_{\mathcal{U}}(\mathcal{U})$. Since we have $|s_{ij}(x)| \leq 1$ for every $x \in d(\widehat{\mathcal{U}})$ and every s_{ij}, the assertion follows.

(2) We may assume that $\mathcal{X}$ is affine. The subset $T(\mathcal{Z},\epsilon)$ is a rational subset of the affinoid adic space $d(\widehat{\mathcal{X}})$, and, hence, it is open and quasicompact. The subset $S(\mathcal{Z},\epsilon)$ is the complement of the union of finitely many rational subsets. It follows that $S(\mathcal{Z},\epsilon)$ is closed and constructible.

(3) We may assume that $\mathcal{X}$ and $\mathcal{Y}$ are affine. Then the assertion follows from the descriptions given in (1). □

The subsets $T(\mathcal{Z},\epsilon)$ and $S(\mathcal{Z},\epsilon)$ in Proposition 4.1 are called an *open tubular neighbourhood* and a *closed tubular neighbourhood* of $d(\widehat{\mathcal{Z}})$ in $d(\widehat{\mathcal{X}})$, respectively. For an element $\epsilon \in |K^\times|$, we also consider the following subsets:

$$Q(\mathcal{Z},\epsilon) := d(\widehat{\mathcal{X}}) \backslash S(\mathcal{Z},\epsilon).$$

This is a quasicompact open subset of $d(\widehat{\mathcal{X}})$.

For a locally closed subset S of an adic space X, the pseudo-adic space (X,S) is often denoted by S for simplicity. For example, the pseudo-adic spaces $(d(\widehat{\mathcal{X}}),S(\mathcal{Z},\epsilon))$ and $(d(\widehat{\mathcal{X}}),T(\mathcal{Z},\epsilon))$ are denoted by $S(\mathcal{Z},\epsilon)$ and $T(\mathcal{Z},\epsilon)$, respectively.

Remark 4.2. For a formal scheme $\mathscr{X}$ of finite type over $\operatorname{Spf} \mathcal{O}$ and a closed immersion $\mathscr{Z} \hookrightarrow \mathscr{X}$ of finite presentation (in the sense of [10, Chapter I, Definition 2.2.1]), we can also define tubular neighbourhoods of $d(\mathscr{Z})$ in $d(\mathscr{X})$ in the same way. However, we will always work with algebraizable formal schemes of finite type over $\mathcal{O}$ in this paper.

We end this subsection with the following lemma.

Lemma 4.3. *Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. For a constructible subset $W \subset d(\widehat{\mathcal{X}})$ containing $d(\widehat{\mathcal{Z}})$, there is an element $\epsilon \in |K^\times|$ such that $T(\mathcal{Z},\epsilon) \subset W$.*

Proof. We may assume that $\mathcal{X}$ is affine. Then the underlying topological space of $d(\widehat{\mathcal{X}})$ is a spectral space. We have

$$d(\widehat{\mathcal{Z}}) = \bigcap_{\epsilon \in |K^\times|} T(\mathcal{Z},\epsilon).$$

Hence, the intersection

$$\bigcap_{\epsilon \in |K^\times|} T(\mathcal{Z},\epsilon) \cap (d(\widehat{\mathcal{X}}) \backslash W)$$

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is empty. In the constructible topology, the subsets $T(\mathcal{Z},\epsilon)$ and $d(\widehat{\mathcal{X}})\backslash W$ are closed and $d(\widehat{\mathcal{X}})$ is quasicompact. It follows that there is an element $\epsilon \in |K^\times|$ such that the intersection $T(\mathcal{Z},\epsilon) \cap d(\widehat{\mathcal{X}})\backslash W$ is empty, whence $T(\mathcal{Z},\epsilon) \subset W$. □

4.3. Main results on tubular neighbourhoods

In this subsection, let K be an algebraically closed complete non-archimedean field with ring of integers $\mathcal{O}$.

To state our main results on tubular neighbourhoods, we need étale cohomology and étale cohomology with proper support of pseudo-adic spaces (see [14, Section 2.3] for definition of the étale site of a pseudo-adic space). As shown in [14, Proposition 2.3.7], for an adic space X and an open subset $U \subset X$, the étale topos of the adic space U is naturally equivalent to the étale topos of the pseudo-adic space (X,U). For a commutative ring Λ, let $D^+(X,\Lambda)$ denote the derived category of bounded below complexes of étale sheaves of Λ-modules on a pseudo-adic space X.

Let $f\colon X \to Y$ be a morphism of analytic pseudo-adic spaces. We assume that f is separated, locally of finite type and *taut* (see [14, Definition 5.1.2] for the definitions of a taut pseudo-adic space and a taut morphism of pseudo-adic spaces; for example, if f is separated and quasicompact, then f is taut). For such a morphism f, the direct image functor with proper support

$$Rf_!\colon D^+(X,\Lambda) \to D^+(Y,\Lambda)$$

is defined in [14, Definition 5.4.4], where Λ is a torsion commutative ring. Moreover, if $Y = \mathrm{Spa}(K,\mathcal{O})$, we obtain for a complex $\mathcal{K} \in D^+(X,\Lambda)$ the cohomology group with proper support

$$H^i_c(X,\mathcal{K}).$$

Example 4.4. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation.

(1) The adic spaces $d(\widehat{\mathcal{Z}})$ and $d(\widehat{\mathcal{X}})$ are separated and of finite type over $\mathrm{Spa}(K,\mathcal{O})$. The morphism $d(\widehat{\mathcal{X}})\backslash d(\widehat{\mathcal{Z}}) \to \mathrm{Spa}(K,\mathcal{O})$ is separated, locally of finite type and taut (see [14, Lemma 5.1.4]).

(2) The pseudo-adic spaces $S(\mathcal{Z},\epsilon)$, $T(\mathcal{Z},\epsilon)$ and $Q(\mathcal{Z},\epsilon)$ are separated and of finite type (and, hence, taut) over $\mathrm{Spa}(K,\mathcal{O})$.

(3) For a subset S of an analytic adic space X, the interior of S in X is denoted by S°. The morphism $S(\mathcal{Z},\epsilon)^\circ \to \mathrm{Spa}(K,\mathcal{O})$ is separated, locally of finite type and taut [15, Lemma 1.3 iii)].

Let us recall the following results due to Huber in our setting.

Theorem 4.5 (Huber [15, Theorem 2.5], [16, Theorem 3.6]). *We assume that K is of characteristic zero. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. Let n be a positive integer invertible in $\mathcal{O}$, and let $\mathcal{F}$*

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be a constructible étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $d(\widehat{\mathcal{X}})$ in the sense of [14, Definition 2.7.2].

(1) *There exists an element $\epsilon_0 \in |K^\times|$ such that, for every $\epsilon \in |K^\times|$ with $\epsilon \le \epsilon_0$, the following natural maps are isomorphisms for every i:*
   (a) $H_c^i(S(\mathcal{Z},\epsilon),\mathcal{F}|_{S(\mathcal{Z},\epsilon)}) \overset{\cong}{\to} H_c^i(d(\widehat{\mathcal{Z}}),\mathcal{F}|_{d(\widehat{\mathcal{Z}})})$.
   (b) $H_c^i(T(\mathcal{Z},\epsilon),\mathcal{F}) \overset{\cong}{\to} H_c^i(T(\mathcal{Z},\epsilon_0),\mathcal{F})$.
   (c) $H_c^i(Q(\mathcal{Z},\epsilon),\mathcal{F}) \overset{\cong}{\to} H_c^i(d(\widehat{\mathcal{X}})\backslash d(\widehat{\mathcal{Z}}),\mathcal{F})$.

(2) *We assume further that $\mathcal{F}$ is locally constant. Then there exists an element $\epsilon_0 \in |K^\times|$ such that, for every $\epsilon \in |K^\times|$ with $\epsilon \le \epsilon_0$, the restriction maps*

$$H^i(T(\mathcal{Z},\epsilon),\mathcal{F}) \overset{\cong}{\to} H^i(S(\mathcal{Z},\epsilon),\mathcal{F}|_{S(\mathcal{Z},\epsilon)}) \overset{\cong}{\to} H^i(S(\mathcal{Z},\epsilon)^\circ,\mathcal{F}) \overset{\cong}{\to} H^i(d(\widehat{\mathcal{Z}}),\mathcal{F}|_{d(\widehat{\mathcal{Z}})})$$

on étale cohomology groups are isomorphisms for every i.

Proof. See [15, Theorem 2.5] for the proof of (1) and a more general result; see [14, Remark 5.5.11] for the constructions of the natural maps. See [13, Theorem 3.6] for the proof (2) and a more general result. □

Remark 4.6. For an algebraically closed complete non-archimedean field K of positive characteristic, an analogous statement to Theorem 4.5 (1) is proved in [18, Corollary 5.8].

Remark 4.7. If the residue field of $\mathcal{O}$ is of positive characteristic $p > 0$, the assumption that n is invertible in $\mathcal{O}$ in Theorem 4.5 is essential. For example, the étale cohomology group $H^1(\mathbb{B}(1),\mathbb{Z}/p\mathbb{Z})$ is an infinite dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space (see the computations in [2, Remark 6.4.2]). However, we have $H^1(\{0\},\mathbb{Z}/p\mathbb{Z}) = 0$ for the origin $0 \in \mathbb{B}(1)$.

The main objective of this paper is to prove uniform variants of Theorem 4.5 for constant sheaves. Our main result on étale cohomology groups with proper support of tubular neighbourhoods is as follows.

Theorem 4.8. *Let K be an algebraically closed complete non-archimedean field with ring of integers $\mathcal{O}$. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. Then there exists an element $\epsilon_0 \in |K^\times|$ such that, for every $\epsilon \in |K^\times|$ with $\epsilon \le \epsilon_0$ and for every positive integer n invertible in $\mathcal{O}$, the following natural maps are isomorphisms for every i:*

(1) $H_c^i(S(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \overset{\cong}{\to} H_c^i(d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z})$.
(2) $H_c^i(T(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \overset{\cong}{\to} H_c^i(T(\mathcal{Z},\epsilon_0),\mathbb{Z}/n\mathbb{Z})$.
(3) $H_c^i(Q(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \overset{\cong}{\to} H_c^i(d(\widehat{\mathcal{X}})\backslash d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z})$.

Our main result on étale cohomology groups of tubular neighbourhoods is as follows.

Theorem 4.9. *Let K be an algebraically closed complete non-archimedean field with ring of integers $\mathcal{O}$. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed*

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immersion of finite presentation. Then there exists an element $\epsilon_0 \in |K^\times|$ *such that, for every* $\epsilon \in |K^\times|$ *with* $\epsilon \leq \epsilon_0$ *and for every positive integer* n *invertible in* $\mathcal{O}$*, the restriction maps*

$$H^i(T(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(S(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(S(\mathcal{Z},\epsilon)^\circ,\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z})$$

are isomorphisms for every i*.*

Theorem 1.2 follows from Theorem 4.9 (see also the following remark).

Remark 4.10. In Theorem 4.8 and Theorem 4.9, the assumption that the closed immersion $\mathcal{Z} \hookrightarrow \mathcal{X}$ is of finite presentation is not important in practice. Indeed, if we are only interested in the adic spaces $d(\widehat{\mathcal{Z}})$ and $d(\widehat{\mathcal{X}})$, then by replacing $\mathcal{Z}$ with the closed subscheme $\mathcal{Z}' \hookrightarrow \mathcal{Z}$ defined by the sections killed by a power of a nonzero element of the maximal ideal of $\mathcal{O}$, we can reduce to the case where $\mathcal{Z}$ is flat over $\mathcal{O}$ without changing $d(\widehat{\mathcal{Z}})$. Then $\mathcal{Z}$ is of finite presentation over $\mathcal{O}$ by [28, Première partie, Corollaire 3.4.7], and, hence, $\mathcal{Z} \hookrightarrow \mathcal{X}$ is also of finite presentation.

The proofs of Theorem 4.8 and Theorem 4.9 will be given in Section 7. In the rest of this section, we will restate Theorem 4.9 for proper schemes over K.

Let $L \subset K$ be a subfield of K which is a complete non-archimedean field with the induced topology. Let $\mathcal{O}_L$ be the ring of integers of L. For a scheme X of finite type over L, the adic space associated with X is denoted by

$$X^{\mathrm{ad}} := X \times_{\operatorname{Spec} L} \operatorname{Spa}(L,\mathcal{O}_L)$$

(see [13, Proposition 3.8]). For an adic space Y locally of finite type over $\operatorname{Spa}(L,\mathcal{O}_L)$, we denote by

$$Y_K := Y \times_{\operatorname{Spa}(L,\mathcal{O}_L)} \operatorname{Spa}(K,\mathcal{O})$$

the base change of Y to $\operatorname{Spa}(K,\mathcal{O})$, which exists by [14, Proposition 1.2.2].

Corollary 4.11. *Let* X *be a proper scheme over* L *and* $Z \hookrightarrow X$ *a closed immersion. We have a closed immersion* $Z^{\mathrm{ad}} \hookrightarrow X^{\mathrm{ad}}$ *of adic spaces over* $\operatorname{Spa}(L,\mathcal{O}_L)$*. Then, there is a quasicompact open subset* V *of* X^{ad} *containing* Z^{ad} *such that, for every positive integer* n *invertible in* $\mathcal{O}$*, the restriction map*

$$H^i(V_K,\mathbb{Z}/n\mathbb{Z}) \to H^i((Z^{\mathrm{ad}})_K,\mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for every i*.*

Proof. There exist a proper scheme $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}_L$ and a closed immersion $\mathcal{Z} \hookrightarrow \mathcal{X}$ such that the base change of it to $\operatorname{Spec} L$ is isomorphic to the closed immersion $Z \hookrightarrow X$ by Nagata's compactification theorem (see [10, Chapter II, Theorem F.1.1] for example). As in Remark 4.10, we may assume that $\mathcal{Z} \hookrightarrow \mathcal{X}$ is of finite presentation. Let

$$\overline{\mathcal{X}} := \mathcal{X} \times_{\operatorname{Spec}\mathcal{O}_L} \operatorname{Spec}\mathcal{O} \quad \text{and} \quad \overline{\mathcal{Z}} := \mathcal{Z} \times_{\operatorname{Spec}\mathcal{O}_L} \operatorname{Spec}\mathcal{O}$$

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denote the fibre products. We have $d(\widehat{\overline{\mathcal{Z}}}) \cong d(\widehat{\mathcal{Z}})_K$ and $d(\widehat{\overline{\mathcal{X}}}) \cong d(\widehat{\mathcal{X}})_K$. For an element $\epsilon \in |L^\times|$, we have $T(\mathcal{Z},\epsilon)_K = T(\overline{\mathcal{Z}},\epsilon)$ in $d(\widehat{\overline{\mathcal{X}}})$. By [14, Proposition 1.9.6], we have $d(\widehat{\mathcal{Z}}) = Z^{\mathrm{ad}}$ and $d(\widehat{\mathcal{X}}) = X^{\mathrm{ad}}$. Therefore, the assertion follows from Theorem 4.9. □

5. Étale cohomology with proper support of adic spaces and nearby cycles

In this section, we study the relation between the compatibility of the sliced nearby cycles functors with base change and the bijectivity of specialization maps on stalks of higher direct image sheaves with proper support for adic spaces by using a comparison theorem of Huber [14, Theorem 5.7.8].

5.1. Analytic adic spaces associated with formal schemes

In this subsection, we recall the functor $d(-)$ from a certain category of formal schemes to the category of analytic adic spaces defined in [14, Section 1.9].

Following [14], for a commutative ring A and an element $s \in A$, let

$$A(s/s)$$

denote the localization $A[1/s]$ equipped with the structure of a Tate ring such that the image A_0 of the map $A \to A[1/s]$ is a ring of definition and sA_0 is an ideal of definition.

We record the following well known results.

Lemma 5.1. *Let A be a commutative ring endowed with the ϖ-adic topology for an element $\varpi \in A$ satisfying the following two properties:*

(i) *A is ϖ-adically complete, that is, the following natural map is an isomorphism:*

$$A \to \widehat{A} := \varprojlim_n A/\varpi^n A.$$

(ii) *Let $A\langle X_1,\ldots,X_n\rangle$ be the ϖ-adic completion of $A[X_1,\ldots,X_n]$, called the restricted formal power series ring. Then $A\langle X_1,\ldots,X_n\rangle[1/\varpi]$ is Noetherian for every $n \geq 0$.*

Then the following assertions hold:

(1) *For every ideal $I \subset A\langle X_1,\ldots,X_n\rangle$, the quotient $A\langle X_1,\ldots,X_n\rangle/I$ is ϖ-adically complete.*

(2) *Let B be an A-algebra such that the ϖ-adic completion $\widehat{B}$ of B is isomorphic to $A\langle X_1,\ldots,X_n\rangle$. Let $I \subset B$ be an ideal. Then, the ϖ-adic completion $\widehat{B/I}$ of B/I is isomorphic to $\widehat{B}/I\widehat{B}$.*

(3) *The Tate ring $A(\varpi/\varpi)$ is complete, and we have for every $n \geq 0$*

$$A\langle X_1,\ldots,X_n\rangle[1/\varpi] \cong A(\varpi/\varpi)\langle X_1,\ldots,X_n\rangle.$$

Here, $A(\varpi/\varpi)\langle Y_1,\ldots,Y_m\rangle$ is the ring defined in [14, Section 1.1] for the Tate ring $A(\varpi/\varpi)$. In particular, the Tate ring $A(\varpi/\varpi)$ is strongly Noetherian.

Proof. See [10, Chapter 0, Proposition 8.4.4] for (1). The rest of the proposition is an immediate consequence of (1). We will sketch the proof for the reader's convenience.

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(2) By (1), the ring $\widehat{B}/I\widehat{B}$ is ϖ-adically complete. Hence, we have

$$\widehat{B/I} = \varprojlim_n (B/I)/\varpi^n \cong \varprojlim_n (\widehat{B}/I\widehat{B})/\varpi^n \cong \widehat{B}/I\widehat{B}.$$

(3) Let A_0 be the image of the map $A \to A[1/\varpi]$. By (1), the ring A_0 is ϖ-adically complete, and, hence, $A(\varpi/\varpi)$ is complete. It is clear from the definitions that

$$A_0\langle Y_1,\dots,Y_m\rangle[1/\varpi] \cong A(\varpi/\varpi)\langle Y_1,\dots,Y_m\rangle.$$

Let N be the kernel of the surjection $B := A[X_1,\dots,X_n] \to A_0[X_1,\dots,X_n]$. By using (2), we have the following exact sequence:

$$N \otimes_B \widehat{B} \to \widehat{B} \to A_0\langle Y_1,\dots,Y_m\rangle \to 0.$$

Since $N[1/\varpi] = 0$, we have $(N \otimes_B \widehat{B})[1/\varpi] = 0$, and, hence,

$$\widehat{B}[1/\varpi] \cong A_0\langle Y_1,\dots,Y_m\rangle[1/\varpi].$$

This completes the proof of (3). □

Let $\mathcal{C}$ be the category whose objects are formal schemes which are locally isomorphic to $\operatorname{Spf} A$ for an adic ring A with an ideal of definition ϖA such that the pair (A,ϖ) satisfies the conditions in Lemma 5.1. The morphisms in $\mathcal{C}$ are the adic morphisms. A formal scheme in $\mathcal{C}$ satisfies the condition (S) in [14, Section 1.9] by Lemma 5.1 (3). In [14, Proposition 1.9.1], Huber defined a functor

$$d(-)$$

from $\mathcal{C}$ to the category of analytic adic spaces. For a formal scheme $\mathscr{X}$ in $\mathcal{C}$, the adic space $d(\mathscr{X})$ is equipped with a morphism of ringed spaces

$$\lambda\colon d(\mathscr{X}) \to \mathscr{X}.$$

This map is called a specialization map. If A and $\varpi \in A$ satisfy the conditions in Lemma 5.1, then we have

$$d(\operatorname{Spf} A) = \operatorname{Spa}(A(\varpi/\varpi), A^+),$$

where A^+ is the integral closure of A in $A(\varpi/\varpi) = A[1/\varpi]$. The map $\lambda\colon d(\operatorname{Spf} A) \to \operatorname{Spf} A$ sends $x \in d(\operatorname{Spf} A)$ to the prime ideal $\{a \in A \,|\, |a(x)| < 1\} \subset A$. If $f\colon \mathscr{X} \to \mathscr{Y}$ is an adic morphism of formal schemes in $\mathcal{C}$, then the induced morphism $d(f)\colon d(\mathscr{X}) \to d(\mathscr{Y})$ fits into the following commutative diagram:

$$\begin{array}{ccc} d(\mathscr{X}) & \xrightarrow{\lambda} & \mathscr{X} \\ \downarrow{\scriptstyle d(f)} & & \downarrow{\scriptstyle f} \\ d(\mathscr{Y}) & \xrightarrow{\lambda} & \mathscr{Y}. \end{array}$$

For the sake of completeness, we include a proof of the following result on the compatibility of the functor $d(-)$ with fibre products.

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Proposition 5.2. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism locally of finite type of formal schemes in $\mathcal{C}$. Let $\mathscr{Z} \to \mathscr{Y}$ be an adic morphism of formal schemes in $\mathcal{C}$. Then the morphism*

$$d(\mathscr{X} \times_{\mathscr{Y}} \mathscr{Z}) \to d(\mathscr{X}) \times_{d(\mathscr{Y})} d(\mathscr{Z})$$

induced by the universal property of the fibre product is an isomorphism.

Proof. First, we note that the fibre product $d(\mathscr{X}) \times_{d(\mathscr{Y})} d(\mathscr{Z})$ exists by [14, Proposition 1.2.2] since $d(f)\colon d(\mathscr{X}) \to d(\mathscr{Y})$ is locally of finite type.

We may assume that $\mathscr{X} = \operatorname{Spf} A$, $\mathscr{Y} = \operatorname{Spf} B$ and $\mathscr{Z} = \operatorname{Spf} C$ are affine, where B and C satisfy the conditions in Lemma 5.1 for some element $\varpi \in B$ and for its image in C, respectively. We may assume further that A is of the form $B\langle X_1,\dots,X_n\rangle/I$. We write $D := A \otimes_B C$. The source of the morphism in question is isomorphic to

$$\operatorname{Spa}((\widehat{D})(\varpi/\varpi), E^+),$$

where $\widehat{D}$ is the ϖ-adic completion of D and E^+ is the integral closure of $\widehat{D}$ in $(\widehat{D})[1/\varpi]$. On the other hand, the target of the morphism in question is isomorphic to

$$\operatorname{Spa}(D(\varpi/\varpi), F^+),$$

where F^+ is the integral closure of D in $D[1/\varpi]$. Let D_0 be the image of the map $D \to D[1/\varpi]$. Clearly, the completion of $D(\varpi/\varpi)$ is isomorphic to $(\widehat{D}_0)(\varpi/\varpi)$. By a similar argument as in the proof of Lemma 5.1 (3), we have $(\widehat{D})(\varpi/\varpi) \cong (\widehat{D}_0)(\varpi/\varpi)$. This completes the proof of the proposition since the adic spaces associated with an affinoid ring and its completion are naturally isomorphic (see [13, Lemma 1.5]). □

A valuation ring R is called a *microbial valuation ring* if the field of fractions L of R admits a topologically nilpotent unit ϖ with respect to the valuation topology (see [14, Definition 1.1.4]). We equip R with the valuation topology unless explicitly mentioned otherwise. In this case, the element ϖ is contained in R, the ideal ϖR is an ideal of definition of R and we have $L = R[1/\varpi]$. The completion $\widehat{R}$ of R is also a microbial valuation ring.

Let R be a complete microbial valuation ring. It is well known that

$$R\langle X_1,\dots,X_n\rangle[1/\varpi] \cong L\langle X_1,\dots,X_n\rangle$$

is Noetherian for every $n \geq 0$ (see [3, 5.2.6, Theorem 1]). A formal scheme $\mathscr{X}$ locally of finite type over $\operatorname{Spf} R$ is in the category $\mathcal{C}$.

5.2. Étale cohomology with proper support of adic spaces and nearby cycles

We shall recall a comparison theorem of Huber. To formulate his result, we need some preparations.

Let R be a microbial valuation ring with field of fractions L. We assume that R is a strictly Henselian local ring. Let ϖ be a topologically nilpotent unit in L. Let $\overline{L}$ be a separable closure of L, and let $\overline{R}$ be the valuation ring of $\overline{L}$ which extends R.

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We will use the following notation. Let $\eta \in \operatorname{Spec} R$ and $\overline{\eta} \in \operatorname{Spec} \overline{R}$ be the generic points. For a scheme $\mathcal{X}$ over R, we write

$$\mathcal{X}_\eta := \mathcal{X} \times_{\operatorname{Spec} R} \eta \quad \text{and} \quad \mathcal{X}_{\overline{\eta}} := \mathcal{X} \times_{\operatorname{Spec} R} \overline{\eta}.$$

The ϖ-adic formal completion of a scheme (or a ring) $\mathcal{X}$ over R is denoted by $\widehat{\mathcal{X}}$. Let $s \in \operatorname{Spec} R$ be the closed point and $\mathcal{X}_s$ the special fibre of $\mathcal{X}$.

We write

$$S := \operatorname{Spa}(L,R) = d(\operatorname{Spf} \widehat{R}) \quad \text{and} \quad \overline{S} := \operatorname{Spa}(\overline{L},\overline{R}) = d(\operatorname{Spf} \widehat{\overline{R}}).$$

Let $t \in S$ and $\bar{t} \in \overline{S}$ be the closed points corresponding to the valuation rings R and $\overline{R}$, respectively. The pseudo-adic space $(\overline{S},\{\bar{t}\})$ is also denoted by $\bar{t}$. The natural morphism $\xi\colon \bar{t} \to S$ is a geometric point with support $t \in S$ in the sense of [14, Definition 2.5.1].

Let $f\colon \mathcal{X} \to \operatorname{Spec} R$ be a separated morphism of finite type of schemes. The induced morphism

$$d(f)\colon d(\widehat{\mathcal{X}}) \to S$$

is separated and of finite type; the separatedness can be checked, for example, by using Proposition 5.2 (we often write $d(f)$ instead of $d(\widehat{f})$). There is a natural morphism $d(\widehat{\mathcal{X}}) \to \mathcal{X}_\eta$ of locally ringed spaces (see [14, (1.9.4)]). An étale morphism $Y \to \mathcal{X}_\eta$ defines an adic space $d(\widehat{\mathcal{X}}) \times_{\mathcal{X}_\eta} Y$, which is étale over $d(\widehat{\mathcal{X}})$ (see [13, Proposition 3.8] and [14, Corollary 1.7.3 i)]). In this way, we get a morphism of étale sites

$$a\colon d(\widehat{\mathcal{X}})_{\text{ét}} \to (\mathcal{X}_\eta)_{\text{ét}}.$$

Let Λ be a torsion commutative ring. Let $\mathcal{F}$ be an étale sheaf of Λ-modules on $\mathcal{X}$. Let $\mathcal{F}^a$ denote the pullback of $\mathcal{F}$ by the composition

$$d(\widehat{\mathcal{X}})_{\text{ét}} \xrightarrow{a} (\mathcal{X}_\eta)_{\text{ét}} \to \mathcal{X}_{\text{ét}}.$$

Recall that we have the direct image functor with proper support

$$Rd(f)_!\colon D^+(d(\widehat{\mathcal{X}}),\Lambda) \to D^+(S,\Lambda)$$

for $d(f)$ by [14, Definition 5.4.4]. We define $R^n d(f)_! \mathcal{F}^a := H^n(Rd(f)_! \mathcal{F}^a)$. We will describe the stalk

$$(R^n d(f)_! \mathcal{F}^a)_{\bar{t}} := \Gamma(\bar{t},\xi^* R^n d(f)_! \mathcal{F}^a)$$

at the geometric point $\xi\colon \bar{t} \to S$ in terms of the sliced nearby cycles functor relative to f. We defined the sliced nearby cycles functor

$$R\Psi_{f,\overline{\eta}} := i^* Rj_* j^*\colon D^+(\mathcal{X},\Lambda) \to D^+(\mathcal{X}_s,\Lambda)$$

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in Section 2. Here, we fix the notation by the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{X}_{\overline{\eta}} & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & \mathcal{X}_s \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\eta} & \longrightarrow & \operatorname{Spec} R & \longleftarrow & s. \end{array}$$

Now we can state the following result due to Huber:

Theorem 5.3 (Huber [14, Theorem 5.7.8]). *There is an isomorphism*

$$(R^n d(f)_! \mathcal{F}^a)_{\overline{t}} \cong H^n_c(\mathcal{X}_s, R\Psi_{f,\overline{\eta}}(\mathcal{F}))$$

for every n. This isomorphism is compatible with the natural actions of $\mathrm{Gal}(\overline{L}/L)$ *on both sides.*

Proof. See [14, Theorem 5.7.8]. The construction of the isomorphism shows that it is $\mathrm{Gal}(\overline{L}/L)$-equivariant. □

5.3. Specialization maps on the stalks of $Rd(f)_!$

In this subsection, we work over a complete non-archimedean field K with ring of integers $\mathcal{O}$ for simplicity. We fix a topologically nilpotent unit ϖ in K.

For an adic space X over $\mathrm{Spa}(K,\mathcal{O})$, we will use the following notation. For a point $x \in X$, let $k(x)$ be the residue field of the local ring $\mathcal{O}_{X,x}$ and $k(x)^+$ the valuation ring corresponding to the valuation v_x. We note that $k(x)^+$ is a microbial valuation ring and the image of ϖ, also denoted by ϖ, is a topologically nilpotent unit in $k(x)$. For a geometric point ξ of X, let $\mathrm{Supp}(\xi) \in X$ denote the support of it.

We recall strict localizations of analytic adic spaces. Let $\xi\colon s \to X$ be a geometric point. The strict localization

$$X(\xi)$$

of X at ξ is defined in [14, Section 2.5.11]. It is an adic space over X with an X-morphism $s \to X(\xi)$. We write $x := \mathrm{Supp}(\xi)$. By [14, Proposition 2.5.13], the strict localization $X(\xi)$ is isomorphic to

$$\mathrm{Spa}(\overline{k}(x), \overline{k}(x)^+)$$

over X, where $\overline{k}(x)$ is a separable closure of $k(x)$ and $\overline{k}(x)^+$ is a valuation ring extending $k(x)^+$. A specialization morphism $\xi_1 \to \xi_2$ of geometric points of X is by definition a morphism $X(\xi_1) \to X(\xi_2)$ over X, and such a morphism exists if and only if we have

$$\mathrm{Supp}(\xi_2) \in \overline{\{\mathrm{Supp}(\xi_1)\}}.$$

Let $\mathcal{F}$ be an abelian étale sheaf on X. A specialization morphism $\xi_1 \to \xi_2$ of geometric points of X induces a mapping

$$\mathcal{F}_{\xi_2} \to \mathcal{F}_{\xi_1}$$

on the stalks in the usual way (see [14, (2.5.16)]).

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Definition 5.4. Let X be an adic space over $\mathrm{Spa}(K,\mathcal{O})$ (or, more generally, an analytic adic space). Let $\mathcal{F}$ be an abelian étale sheaf on X. For a subset $W \subset X$, we say that $\mathcal{F}$ is *overconvergent* on W if for every specialization morphism $\xi_1 \to \xi_2$ of geometric points of X whose supports are contained in W, the induced map $\mathcal{F}_{\xi_2} \to \mathcal{F}_{\xi_1}$ is bijective.

Let $\mathcal{X}$ be a scheme of finite type over $\mathrm{Spec}\,\mathcal{O}$. We write $\mathcal{X}_K := \mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,K$. For an étale sheaf $\mathcal{F}$ on $\mathcal{X}$, let $\mathcal{F}^a$ denote the pullback of $\mathcal{F}$ by the composition

$$d(\widehat{\mathcal{X}})_{\text{ét}} \xrightarrow{a} (\mathcal{X}_K)_{\text{ét}} \to \mathcal{X}_{\text{ét}}.$$

Let Λ be a torsion commutative ring.

Proposition 5.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be separated schemes of finite type over $\mathcal{O}$. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism over $\mathcal{O}$. Let $\mathcal{F}$ be an étale sheaf of Λ-modules on $\mathcal{X}$. We assume that the sliced nearby cycles complexes for f and $\mathcal{F}$ are compatible with any base change (see Definition 2.3 (1)). Let $s \in \widehat{\mathcal{Y}}$ be a point. We consider the inverse image $\lambda^{-1}(s)$ under the specialization map*

$$\lambda\colon d(\widehat{\mathcal{Y}}) \to \widehat{\mathcal{Y}}.$$

Then, the sheaf $R^n d(f)_! \mathcal{F}^a$ is overconvergent on $\lambda^{-1}(s)$ for every n.

Proof. Let $\xi_1 \to \xi_2$ be a specialization morphism of geometric points of $d(\widehat{\mathcal{Y}})$ whose supports are contained in $\lambda^{-1}(s)$. We write $y_m := \mathrm{Supp}(\xi_m)$ $(m = 1,2)$. Let $\overline{k}(y_m)$ be a separable closure of $k(y_m)$, and let $\overline{k}(y_m)^+$ be a valuation ring extending $k(y_m)^+$. We identify $d(\widehat{\mathcal{Y}})(\xi_m)$ with $\mathrm{Spa}(\overline{k}(y_m),\overline{k}(y_m)^+)$. Let R_m be the completion of $\overline{k}(y_m)^+$, and we put $U_m := \mathrm{Spec}\,R_m$. The morphism $\mathrm{Spa}(\overline{k}(y_m),\overline{k}(y_m)^+) \to d(\widehat{\mathcal{Y}})$ induces a natural morphism

$$q_m\colon U_m \to \mathcal{Y}$$

over $\mathrm{Spec}\,\mathcal{O}$, and the specialization morphism $d(\widehat{\mathcal{Y}})(\xi_1) \to d(\widehat{\mathcal{Y}})(\xi_2)$ induces a natural $\mathcal{Y}$-morphism

$$r\colon U_1 \to U_2.$$

By the assumption, we have $q_m(s_m) = s$ for the closed point $s_m \in U_m$, where the image of $s \in \widehat{\mathcal{Y}}$ in $\mathcal{Y}$ is denoted by the same letter. Let $\overline{s} \to \mathcal{Y}$ be an algebraic geometric point lying above s, and let U be the strict localization of $\mathcal{Y}$ at $\overline{s}$. There are local $\mathcal{Y}$-morphisms $\widetilde{q}_m\colon U_m \to U$ $(m = 1,2)$ such that the following diagram commutes:

$$\begin{array}{ccc} U_1 & \xrightarrow{\;r\;} & U_2 \\ & \underset{\widetilde{q}_1}{\searrow} \quad \underset{\widetilde{q}_2}{\swarrow} & \\ & U. & \end{array}$$

We remark that r is not a local morphism if $y_1 \neq y_2$. Let η_m be the generic point of U_m. Then we have $r(\eta_1) = \eta_2$. We write $\eta := \widetilde{q}_1(\eta_1) = \widetilde{q}_2(\eta_2)$. Let $\overline{\eta} \to U$ denote the algebraic geometric point which is the image of η_2.

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Let $\mathcal{F}_m$ be the pullback of $\mathcal{F}$ to $\mathcal{X}\times_{\mathcal{Y}}U_m$ and $\mathcal{F}_U$ the pullback of $\mathcal{F}$ to $\mathcal{X}\times_{\mathcal{Y}}U$. For $m=1,2$, we have

$$(R^n d(f)_!\mathcal{F}^a)_{\xi_m}\cong H^n_c(\mathcal{X}\times_{\mathcal{Y}}s_m, R\Psi_{f_{U_m},\eta_m}(\mathcal{F}_m))$$

by Theorem 5.3. By our assumption that the sliced nearby cycles complexes for f and $\mathcal{F}$ are compatible with any base change, the map

$$H^n_c(\mathcal{X}\times_{\mathcal{Y}}\overline{s}, R\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U))\to H^n_c(\mathcal{X}\times_{\mathcal{Y}}s_m, R\Psi_{f_{U_m},\eta_m}(\mathcal{F}_m))$$

is bijective for both $m=1$ and $m=2$. Hence, the proposition follows from the following commutative diagram:

$$\begin{array}{ccc}
(R^n d(f)_!\mathcal{F}^a)_{\xi_2} & \longrightarrow & (R^n d(f)_!\mathcal{F}^a)_{\xi_1} \\
\downarrow \cong & & \downarrow \cong \\
H^n_c(\mathcal{X}\times_{\mathcal{Y}}s_2, R\Psi_{f_{U_2},\eta_2}(\mathcal{F}_2)) & & H^n_c(\mathcal{X}\times_{\mathcal{Y}}s_1, R\Psi_{f_{U_1},\eta_1}(\mathcal{F}_1)) \\
\uparrow \cong & \nearrow \cong & \\
H^n_c(\mathcal{X}\times_{\mathcal{Y}}\overline{s}, R\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U)). & &
\end{array}$$

The commutativity of the diagram can be verified by tracing the construction of the isomorphism in Theorem 5.3. □

6. Local constancy of higher direct images with proper support

In this section, we study local constancy of higher direct images with proper support for generically smooth morphisms of adic spaces whose target is one dimensional. We will formulate and prove the results not only for constant sheaves but also for nonconstant sheaves satisfying certain conditions related to the sliced nearby cycles functors.

Throughout this section, we fix an algebraically closed complete non-archimedean field K with ring of integers $\mathcal{O}$.

6.1. Tame sheaves on annuli

In this subsection, we recall two theorems on finite étale coverings on annuli and the punctured disc, which basically follow from the results in [22, 23, 27]. We do not impose any conditions on the characteristic of K. Since we can not directly apply some results there and some results are only stated in the case where the base field is of characteristic zero, we give proofs of the theorems in Appendix A.

To state the two theorems, we need some preparations. Recall that we defined $\mathbb{B}(1)=\mathrm{Spa}(K\langle T\rangle,\mathcal{O}\langle T\rangle)$. Let

$$\mathbb{B}(1)^*:=\mathbb{B}(1)\backslash\{0\}$$

be the *punctured disc*, where $0\in\mathbb{B}(1)$ is the K-rational point corresponding to $0\in K$. It is an adic space locally of finite type over $\mathrm{Spa}(K,\mathcal{O})$. We fix a valuation $|\cdot|\colon K\to\mathbb{R}_{\geq 0}$ of rank 1 such that the topology of K is induced by it. For elements $a,b\in|K^\times|$ with

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$a \leq b \leq 1$, we define

$$\begin{aligned}\mathbb{B}(a,b) &:= \{x \in \mathbb{B}(1) \,|\, a \leq |T(x)| \leq b\} \\ &:= \{x \in \mathbb{B}(1) \,|\, |\varpi_a(x)| \leq |T(x)| \leq |\varpi_b(x)|\},\end{aligned}$$

which is called an *open annulus*. Here, $\varpi_a, \varpi_b \in K^\times$ are elements such that $a = |\varpi_a|$ and $b = |\varpi_b|$. It is a rational subset of $\mathbb{B}(1)$, and, hence, it is an affinoid open subspace of $\mathbb{B}(1)$.

Let m be a positive integer invertible in K. The finite étale morphism $\varphi_m \colon \mathbb{B}(1)^* \to \mathbb{B}(1)^*$ defined by $T \mapsto T^m$ is called a *Kummer covering* of degree m. For elements $a,b \in |K^\times|$ with $a \leq b \leq 1$, the restriction

$$\varphi_m \colon \mathbb{B}(a^{1/m}, b^{1/m}) \to \mathbb{B}(a,b)$$

of φ_m is also called a Kummer covering of degree m (we also call a morphism of affinoid adic spaces of finite type over $\mathrm{Spa}(K,\mathcal{O})$ a Kummer covering if it is isomorphic to $\varphi_m \colon \mathbb{B}(a^{1/m}, b^{1/m}) \to \mathbb{B}(a,b)$ for $a,b \in |K^\times|$ and some m with $a \leq b \leq 1$).

In this paper, we use the following notion of tameness for étale sheaves on one-dimensional smooth adic spaces over $\mathrm{Spa}(K,\mathcal{O})$.

Definition 6.1. Let X be a one-dimensional smooth adic space over $\mathrm{Spa}(K,\mathcal{O})$. Let $x \in X$ be a point which has a proper generalization in X, that is, there exists a point $x' \in X$ with $x \in \overline{\{x'\}}$ and $x \neq x'$. Let

$$k(x)^{\wedge h+}$$

be the Henselization of the completion of the valuation ring $k(x)^+$ of x. Let $L(x)$ be a separable closure of the field of fractions $k(x)^{\wedge h}$ of $k(x)^{\wedge h+}$. It induces a geometric point $\overline{x} \to X$ with support x. For an étale sheaf $\mathcal{F}$ on X, we say that $\mathcal{F}$ is *tame* at $x \in X$ if the action of

$$\mathrm{Gal}(L(x)/k(x)^{\wedge h})$$

on the stalk $\mathcal{F}_{\overline{x}}$ at the geometric point $\overline{x}$ factors through a finite group G such that $\sharp G$ is invertible in $\mathcal{O}$, where $\sharp G$ denotes the cardinality of G.

Now we can formulate the results.

Theorem 6.2 ([22, Theorem 2.2], [23, Theorem 4.11], [27, Theorem 2.4.3]). *Let $f \colon X \to \mathbb{B}(1)^*$ be a finite étale morphism of adic spaces. There exists an element $\epsilon \in |K^\times|$ with $\epsilon \leq 1$ such that, for all $a,b \in |K^\times|$ with $a < b \leq \epsilon$, we have*

$$f^{-1}(\mathbb{B}(a,b)) \cong \coprod_{i=1}^{n} \mathbb{B}(c_i, d_i)$$

for some elements $c_i, d_i \in |K^\times|$ with $c_i < d_i \leq 1$ $(1 \leq i \leq n)$. If K is of characteristic zero, after replacing $\epsilon \in |K^\times|$ by a smaller one, the restriction

$$\mathbb{B}(c_i, d_i) \to \mathbb{B}(a,b)$$

of f to every component $\mathbb{B}(c_i, d_i)$ appearing in the above decomposition becomes a Kummer covering.

Theorem 6.3. *Let $a,b \in |K^\times|$ be elements with $a < b \le 1$. Let $\mathcal{F}$ be a locally constant étale sheaf with finite stalks on $\mathbb{B}(a,b)$. We assume that the sheaf $\mathcal{F}$ is tame at every $x \in \mathbb{B}(a,b)$ having a proper generalization in $\mathbb{B}(a,b)$, in the sense of Definition 6.1. Let $t \in |K^\times|$ be an element with $a/b < t^2 < 1$. Then the restriction $\mathcal{F}|_{\mathbb{B}(a/t,tb)}$ of $\mathcal{F}$ to $\mathbb{B}(a/t,tb)$ is trivialized by a Kummer covering φ_m of degree m, that is, the pullback*

$$\varphi_m^*(\mathcal{F}|_{\mathbb{B}(a/t,tb)})$$

is a constant sheaf. Moreover, we can assume that the degree m is invertible in $\mathcal{O}$.

We prove Theorem 6.2 and Theorem 6.3 in Appendix A.

Remark 6.4. If K is of characteristic zero, then Theorem 6.2 is known as the p-adic Riemann existence theorem of Lütkebohmert [22].

6.2. Local constancy of $R^i d(f)_!$ for generic smooth morphisms

As in Section 5, we use the following notation. Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{O}$. We write $\mathcal{X}_K := \mathcal{X} \times_{\operatorname{Spec}\mathcal{O}} \operatorname{Spec} K$. For an étale sheaf $\mathcal{F}$ on $\mathcal{X}$, we denote by $\mathcal{F}^a$ the pullback of $\mathcal{F}$ by the composition $d(\widehat{\mathcal{X}})_{\text{ét}} \xrightarrow{a} (\mathcal{X}_K)_{\text{ét}} \to \mathcal{X}_{\text{ét}}$ (see Section 5.2 for the morphism $a\colon d(\widehat{\mathcal{X}})_{\text{ét}} \to (\mathcal{X}_K)_{\text{ét}}$).

Let us introduce the following slightly technical definition.

Definition 6.5. We consider the following diagram:

$$\begin{array}{ccc} & & \mathcal{Z} \\ & & \downarrow{\scriptstyle \pi} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}, \end{array}$$

where

- $\mathcal{Y} = \operatorname{Spec} A$ is an integral affine scheme of finite type over $\mathcal{O}$ such that $\mathcal{Y}_K$ is one dimensional and smooth over K,
- $f\colon \mathcal{X} \to \mathcal{Y}$ is a separated morphism of finite type and
- $\pi\colon \mathcal{Z} \to \mathcal{Y}$ is a proper surjective morphism such that $\mathcal{Z}$ is an integral scheme whose generic fibre $\mathcal{Z}_K$ is smooth over K, and the base change $\pi_K\colon \mathcal{Z}_K \to \mathcal{Y}_K$ is a finite morphism.

Let n be a positive integer invertible in $\mathcal{O}$ and $\mathcal{F}$ a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$. We say that $\mathcal{F}$ is *adapted* to the pair (f,π) if the following conditions are satisfied:

(1) The étale sheaf $\mathcal{F}^a$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $d(\widehat{\mathcal{X}})$ is constructible in the sense of [14, Definition 2.7.2].

(2) The sliced nearby cycles complexes for $f_{\mathcal{Z}}$ and $\mathcal{F}_{\mathcal{Z}}$ are compatible with any base change.

(3) The sliced nearby cycles complexes for $f_{\mathcal{Z}}$ and $\mathcal{F}_{\mathcal{Z}}$ are unipotent.

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See Definition 2.3 for the terminology used in the conditions (2) and (3). Here, we retain the notation of Section 2. For example, $f_{\mathcal{Z}}$ denotes the base change $f_{\mathcal{Z}}\colon \mathcal{X}\times_{\mathcal{Y}}\mathcal{Z}\to\mathcal{Z}$ of f and $\mathcal{F}_{\mathcal{Z}}$ denotes the pullback of the sheaf $\mathcal{F}$ to $\mathcal{X}\times_{\mathcal{Y}}\mathcal{Z}$.

We need the following proposition, which is a consequence of Corollary 2.9.

Proposition 6.6. *Let $\mathcal{Y}=\operatorname{Spec}A$ be an integral affine scheme of finite type over $\mathcal{O}$ such that $\mathcal{Y}_K$ is one dimensional and smooth over K. Let $f\colon\mathcal{X}\to\mathcal{Y}$ be a separated morphism of finite presentation. Then, there exists a proper surjective morphism $\pi\colon\mathcal{Z}\to\mathcal{Y}$ as in Definition 6.5 such that, for every positive integer n invertible in $\mathcal{O}$, the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ on $\mathcal{X}$ is adapted to (f,π).*

Proof. The assertion follows from Corollary 2.9 and a spreading out argument. □

By using the results in Section 5, we prove the following proposition:

Proposition 6.7. *Let $f\colon\mathcal{X}\to\mathcal{Y}$ and $\pi\colon\mathcal{Z}\to\mathcal{Y}$ be morphisms as in Definition 6.5. We have the following diagram:*

$$\begin{array}{ccc} & & d(\widehat{\mathcal{Z}}) \\ & & \big\downarrow{\scriptstyle d(\pi)} \\ d(\widehat{\mathcal{X}}) & \xrightarrow{d(f)} & d(\widehat{\mathcal{Y}}). \end{array}$$

Then the following assertions hold:

(1) *Let $\mathcal{F}$ be an étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to (f,π) with $n\in\mathcal{O}^{\times}$. For every i, the sheaf*

$$d(\pi)^*R^i d(f)_!\mathcal{F}^a$$

on $d(\widehat{\mathcal{Z}})$ is tame at every $z\in d(\widehat{\mathcal{Z}})$ having a proper generalization in $d(\widehat{\mathcal{Z}})$, in the sense of Definition 6.1.

(2) *Let $y\in d(\widehat{\mathcal{Y}})$ be a K-rational point. There exists an open subset $V\subset d(\widehat{\mathcal{Y}})$ containing y such that, for every étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to (f,π) with $n\in\mathcal{O}^{\times}$, the sheaf $d(\pi)^*R^i d(f)_!\mathcal{F}^a$ is overconvergent on $d(\pi)^{-1}(V)$ for every i.*

Proof. For an étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ with $n\in\mathcal{O}^{\times}$, the pullback of $\mathcal{F}^a$ by $d(\widehat{\mathcal{X}_{\mathcal{Z}}})\to d(\widehat{\mathcal{X}})$ is isomorphic to $(\mathcal{F}_{\mathcal{Z}})^a$, and, hence, by using the base change theorem [14, Theorem 5.4.6] for $Rd(f)_!$, we have

$$d(\pi)^*R^i d(f)_!\mathcal{F}^a\cong R^i d(f_{\mathcal{Z}})_!(\mathcal{F}_{\mathcal{Z}})^a.$$

(1) This is an immediate consequence of Theorem 5.3. Indeed, let $z\in d(\widehat{\mathcal{Z}})$ be an element having a proper generalization in $d(\widehat{\mathcal{Z}})$. Let

$$R:=k(z)^{\wedge h+}$$

be the Henselization of the completion of the valuation ring $k(z)^+$ of z. By [17, Corollary 5.4], the residue field of R is algebraically closed. We write $U:=\operatorname{Spec}R$. Let L be the field

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of fractions of R. The composite

$$\operatorname{Spa}(L,R) \to \operatorname{Spa}(k(z),k(z)^+) \to d(\widehat{\mathcal{Z}})$$

is induced by a natural morphism $q\colon U \to \mathcal{Z}$ of schemes over $\mathcal{O}$. Let $\overline{L}$ be a separable closure of L, which induces a geometric point $\overline{t} \to \operatorname{Spa}(L,R)$ and a geometric point $\overline{z} \to d(\widehat{\mathcal{Z}})$ in the usual way.

Let $\mathcal{F}$ be an étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to (f,π) with $n \in \mathcal{O}^\times$. By applying Theorem 5.3 to $f_U\colon \mathcal{X}_U \to U$, we have $\operatorname{Gal}(\overline{L}/L)$-equivariant isomorphisms

$$(R^i d(f_{\mathcal{Z}})_!(\mathcal{F}_{\mathcal{Z}})^a)_{\overline{z}} \cong (R^i d(f_U)_!(\mathcal{F}_U)^a)_{\overline{t}} \cong H^i_c((\mathcal{X}_U)_s, R\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U)),$$

where $s \in U$ is the closed point and $\overline{\eta} = \operatorname{Spec}\overline{L} \to U$ is the algebraic geometric point. By [14, Corollary 5.4.8 and Proposition 6.2.1 i)], the left-hand side is a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module. Moreover, the action of $\operatorname{Gal}(\overline{L}/L)$ on it factors through a finite group. Since the complex $R\Psi_{f_U,\overline{\eta}}(\mathcal{F}_U)$ is $\operatorname{Gal}(\overline{L}/L)$-unipotent and the integer n is invertible in $\mathcal{O}$, it follows that the action of $\operatorname{Gal}(\overline{L}/L)$ on the right-hand side factors through a finite group G such that $\sharp G$ is invertible in $\mathcal{O}$. This proves (1).

(2) Since $\pi\colon \mathcal{Z} \to \mathcal{Y}$ is proper, by [14, Proposition 1.9.6], we have $d(\widehat{\mathcal{Z}}) \cong d(\widehat{\mathcal{Y}}) \times_{\mathcal{Y}} \mathcal{Z}$, where $d(\widehat{\mathcal{Y}}) \times_{\mathcal{Y}} \mathcal{Z}$ is the adic space over $d(\widehat{\mathcal{Y}})$ associated with $d(\widehat{\mathcal{Y}}) \to \mathcal{Y}_K \to \mathcal{Y}$ and $\pi\colon \mathcal{Z} \to \mathcal{Y}$ (see [13, Proposition 3.8]). Since $\pi_K\colon \mathcal{Z}_K \to \mathcal{Y}_K$ is a finite morphism, it follows that

$$d(\widehat{\mathcal{Z}}) \cong d(\widehat{\mathcal{Y}}) \times_{\mathcal{Y}} \mathcal{Z} \cong d(\widehat{\mathcal{Y}}) \times_{\mathcal{Y}_K} \mathcal{Z}_K$$

is finite over $d(\widehat{\mathcal{Y}})$. The inverse image

$$d(\pi)^{-1}(y) = \{z_1,\dots,z_m\}$$

of $y \in d(\widehat{\mathcal{Y}})$ consists of finitely many K-rational points. Let

$$\lambda\colon d(\widehat{\mathcal{Z}}) \to \widehat{\mathcal{Z}}$$

be the specialization map associated with the formal scheme $\widehat{\mathcal{Z}}$. Since the inverse image $\lambda^{-1}(\lambda(z_j))$ of $\lambda(z_j)$ is a closed constructible subset of $d(\widehat{\mathcal{Z}})$ and z_j is a K-rational point, there exists an open neighbourhood $V_j \subset d(\widehat{\mathcal{Z}})$ of z_j with

$$V_j \subset \lambda^{-1}(\lambda(z_j))$$

for every j (see Lemma 4.3). Since $d(\pi)$ is a finite morphism, there is an open neighbourhood $V \subset d(\widehat{\mathcal{Y}})$ of y with $d(\pi)^{-1}(V) \subset \cup_j V_j$. By using Proposition 5.5, we see that, for every étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to (f,π) with $n \in \mathcal{O}^\times$, the sheaf $R^i d(f_{\mathcal{Z}})_!(\mathcal{F}_{\mathcal{Z}})^a$ is overconvergent on $d(\pi)^{-1}(V)$. □

We need the following finiteness result due to Huber.

Theorem 6.8 (Huber [14, Theorem 6.2.2]). *Let Y be an adic space over* $\operatorname{Spa}(K,\mathcal{O})$. *Let $f\colon X \to Y$ be a morphism of adic spaces which is smooth, separated and quasicompact. Let $\mathcal{F}$ be a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X with $n \in \mathcal{O}^\times$. Then, the sheaf $R^i f_!\mathcal{F}$ on Y is a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules for every i.*

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Proof. See [14, Theorem 6.2.2]. □

Remark 6.9. The assertion of Theorem 6.8 can fail for nonsmooth morphisms. See the introduction of [15] for details (see also [18, Proposition 7.1] for a more general result for smooth, separated and quasicompact morphisms of analytic pseudo-adic spaces).

As in the previous sections, we write $\mathbb{B}(1) = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}\langle T\rangle)$. For an element $\epsilon \in |K^\times|$ with $\epsilon \leq 1$, we define

$$\mathbb{B}(\epsilon) := \{x \in \mathbb{B}(1) \,|\, |T(x)| \leq \epsilon\}$$

and $\mathbb{B}(\epsilon)^* := \mathbb{B}(\epsilon)\backslash\{0\}$. Let X be a one-dimensional adic space of finite type over $\mathrm{Spa}(K,\mathcal{O})$. We define an *open disc* $V \subset X$ as an open subset V of X equipped with an isomorphism

$$\phi\colon \mathbb{B}(1) \cong V$$

over $\mathrm{Spa}(K,\mathcal{O})$. For an open disc $V \subset X$, we write

$$V(\epsilon) := \phi(\mathbb{B}(\epsilon)) \quad \text{and} \quad V(\epsilon)^* := \phi(\mathbb{B}(\epsilon)^*).$$

Similarly, we write

$$V(a,b) := \phi(\mathbb{B}(a,b))$$

for an open annulus $\mathbb{B}(a,b) \subset \mathbb{B}(1)$.

The main result of this section is the following theorem.

Theorem 6.10. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $\pi\colon \mathcal{Z} \to \mathcal{Y}$ be morphisms as in Definition 6.5. We assume that there is an open disc $V \subset d(\widehat{\mathcal{Y}})$ such that*

$$d(f)\colon d(\widehat{\mathcal{X}}) \to d(\widehat{\mathcal{Y}})$$

is smooth over $V(1)^*$. Then there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \leq 1$ such that, for every étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to (f,π) with $n \in \mathcal{O}^\times$, the following two assertions hold:

(1) *The restriction*

$$(R^i d(f)_! \mathcal{F}^a)|_{V(\epsilon_0)^*}$$

of $R^i d(f)_! \mathcal{F}^a$ to $V(\epsilon_0)^*$ is a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules for every i.

(2) *For elements $a,b \in |K^\times|$ with $a < b \leq \epsilon_0$, there exists a composition*

$$h\colon \mathbb{B}(c^{1/m}, d^{1/m}) \xrightarrow{\varphi_m} \mathbb{B}(c,d) \xrightarrow{g} V(a,b)$$

of a Kummer covering φ_m of degree m, where m is invertible in $\mathcal{O}$, with a finite Galois étale morphism g, such that the pullback

$$h^*((R^i d(f)_! \mathcal{F}^a)|_{V(a,b)})$$

is a constant sheaf associated with a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module for every i. If K is of characteristic zero, then we can take g as a Kummer covering.

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Proof. Clearly, the first assertion (1) follows from the second assertion (2). We shall prove (2).

Step 1. We may assume that, for the dominant morphism $\pi\colon \mathcal{Z} \to \mathcal{Y}$, the separable closure $k(\mathcal{Y})^{\mathrm{sep}}$ of the function field $k(\mathcal{Y})$ of $\mathcal{Y}$ in the function field of $\mathcal{Z}$ is Galois over $k(\mathcal{Y})$.

Indeed, there is a finite surjective morphism $Z' \to \mathcal{Z}_K$ from an integral scheme Z' which is smooth over K such that the separable closure of the function field $k(\mathcal{Y})$ of $\mathcal{Y}$ in the function field of Z' is Galois over $k(\mathcal{Y})$. There exists a proper surjective morphism $\mathcal{Z}' \to \mathcal{Z}$ such that $\mathcal{Z}'$ is integral and the base change $\mathcal{Z}'_K \to \mathcal{Z}_K$ is isomorphic to $Z' \to \mathcal{Z}_K$. We define π' as the composition $\pi'\colon \mathcal{Z}' \to \mathcal{Z} \to \mathcal{Y}$. If a sheaf $\mathcal{F}$ is adapted to (f,π), then it is also adapted to (f,π'). Thus, it suffices to prove Theorem 6.10 for (f,π').

Step 2. We will choose an appropriate $\epsilon_0 \in |K^\times|$.

Let $W \to \mathcal{Y}_K$ be the normalization of $\mathcal{Y}_K$ in $k(\mathcal{Y})^{\mathrm{sep}}$. Then the induced morphism $\mathcal{Z}_K \to W$ is finite, radicial and surjective and there is a dense open subset $U \subset \mathcal{Y}_K$ over which $W \to \mathcal{Y}_K$ is a finite Galois étale morphism. Let

$$W' := d(\widehat{\mathcal{Y}}) \times_{\mathcal{Y}_K} W$$

be the adic space over $d(\widehat{\mathcal{Y}})$ associated with $d(\widehat{\mathcal{Y}}) \to \mathcal{Y}_K$ and $W \to \mathcal{Y}_K$. Let $g\colon W' \to d(\widehat{\mathcal{Y}})$ denote the structure morphism. The morphism $d(\pi)$ can be written as the composition of finite morphisms

$$d(\widehat{\mathcal{Z}}) \xrightarrow{\alpha} W' \xrightarrow{g} d(\widehat{\mathcal{Y}}).$$

Let $\epsilon_1 \in |K^\times|$ be an element with $\epsilon_1 \leq 1$, such that $V(\epsilon_1)^* \subset d(\widehat{\mathcal{Y}})$ is mapped into U under the map $d(\widehat{\mathcal{Y}}) \to \mathcal{Y}_K$. Then the restriction

$$g^{-1}(V(\epsilon_1)^*) \to V(\epsilon_1)^*$$

is finite and étale. By Theorem 6.2, there exists an element $\epsilon_2 \in |K^\times|$ with $\epsilon_2 \leq \epsilon_1$ such that, for all $a,b \in |K^\times|$ with $a < b \leq \epsilon_2$, we have

$$g^{-1}(V(a,b)) \cong \coprod_{j=1}^{N} \mathbb{B}(c_j,d_j)$$

for some elements $c_j,d_j \in |K^\times|$ with $c_j < d_j \leq 1$ $(1 \leq j \leq N)$. If K is of characteristic zero, after replacing $\epsilon_2 \in |K^\times|$ by a smaller one, the restriction

$$\mathbb{B}(c_j,d_j) \to V(a,b)$$

of g to every component $\mathbb{B}(c_j,d_j)$ appearing in the above decomposition becomes a Kummer covering. By Proposition 6.7 (2), there exists an element $\epsilon_3 \in |K^\times|$ with $\epsilon_3 \leq \epsilon_2$ such that, for every étale sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to (f,π) with $n \in \mathcal{O}^\times$, the sheaf $d(\pi)^* R^i d(f)_! \mathcal{F}^a$ is overconvergent on $d(\pi)^{-1}(V(\epsilon_3))$ for every i. Let $t \in |K^\times|$ be an element with $t < 1$. Then we put $\epsilon_0 := t\epsilon_3$.

Step 3. We shall show that ϵ_0 satisfies the condition.

Indeed, let $\mathcal{F}$ be an étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathcal{X}$ adapted to (f,π) with $n \in \mathcal{O}^\times$. Let $a,b \in |K^\times|$ be elements with $a < b \leq \epsilon_0$. We have $g^{-1}(V(ta,b/t)) \cong \coprod_{i=1}^{N} \mathbb{B}(c_j,d_j)$ for

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some elements $c_j, d_j \in |K^\times|$ with $c_j < d_j \leq 1$ $(1 \leq j \leq N)$. We take a component $\mathbb{B}(c_1, d_1)$ of the decomposition. The restriction

$$g\colon \mathbb{B}(c_1,d_1) \to V(ta,b/t)$$

of g is denoted by the same letter. By the construction, it is a finite Galois étale morphism.

We remark that, since the morphism $\mathcal{Z}_K \to W$ is finite, radicial and surjective, it follows that $\alpha\colon d(\widehat{\mathcal{Z}}) \to W'$ is a homeomorphism and, for every $z \in d(\widehat{\mathcal{Z}})$, the extension $k(\alpha(z))^\wedge \to k(z)^\wedge$ of the completions of the residue fields is a finite purely inseparable extension, and, hence, the extension $k(\alpha(z))^{\wedge h} \to k(z)^{\wedge h}$ of the Henselizations of these fields is also a finite purely inseparable extension.

Since $d(f)\colon d(\widehat{\mathcal{X}}) \to d(\widehat{\mathcal{Y}})$ is smooth over $V(1)^*$, the sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules

$$\mathcal{G} := g^*((R^i d(f)_! \mathcal{F}^a)|_{V(ta,b/t)})$$

on $\mathbb{B}(c_1,d_1)$ is constructible by Theorem 6.8. By the construction, it is overconvergent on $\mathbb{B}(c_1,d_1)$. Therefore, by [14, Lemma 2.7.11], the sheaf $\mathcal{G}$ is locally constant. Moreover, by Proposition 6.7 (1) and the remark above, the sheaf $\mathcal{G}$ is tame at every $x \in \mathbb{B}(c_1,d_1)$ having a proper generalization in $\mathbb{B}(c_1,d_1)$. We have $g^{-1}(V(a,b)) = \mathbb{B}(c,d)$ for some elements $c,d \in |K^\times|$ with $c_1 < c < d < d_1$. Hence, by Theorem 6.3, we conclude that the restriction of $\mathcal{G}$ to $g^{-1}(V(a,b)) = \mathbb{B}(c,d)$ is trivialized by a Kummer covering $\varphi_m\colon \mathbb{B}(c^{1/m}, d^{1/m}) \to \mathbb{B}(c,d)$ with $m \in \mathcal{O}^\times$.

The proof of Theorem 6.10 is now complete. □

Remark 6.11. In Theorem 6.10, the morphism g can be taken independent of $\mathcal{F}$, although the integer m possibly depends on $\mathcal{F}$.

For an element $\epsilon \in |K^\times|$, we define

$$\mathbb{D}(\epsilon) := \{x \in \mathbb{B}(1) \mid |T(x)| < \epsilon\}.$$

This is a closed constructible subset of $\mathbb{B}(1)$. For later use, we record the following results.

Lemma 6.12. *Let n be a positive integer invertible in $\mathcal{O}$.*

(1) *Let $a,b \in |K^\times|$ be elements with $a < b \leq 1$. Let $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathbb{B}(a,b)$. Assume that there exists a finite étale morphism $h\colon \mathbb{B}(c,d) \to \mathbb{B}(a,b)$ such that h is a composition of finite Galois étale morphisms and the pullback $h^*\mathcal{F}$ is a constant sheaf. Then the following assertions hold:*

   (a) *We have*

$$H^i_c(\mathbb{B}(b)\backslash\mathbb{B}(a), \mathcal{F}|_{\mathbb{B}(b)\backslash\mathbb{B}(a)}) = 0 \quad \text{and} \quad H^i_c(\mathbb{D}(b)\cap\mathbb{B}(a,b), \mathcal{F}|_{\mathbb{D}(b)\cap\mathbb{B}(a,b)}) = 0$$

   for every i.

   (b) *The restriction map*

$$H^i(\mathbb{B}(a,b),\mathcal{F}) \to H^i(\mathbb{D}(b)^\circ \cap \mathbb{B}(a,b), \mathcal{F})$$

   is an isomorphism for every i, where $\mathbb{D}(b)^\circ$ is the interior of $\mathbb{D}(b)$ in $\mathbb{B}(1)$.

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(2) *Let $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathbb{B}(1)^*$. Assume that for all $a,b \in |K^\times|$ with $a < b \leq 1$, there exists a finite étale morphism $h\colon \mathbb{B}(c,d) \to \mathbb{B}(a,b)$ such that h is a composition of finite Galois étale morphisms and the pullback $h^*(\mathcal{F}|_{\mathbb{B}(a,b)})$ is a constant sheaf. Then we have*

$$H^i_c(\mathbb{D}(1)\backslash\{0\}, \mathcal{F}|_{\mathbb{D}(1)\backslash\{0\}}) = 0$$

for every i.

Proof. (1) After possibly changing the coordinate function of $\mathbb{B}(c,d)$, we may assume that $h^{-1}(\mathbb{B}(a,a)) = \mathbb{B}(c,c)$ and $h^{-1}(\mathbb{B}(b,b)) = \mathbb{B}(d,d)$. Then, we have

$$h^{-1}(\mathbb{B}(b)\backslash\mathbb{B}(a)) = \mathbb{B}(d)\backslash\mathbb{B}(c) \quad \text{and} \quad h^{-1}(\mathbb{D}(b)\cap\mathbb{B}(a,b)) = \mathbb{D}(d)\cap\mathbb{B}(c,d).$$

We shall show the first equality of (a). By the spectral sequence in [15, 4.2 ii)] and the fact that h is a composition of finite Galois étale morphisms, it is enough to show that, for a constant sheaf M on $\mathbb{B}(d)\backslash\mathbb{B}(c)$ associated with a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module, we have for every i

$$H^i_c(\mathbb{B}(d)\backslash\mathbb{B}(c), M) = 0. \tag{6.1}$$

This is proved in [15, (II) in the proof of Theorem 2.5] (although the characteristic of the base field is always assumed to be zero in [15], we need not assume that the characteristic of K is zero here).

We shall show the second equality of (a). Similarly as above, it suffices to show that, for a constant sheaf M on $\mathbb{D}(d)\cap\mathbb{B}(c,d)$ associated with a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module, we have $H^i_c(\mathbb{D}(d)\cap\mathbb{B}(c,d), M) = 0$ for every i. But this follows from (6.1) since $\mathbb{D}(d)\cap\mathbb{B}(c,d)$ is isomorphic to $\mathbb{B}(d)\backslash\mathbb{B}(c)$ as a pseudo-adic space over $\mathrm{Spa}(K,\mathcal{O})$.

We shall prove (b). By the Čech-to-cohomology spectral sequences and by the fact that h is a composition of finite Galois étale morphisms, it is enough to show that, for a constant sheaf M on $\mathbb{B}(c,d)$ associated with a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module, the restriction map

$$H^i(\mathbb{B}(c,d), M) \to H^i(\mathbb{D}(d)^\circ\cap\mathbb{B}(c,d), M)$$

is an isomorphism for every i. Let $t \in |K^\times|$ be an element with $t < 1$. Then we have

$$\mathbb{D}(d)^\circ\cap\mathbb{B}(c,d) = \bigcup_{m\in\mathbb{Z}_{>0}} \mathbb{B}(c, t^{1/m}d)$$

(see also [15, Lemma 1.3]). Moreover, $H^i(\mathbb{B}(c,t^{1/m}d), M)$ is a finite group for every i (see [14, Proposition 6.1.1]). Therefore, by [14, Lemma 3.9.2 i)], we obtain that

$$H^i(\mathbb{D}(d)^\circ\cap\mathbb{B}(c,d), M) \cong \varprojlim_m H^i(\mathbb{B}(c,t^{1/m}d), M)$$

for every i. Thus, it suffices to prove that, for every $m \in \mathbb{Z}_{>0}$, the restriction map

$$H^i(\mathbb{B}(c,d), M) \to H^i(\mathbb{B}(c,t^{1/m}d), M)$$

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is an isomorphism for every i. By [14, Example 6.1.2], both sides vanish when $i \geq 2$. For $i \leq 1$, the assertion can be proved by using the Kummer sequence (see the last paragraph of the proof of Theorem 6.3 in Appendix A).

(2) Since $\mathbb{D}(1)\backslash\{0\} = \bigcup_{\epsilon\in|K^\times|,\, \epsilon<1} \mathbb{D}(1)\cap\mathbb{B}(\epsilon,1)$, we have

$$\varinjlim_{\epsilon} H^i_c(\mathbb{D}(1)\cap\mathbb{B}(\epsilon,1), \mathcal{F}|_{\mathbb{D}(1)\cap\mathbb{B}(\epsilon,1)}) \cong H^i_c(\mathbb{D}(1)\backslash\{0\}, \mathcal{F}|_{\mathbb{D}(1)\backslash\{0\}})$$

by [14, Proposition 5.4.5 ii)]. Therefore, the assertion follows from (1). □

7. Proofs of Theorem 4.8 and Theorem 4.9

In this section, we shall prove Theorem 4.8 and Theorem 4.9. Let K be an algebraically closed complete non-archimedean field with ring of integers $\mathcal{O}$.

The main part of the proofs of Theorem 4.8 and Theorem 4.9 is contained in the following lemma.

Lemma 7.1. *Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion defined by one global section $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. Let $f\colon \mathcal{X} \to \operatorname{Spec}\mathcal{O}[T]$ be the morphism defined by $T \mapsto f$, which is also denoted by f. We assume that there is an element $\epsilon_1 \in |K^\times|$ with $\epsilon_1 \leq 1$ such that*

$$d(f)\colon d(\widehat{\mathcal{X}}) \to d((\operatorname{Spec}\mathcal{O}[T])^\wedge) = \mathbb{B}(1)$$

is smooth over $\mathbb{B}(\epsilon_1)^* = \mathbb{B}(\epsilon_1)\backslash\{0\}$. Then, there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \leq \epsilon_1$ such that the following assertions hold for every $\epsilon \in |K^\times|$ with $\epsilon \leq \epsilon_0$, every positive integer n invertible in $\mathcal{O}$ and every integer i.

(1) $H^i_c(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}) = 0$ *and* $H^i_c(T(\mathcal{Z},\epsilon_0)\backslash T(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) = 0$.

(2) $H^i(T(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(S(\mathcal{Z},\epsilon)^\circ, \mathbb{Z}/n\mathbb{Z})$.

(3) $H^i(S(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z})$.

(4) $H^i(S(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^i(S(\mathcal{Z},\epsilon)^\circ, \mathbb{Z}/n\mathbb{Z})$.

We first deduce Theorem 4.8 and Theorem 4.9 from Lemma 7.1. We will prove Lemma 7.1 in Section 7.3.

7.1. Cohomological descent for analytic adic spaces

We will recall some results on cohomological descent for analytic pseudo-adic spaces. Our basic references are [1, Exposé Vbis] and [8, Section 5] (see also [24, Section 3]).

Let $f\colon Y \to X$ be a morphism of finite type of analytic pseudo-adic spaces. Let

$$\beta\colon Y_\bullet := \operatorname{cosq}_0(Y/X) \to X$$

be the augmented simplicial object in the category of analytic pseudo-adic spaces of finite type over X (this category has finite projective limits by [14, Proposition 1.10.6]) defined as in [8, (5.1.4)]. So Y_m is the $(m+1)$-times fibre product $Y \times_X \cdots \times_X Y$ for $m \geq 0$. As in [8, (5.1.6)–(5.1.8)], one can associate to the étale topoi $(Y_m)^\sim_{\text{ét}}$ $(m \geq 0)$ a topos $(Y_\bullet)^\sim$. Moreover, as in [8, (5.1.11)], we have a morphism of topoi

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$$(\beta_*,\beta^*)\colon (Y_\bullet)^\sim \to X^\sim_{\text{ét}}$$

from $(Y_\bullet)^\sim$ to the étale topos $X^\sim_{\text{ét}}$ of X. We say that $f\colon Y \to X$ is a *morphism of cohomological descent* for torsion abelian étale sheaves if, for every torsion abelian étale sheaf $\mathcal{F}$ on X, the natural morphism

$$\mathcal{F} \to R\beta_*\beta^*\mathcal{F}$$

in the derived category $D^+(X^\sim_{\text{ét}})$ is an isomorphism (see [1, Exposé Vbis, Section 2] for details). As a consequence of the proper base change theorem for analytic pseudo-adic spaces [14, Theorem 4.4.1], we have the following proposition. We formulate it in the generality we need.

Proposition 7.2. *Let $f\colon Y \to X$ be a morphism of analytic adic spaces which is proper, of finite type and surjective. Then for every morphism $Z \to X$ of analytic pseudo-adic spaces, the base change $f\colon Y \times_X Z \to Z$ is of cohomological descent for torsion abelian étale sheaves.*

Proof. First, we note that the fibre product $Y \times_X Z \to Z$ exists by [14, Proposition 1.10.6]. By the proper base change theorem for analytic pseudo-adic spaces [14, Theorem 4.4.1 (b)], it suffices to prove that, for every geometric point $S \to X$, the base change $Y \times_X S \to S$ is of cohomological descent for torsion abelian étale sheaves. It is enough to show that there exists a section $S \to Y \times_X S$ (see [SGA 5 II, Exposé Vbis, Proposition 3.3.1] for example). The existence of a section can be easily proved in our case: By the properness of f and [14, Corollary 1.3.9], we may assume that S is of rank 1. Then it is well known. □

For future reference, we state the following corollaries of Proposition 7.2.

Corollary 7.3. *Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$. Let $\beta_0\colon \mathcal{Y} \to \mathcal{X}$ be a proper surjective morphism. We put $\beta\colon \mathcal{Y}_\bullet = \operatorname{cosq}_0(\mathcal{Y}/\mathcal{X}) \to \mathcal{X}$. Let $Z \subset d(\widehat{\mathcal{X}})$ be a taut locally closed subset. Then we have the following spectral sequence:*

$$E_1^{i,j} = H_c^j(Z_i,\mathbb{Z}/n\mathbb{Z}) \Rightarrow H_c^{i+j}(Z,\mathbb{Z}/n\mathbb{Z}),$$

where Z_i is the inverse image of Z under the morphism $d(\beta_i)\colon d(\widehat{\mathcal{Y}}_i) \to d(\widehat{\mathcal{X}})$.

Proof. This is an immediate consequence of Proposition 7.2 and [1, Exposé Vbis, Proposition 2.5.5]. □

Corollary 7.4. *Let $\beta_0\colon Y \to X$ be a morphism of analytic adic spaces which is proper, of finite type and surjective. We put $\beta\colon Y_\bullet = \operatorname{cosq}_0(Y/X) \to X$. Let $i\colon Z \hookrightarrow W$ be an inclusion of locally closed subsets of X. For $m \geq 0$, let $i_m\colon Z_m \hookrightarrow W_m$ be the pullback of i by $\beta_m\colon Y_m \to X$. Let*

$$R\Gamma(W,\mathbb{Z}/n\mathbb{Z}) \to R\Gamma(Z,\mathbb{Z}/n\mathbb{Z}) \to \mathcal{K} \to$$

and

$$R\Gamma(W_m,\mathbb{Z}/n\mathbb{Z}) \to R\Gamma(Z_m,\mathbb{Z}/n\mathbb{Z}) \to \mathcal{K}_m \to$$

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be distinguished triangles. Let $\rho \geq -1$ be an integer. If $\tau_{\leq(\rho-m)}\mathcal{K}_m = 0$ for every $0 \leq m \leq \rho+1$, then we have $\tau_{\leq\rho}\mathcal{K} = 0$.

Proof. The assertion follows from Proposition 7.2 and an argument similar to that of the proof of [25, Lemma 4.1] (see also the proof of Lemma 3.5). □

7.2. Reduction to the key case

In this subsection, we deduce Theorem 4.8 and Theorem 4.9 from Lemma 7.1. A theorem of de Jong [6, Theorem 4.1] will, again, play a key role.

Lemma 7.5. *To prove Theorem 4.8, it suffices to prove the following statement $\boldsymbol{P}_c(i)$ for every integer i.*

> $\boldsymbol{P}_c(i)$*: For every separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}$ and every closed immersion $\mathcal{Z} \hookrightarrow \mathcal{X}$ defined by one global section $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$, there exists an element $\epsilon_0 \in |K^\times|$ such that, for every $\epsilon \in |K^\times|$ with $\epsilon \leq \epsilon_0$ and every positive integer n invertible in $\mathcal{O}$, we have*
>
> $$H^j_c(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{and} \quad H^j_c(T(\mathcal{Z},\epsilon_0)\backslash T(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) = 0$$
>
> *for every $j \leq i$.*

Proof. *Step 1.* To prove Theorem 4.8, it suffices to prove the following statement $\mathbf{P}'_c(i)$ for every i.

> $\mathbf{P}'_c(i)$: For every separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}$ and every closed immersion $\mathcal{Z} \hookrightarrow \mathcal{X}$ of finite presentation, there exists an element $\epsilon_0 \in |K^\times|$ such that, for every $\epsilon \in |K^\times|$ with $\epsilon \leq \epsilon_0$ and every positive integer n invertible in $\mathcal{O}$, we have
>
> $$H^j_c(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{and} \quad H^j_c(T(\mathcal{Z},\epsilon_0)\backslash T(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) = 0$$
>
> for every $j \leq i$.

Indeed, let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion of finite presentation. By applying [14, Remark 5.5.11 iv)] to the following diagram

$$S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}) \hookrightarrow S(\mathcal{Z},\epsilon) \hookleftarrow d(\widehat{\mathcal{Z}}),$$

we have the following long exact sequence:

$$\cdots \to H^i_c(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z}) \to H^i_c(S(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \\ \to H^i_c(d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z}) \to H^{i+1}_c(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z}) \to \cdots.$$

We note that $S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}) = (d(\widehat{\mathcal{X}})\backslash d(\widehat{\mathcal{Z}}))\backslash Q(\mathcal{Z},\epsilon)$. Hence, we have a similar spectral sequence for the diagram

$$Q(\mathcal{Z},\epsilon) \hookrightarrow d(\widehat{\mathcal{X}})\backslash d(\widehat{\mathcal{Z}}) \hookleftarrow S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}).$$

Moreover, for elements $\epsilon,\epsilon' \in |K^\times|$ with $\epsilon \leq \epsilon'$, we have a similar spectral sequence for the diagram

$$T(\mathcal{Z},\epsilon) \hookrightarrow T(\mathcal{Z},\epsilon') \hookleftarrow T(\mathcal{Z},\epsilon')\backslash T(\mathcal{Z},\epsilon).$$

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By [14, Proposition 5.5.8], there exists an integer N, which is independent of n and $\epsilon,\epsilon' \in |K^\times|$, such that we have

$$H_c^i(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{and} \quad H_c^i(T(\mathcal{Z},\epsilon')\backslash T(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) = 0$$

for every $i \geq N$. Our claim follows from these results.

Step 2. We suppose that $\mathbf{P}_c(i)$ holds for every i. We will prove $\mathbf{P}'_c(i)$ by induction on i. The assertion holds trivially for $i = -1$. We assume that $\mathbf{P}'_c(i_0 - 1)$ holds. First, we claim that, to prove $\mathbf{P}'_c(i_0)$, we may assume that $\mathcal{X}$ is integral.

We may assume that $\mathcal{X}$ is flat over $\mathcal{O}$. Then every irreducible component of $\mathcal{X}$ dominates $\operatorname{Spec}\mathcal{O}$, and, hence, $\mathcal{X}$ has finitely many irreducible components. Let $\mathcal{X}'$ be the disjoint union of the irreducible components of $\mathcal{X}$. Then $\mathcal{X}' \to \mathcal{X}$ is proper and surjective. By $\mathbf{P}'_c(i_0 - 1)$ and Corollary 7.3, it suffices to prove $\mathbf{P}'_c(i_0)$ for $\mathcal{X}'$ and $\mathcal{Z} \times_{\mathcal{X}} \mathcal{X}'$. By considering each component of $\mathcal{X}'$ separately, our claim follows.

Step 3. We assume that $\mathcal{X}$ is integral. We may assume further that $\mathcal{Z}$ is not equal to $\mathcal{X}$. Let $\mathcal{Y} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $\mathcal{Z}$, which is proper and surjective. By $\mathbf{P}'_c(i_0 - 1)$ and Corollary 7.3, it suffices to prove $\mathbf{P}'_c(i_0)$ for $\mathcal{Y}$ and $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$. Consequently, to prove $\mathbf{P}'_c(i_0)$, we may assume further that $\mathcal{Z} \hookrightarrow \mathcal{X}$ is locally defined by one function.

Finally, let $\mathcal{X} = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$ be a finite affine covering such that $\mathcal{Z} \cap \mathcal{U}_\alpha \hookrightarrow \mathcal{U}_\alpha$ is defined by one global section in $\mathcal{O}_{\mathcal{X}}(\mathcal{U}_\alpha)$ for every $\alpha \in I$. We have the following spectral sequence by [14, Remark 5.5.12 iii)]:

$$E_1^{i,j} = \bigoplus_{J \subset I,\, \sharp J = -i+1} H_c^j(S(\mathcal{Z}_J,\epsilon)\backslash d(\widehat{\mathcal{Z}}_J),\mathbb{Z}/n\mathbb{Z}) \Rightarrow H_c^{i+j}(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z}).$$

Here, we write $\mathcal{U}_J := \cap_{\alpha \in J} \mathcal{U}_\alpha$ and $\mathcal{Z}_J := \mathcal{Z} \times_{\mathcal{X}} \mathcal{U}_J \hookrightarrow \mathcal{U}_J$. We have a similar spectral sequence for $T(\mathcal{Z},\epsilon')\backslash T(\mathcal{Z},\epsilon)$. Since $\mathbf{P}_c(i)$ holds for every i by hypothesis, it follows that $\mathbf{P}'_c(i_0)$ holds for $\mathcal{X}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$. □

Lemma 7.6. *To prove Theorem 4.9, it suffices to prove the following statement* $\boldsymbol{P}(i)$ *for every integer* i.

> $\boldsymbol{P}(i)$: *For a separated scheme* $\mathcal{X}$ *of finite type over* $\mathcal{O}$ *and a closed immersion* $\mathcal{Z} \hookrightarrow \mathcal{X}$ *defined by one global section* $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$, *we consider the following distinguished triangles:*
>
> $$R\Gamma(T(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \to R\Gamma(S(\mathcal{Z},\epsilon)^\circ,\mathbb{Z}/n\mathbb{Z}) \to \mathcal{K}_1(\epsilon,n) \to,$$
>
> $$R\Gamma(S(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \to R\Gamma(d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z}) \to \mathcal{K}_2(\epsilon,n) \to,$$
>
> $$R\Gamma(S(\mathcal{Z},\epsilon),\mathbb{Z}/n\mathbb{Z}) \to R\Gamma(S(\mathcal{Z},\epsilon)^\circ,\mathbb{Z}/n\mathbb{Z}) \to \mathcal{K}_3(\epsilon,n) \to.$$
>
> *For every* $\mathcal{X}$ *and* $\mathcal{Z} \hookrightarrow \mathcal{X}$ *as above, there exists an element* $\epsilon_0 \in |K^\times|$ *such that, for every* $\epsilon \in |K^\times|$ *with* $\epsilon \leq \epsilon_0$ *and every positive integer* n *invertible in* $\mathcal{O}$, *we have* $\tau_{\leq i}\mathcal{K}_m(\epsilon,n) = 0$ *for every* $m \in \{1,2,3\}$.

Proof. The assertion can be proved by a similar argument as in the proof of Lemma 7.5. □

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We will now prove the desired statement:

Lemma 7.7. *To prove Theorem 4.8 and Theorem 4.9, it suffices to prove Lemma 7.1.*

Proof. We suppose that Lemma 7.1 holds. By Lemma 7.5 and Lemma 7.6, it suffices to prove that $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ hold for every i. Let us show the assertions by induction on i. The assertions $\mathbf{P}_c(-1)$ and $\mathbf{P}(-2)$ hold trivially. Assume that $\mathbf{P}_c(i-1)$ and $\mathbf{P}(i-1)$ hold. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}$ and $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed immersion defined by one global section $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. We shall show that $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ hold for $\mathcal{X}$ and $\mathcal{Z}$. As in the proof of Lemma 7.5, we may assume that $\mathcal{X}$ is integral. First, we prove the assertions in the case where K is of characteristic zero. By [6, Theorem 4.1], there is an integral alteration $Y \to \mathcal{X}_K$ such that Y is smooth over K. By Nagata's compactification theorem, there is a proper surjective morphism $\mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y}_K \cong Y$ over $\mathcal{X}_K$ and $\mathcal{Y}$ is integral. By the induction hypothesis, Corollary 7.3, and Corollary 7.4, it suffices to prove $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ for $\mathcal{Y}$ and $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$. Therefore, we may assume that $\mathcal{X}_K$ is smooth over K. Let

$$f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}[T]$$

be the morphism defined by $T \mapsto f$. Since K is of characteristic zero, there is an open dense subset $W \subset \operatorname{Spec} K[T]$ such that f_K is smooth over W. It follows from [14, Proposition 1.9.6] that there exists an open subset $V \subset \mathbb{B}(1)$ whose complement consists of finitely many K-rational points of $\mathbb{B}(1)$ such that $d(f)\colon d(\widehat{\mathcal{X}}) \to \mathbb{B}(1)$ is smooth over V. Thus, $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ hold for $\mathcal{X}$ and $\mathcal{Z}$ since we suppose that Lemma 7.1 holds.

Let us now suppose that K is of characteristic $p > 0$. As above, let $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}[T]$ be the morphism defined by $T \mapsto f$. If f is not dominant, then $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ hold trivially for $\mathcal{X}$ and $\mathcal{Z}$. Thus, we may assume that f is dominant. By applying [6, Theorem 4.1] to the underlying reduced subscheme of $\mathcal{X} \times_{\operatorname{Spec} \mathcal{O}[T]} \operatorname{Spec} K(T^{1/p^{\infty}})$, where $K(T^{1/p^{\infty}})$ is the perfection of $K(T)$, we find an alteration

$$g_K\colon Y \to \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}[T]} \operatorname{Spec} K(T^{1/p^N})$$

such that Y is integral and smooth over $K(T^{1/p^N})$ for some integer $N \geq 0$. By Nagata's compactification theorem, there is a proper surjective morphism

$$g\colon \mathcal{Y} \to \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}[T]} \operatorname{Spec} \mathcal{O}[T^{1/p^N}]$$

whose base change to $\operatorname{Spec} K(T^{1/p^N})$ is isomorphic to g_K. As above, it suffices to prove $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ for $\mathcal{Y}$ and $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$.

Let f' be the image of $T^{1/p^N} \in \mathcal{O}[T^{1/p^N}]$ in $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$, and let $\mathcal{Z}' \hookrightarrow \mathcal{Y}$ be the closed subscheme defined by f'. Then we have $(f')^{p^N} = f$, where the image of f in $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$ is also denoted by f. Hence, the closed immersion $d(\widehat{\mathcal{Z}'}) \hookrightarrow d((\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y})^\wedge)$ is a homeomorphism and we have

$$S(\mathcal{Z}', \epsilon) = S(\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}, \epsilon^{p^N}) \quad \text{and} \quad T(\mathcal{Z}', \epsilon) = T(\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}, \epsilon^{p^N})$$

for every $\epsilon \in |K^\times|$. Thus, it suffices to prove that $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ hold for $\mathcal{Y}$ and $\mathcal{Z}'$ (see [14, Proposition 2.3.7]). By the construction, the generic fibre of the morphism $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}[T]$

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defined by $T \mapsto f'$ is smooth over $K(T)$. Therefore, as in the case of characteristic zero, Lemma 7.1 implies $\mathbf{P}_c(i)$ and $\mathbf{P}(i)$ for $\mathcal{Y}$ and $\mathcal{Z}'$.

The proof of Lemma 7.7 is complete. □

7.3. Proof of the key case

In this subsection, we prove Lemma 7.1 and finish the proofs of Theorem 4.8 and Theorem 4.9.

Proof of Lemma 7.1. We may assume that $\mathcal{X}$ is flat over $\operatorname{Spec}\mathcal{O}$. Then $\mathcal{X}$ is of finite presentation over $\operatorname{Spec}\mathcal{O}$ by [28, Première partie, Corollaire 3.4.7]. By Proposition 6.6 and Theorem 6.10, there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \le \epsilon_1$ such that, for all $a,b \in |K^\times|$ with $a < b \le \epsilon_0$ and every positive integer n invertible in $\mathcal{O}^\times$, there exists a finite étale morphism

$$h \colon \mathbb{B}(c,d) \to \mathbb{B}(a,b)$$

such that h is a composition of finite Galois étale morphisms and the pullback

$$h^*((R^i d(f)_! \mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(a,b)})$$

is a constant sheaf associated with a finitely generated $\mathbb{Z}/n\mathbb{Z}$-module for every i. We shall show that ϵ_0 satisfies the desired properties. Let n be a positive integer invertible in $\mathcal{O}$ and $\epsilon \in |K^\times|$ an element with $\epsilon \le \epsilon_0$.

(1) We have

$$S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}) = d(f)^{-1}(\mathbb{D}(\epsilon)\backslash\{0\}) \quad \text{and} \quad T(\mathcal{Z},\epsilon_0)\backslash T(\mathcal{Z},\epsilon) = d(f)^{-1}(\mathbb{B}(\epsilon_0)\backslash\mathbb{B}(\epsilon)).$$

Keeping the base change theorem [14, Theorem 5.4.6] for $Rd(f)_!$ in mind, we have the following spectral sequences by [14, Remark 5.5.12 i)]:

$$E_2^{i,j} = H_c^i(\mathbb{D}(\epsilon)\backslash\{0\}, (R^j d(f)_! \mathbb{Z}/n\mathbb{Z})|_{\mathbb{D}(\epsilon)\backslash\{0\}}) \Rightarrow H_c^{i+j}(S(\mathcal{Z},\epsilon)\backslash d(\widehat{\mathcal{Z}}), \mathbb{Z}/n\mathbb{Z}),$$

$$E_2^{i,j} = H_c^i(\mathbb{B}(\epsilon_0)\backslash\mathbb{B}(\epsilon), (R^j d(f)_! \mathbb{Z}/n\mathbb{Z})|_{\mathbb{B}(\epsilon_0)\backslash\mathbb{B}(\epsilon)}) \Rightarrow H_c^{i+j}(T(\mathcal{Z},\epsilon_0)\backslash T(\mathcal{Z},\epsilon), \mathbb{Z}/n\mathbb{Z}).$$

Hence, the assertion (1) follows from Lemma 6.12.

(2) We have

$$T(\mathcal{Z},\epsilon) = d(f)^{-1}(\mathbb{B}(\epsilon)) \quad \text{and} \quad S(\mathcal{Z},\epsilon)^\circ = d(f)^{-1}(\mathbb{D}(\epsilon)^\circ).$$

By the Leray spectral sequences, it suffices to prove that the restriction map

$$H^i(\mathbb{B}(\epsilon), R^j d(f)_* \mathbb{Z}/n\mathbb{Z}) \to H^i(\mathbb{D}(\epsilon)^\circ, R^j d(f)_* \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for all i,j. Let $\epsilon' \in |K^\times|$ be an element with $\epsilon' < \epsilon$. Then $\{\mathbb{B}(\epsilon',\epsilon), \mathbb{D}(\epsilon)^\circ\}$ is an open covering of $\mathbb{B}(\epsilon)$. By the Čech-to-cohomology spectral sequences, it is enough to prove that the restriction map

$$H^i(\mathbb{B}(\epsilon',\epsilon), R^j d(f)_* \mathbb{Z}/n\mathbb{Z}) \to H^i(\mathbb{D}(\epsilon)^\circ \cap \mathbb{B}(\epsilon',\epsilon), R^j d(f)_* \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for all i,j.

The inverse image $d(f)^{-1}(\mathbb{B}(\epsilon',\epsilon))$ has finitely many connected components. It is enough to show that, for every connected component $W \subset d(f)^{-1}(\mathbb{B}(\epsilon',\epsilon))$ and the restriction

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$g\colon W \to \mathbb{B}(\epsilon',\epsilon)$ of $d(f)$, the restriction map

$$H^i(\mathbb{B}(\epsilon',\epsilon), R^j g_* \mathbb{Z}/n\mathbb{Z}) \to H^i(\mathbb{D}(\epsilon)^\circ \cap \mathbb{B}(\epsilon',\epsilon), R^j g_* \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for all i,j. The morphism g is of pure dimension N for some integer $N \geq 0$ (see [14, Section 1.8] for the definition of the dimension of a morphism of adic spaces). Since g is smooth and $R^i g_! \mathbb{Z}/n\mathbb{Z}$ is a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules, Poincaré duality [14, Corollary 7.5.5] implies that

$$R^j g_*(\mathbb{Z}/n\mathbb{Z}(N)) \cong (R^{2N-j} g_! \mathbb{Z}/n\mathbb{Z})^\vee$$

for every j, where (N) denotes the Tate twist and $()^\vee$ denotes the $\mathbb{Z}/n\mathbb{Z}$-dual (here, we use the fact that $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$-module). The right-hand side satisfies the assumption of Lemma 6.12 (1), and, hence, the assertion follows from the lemma.

(3) We have $S(\mathcal{Z},\epsilon) = d(f)^{-1}(\mathbb{D}(\epsilon))$. Let

$$d(f)'\colon S(\mathcal{Z},\epsilon) \to \mathbb{D}(\epsilon)$$

be the base change of $d(f)$ to $\mathbb{D}(\epsilon)$. We write

$$\mathcal{F}_j := R^j d(f)'_* \mathbb{Z}/n\mathbb{Z}.$$

We claim that restriction map $H^i(\mathbb{D}(\epsilon),\mathcal{F}_j) \to H^i(\{0\},\mathcal{F}_j|_{\{0\}})$ is an isomorphism for all i,j. Since we have by [14, Example 2.6.2]

$$H^i(\{0\},\mathcal{F}_j|_{\{0\}}) = \begin{cases} H^j(d(\widehat{\mathcal{Z}}),\mathbb{Z}/n\mathbb{Z}) & (i=0) \\ 0 & (i \neq 0), \end{cases}$$

the claim and the Leray spectral sequence for $d(f)'$ imply the assertion (3). We prove the claim. Since $\mathbb{D}(\epsilon)$ and $\{0\}$ are proper over $\mathrm{Spa}(K,\mathcal{O})$, we have

$$H^i(\mathbb{D}(\epsilon),\mathcal{F}_j) = H^i_c(\mathbb{D}(\epsilon),\mathcal{F}_j) \quad \text{and} \quad H^i(\{0\},\mathcal{F}_j|_{\{0\}}) = H^i_c(\{0\},\mathcal{F}_j|_{\{0\}}),$$

and, hence, it suffices to prove that $H^i_c(\mathbb{D}(\epsilon)\backslash\{0\},\mathcal{F}_j|_{\mathbb{D}(\epsilon)\backslash\{0\}}) = 0$ for all i,j. Moreover, by [14, Proposition 5.4.5 ii)], it suffices to prove that, for any $\epsilon' \in |K^\times|$ with $\epsilon' < \epsilon$, we have $H^i_c(\mathbb{D}(\epsilon) \cap \mathbb{B}(\epsilon',\epsilon),\mathcal{F}_j|_{\mathbb{D}(\epsilon)\cap\mathbb{B}(\epsilon',\epsilon)}) = 0$ for all i,j.

Let $\epsilon' \in |K^\times|$ be an element with $\epsilon' < \epsilon$. Let W and g be as in the proof of (2). Let

$$g'\colon g^{-1}(\mathbb{D}(\epsilon)\cap\mathbb{B}(\epsilon',\epsilon)) \to \mathbb{D}(\epsilon)\cap\mathbb{B}(\epsilon',\epsilon)$$

be the base change of g. It suffices to prove that $H^i_c(\mathbb{D}(\epsilon)\cap\mathbb{B}(\epsilon',\epsilon), R^j g'_* \mathbb{Z}/n\mathbb{Z}) = 0$ for all i,j. By the base change theorem [14, Theorem 5.4.6] for $Rg_!$, we have

$$(R^j g_! \mathbb{Z}/n\mathbb{Z})|_{\mathbb{D}(\epsilon)\cap\mathbb{B}(\epsilon',\epsilon)} \cong R^j g'_! \mathbb{Z}/n\mathbb{Z}.$$

In particular, the right-hand side is a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules. As in the proof of (2), Poincaré duality [14, Corollary 7.5.5] for g' then implies that

$$R^j g'_*(\mathbb{Z}/n\mathbb{Z}(N)) \cong (R^{2N-j} g'_! \mathbb{Z}/n\mathbb{Z})^\vee,$$

and the assertion follows from Lemma 6.12 (1).

(4) Similarly as above, it suffices to prove that the restriction map

$$H^i(\mathbb{D}(\epsilon),\mathcal{F}_j) \to H^i(\mathbb{D}(\epsilon)^\circ,\mathcal{F}_j)$$

is an isomorphism for all i,j. Let $\epsilon' \in |K^\times|$ be an element with $\epsilon' < \epsilon$. As in the proof of (2), it suffices to prove that the restriction map

$$H^i(\mathbb{D}(\epsilon)\cap\mathbb{B}(\epsilon',\epsilon),\mathcal{F}_j) \to H^i(\mathbb{D}(\epsilon)^\circ\cap\mathbb{B}(\epsilon',\epsilon),\mathcal{F}_j)$$

is an isomorphism for all i,j. By the proof of (3), the sheaf $\mathcal{F}_j|_{\mathbb{D}(\epsilon)\cap\mathbb{B}(\epsilon',\epsilon)}$ is a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules. Hence, the assertion follows from the proof of [16, Lemma 2.5] (in [16], the characteristic of the base field is always assumed to be zero; however, [16, Lemma 2.5] holds in positive characteristic without changing the proof).

The proof of Lemma 7.1 is complete. □

Theorem 4.8 and Theorem 4.9 now follow from Lemma 7.1 and Lemma 7.7.

Appendix A. Finite étale coverings of annuli

In this appendix, we prove Theorem 6.2 and Theorem 6.3. We retain the notation of Section 6.1. In particular, we fix an algebraically closed complete non-archimedean field K with ring of integers $\mathcal{O}$. We will follow the methods given in Ramero's paper [27].

Following [27], we will use the following notation in this appendix. Recall that for a morphism of finite type $\mathrm{Spa}(A,A^+) \to \mathrm{Spa}(K,\mathcal{O})$, where (A,A^+) is a complete affinoid ring, the ring A^+ coincides with the ring A° of power-bounded elements of A (see [13, Lemma 4.4] and [14, Section 1.2]). We often omit A^+ and abbreviate $\mathrm{Spa}(A,A^+)$ to $\mathrm{Spa}(A)$. If A is reduced, then A° is topologically of finite type over $\mathcal{O}$, that is, $A^\circ \cong \mathcal{O}\langle T_1,\dots,T_n\rangle/I$ for some ideal $I \subset \mathcal{O}\langle T_1,\dots,T_n\rangle$ by [3, 6.4.1, Corollary 4]. Let $\mathfrak{m} \subset \mathcal{O}$ be the maximal ideal, and let $\kappa := \mathcal{O}/\mathfrak{m}$ be the residue field. The quotient

$$A^\sim := A^\circ/\mathfrak{m}A^\circ$$

is a finitely generated algebra over κ. We note that the ideal $\mathfrak{m}A^\circ$ coincides with the set of topologically nilpotent elements of A. In particular, the ring $A^\sim$ is reduced.

A.1. Open annuli in the unit disc

In this subsection, we prepare some notation and recall some basic properties of open annuli in the unit disc $\mathbb{B}(1) = \mathrm{Spa}(K\langle T\rangle)$.

Let $a,b \in |K^\times|$ be elements with $a \le b \le 1$. Recall that we defined

$$\begin{aligned}\mathbb{B}(a,b) &:= \{x \in \mathbb{B}(1)\,|\,a \le |T(x)| \le b\}\\ &:= \{x \in \mathbb{B}(1)\,|\,|\varpi_a(x)| \le |T(x)| \le |\varpi_b(x)|\},\end{aligned}$$

where $\varpi_a,\varpi_b \in K^\times$ are elements such that $a = |\varpi_a|$ and $b = |\varpi_b|$. We have

$$\mathbb{B}(a,b) \cong \mathrm{Spa}(K\langle T,T_a,T_b\rangle/(T_aT-\varpi_a,T-\varpi_bT_b))$$

as an adic space over $\mathbb{B}(1)$. The adic space $\mathbb{B}(a,b)$ is isomorphic to $\mathbb{B}(a/b,1)$ as an adic space over $\mathrm{Spa}(K)$. We write $A(a,b) := \mathcal{O}_{\mathbb{B}(1)}(\mathbb{B}(a,b))$.

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We will focus on the following points of the unit disc $\mathbb{B}(1)$. Let $r \in |K^\times|$ be an element with $r \le 1$.

- Let
$$\eta(r)^\flat \colon K\langle T\rangle \to |K^\times|, \quad \sum_{i\ge 0} a_i T^i \mapsto \max_{i\ge 0}\{|a_i| r^i\}$$
be the Gauss norm of radius r centred at 0. The corresponding point $\eta(r)^\flat \in \mathbb{B}(1)$ is denoted by the same letter.
- Let $\langle\delta\rangle$ be an infinite cyclic group with generator δ. We equip $|K^\times| \times \langle\delta\rangle$ with a total order such that
$$(s,\delta^m) < (t,\delta^n) \Longleftrightarrow s < t, \quad \text{or} \quad s = t \quad \text{and} \quad m > n.$$
So we have $(1,\delta) < (1,1)$ and $(s,1) < (1,\delta)$ for every $s \in |K^\times|$ with $s < 1$. We identify δ with $(1,\delta)$ and r with $(r,1)$. The valuation
$$\eta(r) \colon K\langle T\rangle \to |K^\times| \times \langle\delta\rangle, \quad \sum_{i\ge 0} a_i T^i \mapsto \max_{i\ge 0}\{|a_i| r^i \delta^i\}$$
gives a point $\eta(r) \in \mathbb{B}(1)$. The point $\eta(r)$ is a specialization of $\eta(r)^\flat$, that is, we have
$$\eta(r) \in \overline{\{\eta(r)^\flat\}}.$$
- Similarly, if $r < 1$, the valuation
$$\eta(r)' \colon K\langle T\rangle \to |K^\times| \times \langle\delta\rangle, \quad \sum_{i\ge 0} a_i T^i \mapsto \max_{i\ge 0}\{|a_i| r^i \delta^{-i}\}$$
gives a point $\eta(r)' \in \mathbb{B}(1)$. The point $\eta(r)'$ is a specialization of $\eta(r)^\flat$.

We can use the points $\eta(r)$ and $\eta(r)'$ to describe the closure of an annulus $\mathbb{B}(a,b)$.

Example A.1. Let $a,b \in |K^\times|$ be elements with $a \le b \le 1$.

(1) For an element $r \in |K^\times|$ with $a \le r \le b$, we have $\eta(r)^\flat \in \mathbb{B}(a,b)$. If $a < r \le b$, we have $\eta(r) \in \mathbb{B}(a,b)$. Similarly, if $a \le r < b$, we have $\eta(r)' \in \mathbb{B}(a,b)$.

(2) We assume that $a \le b < 1$. Let $\mathbb{B}(a,b)^c$ be the closure of $\mathbb{B}(a,b)$ in $\mathbb{B}(1)$. Then we have $\mathbb{B}(a,b)^c \backslash \mathbb{B}(a,b) = \{\eta(a), \eta(b)'\}$. In particular, the complement $\mathbb{B}(a,b)^c \backslash \mathbb{B}(a,b)$ consists of two points.

(3) We have some kind of converse to (2). We define $\mathbb{D}(1) := \{x \in \mathbb{B}(1) \,|\, |T(x)| < 1\}$, which is a closed subset of $\mathbb{B}(1)$. Let $X \subset \mathbb{B}(1)$ be a connected affinoid open subset contained in $\mathbb{D}(1)$. Let X^c be the closure of X in $\mathbb{B}(1)$. If the complement $X^c \backslash X$ consists of two points, then there exists an isomorphism
$$X \cong \mathbb{B}(a,1)$$
of adic spaces over $\mathrm{Spa}(K)$ for some element $a \in |K^\times|$ with $a \le 1$. This can be proved by using [3, 9.7.2, Theorem 2].

We recall the following example from [27], which is useful to study finite étale coverings of $\mathbb{B}(a,b)$.

Example A.2 [27, Example 2.1.12]. We assume that $a < b$. Let

$$\Psi\colon \mathbb{B}(a,b) = \mathrm{Spa}(A(a,b)) \to \mathbb{B}(1) = \mathrm{Spa}(K\langle S\rangle)$$

be the morphism over $\mathrm{Spa}(K)$ defined by the following homomorphism

$$\psi\colon \mathcal{O}\langle S\rangle \to \mathcal{O}\langle T_a,T_b\rangle/(T_aT_b - \varpi_a/\varpi_b) \cong A(a,b)^\circ, \quad S \mapsto T_a + T_b.$$

The homomorphism ψ makes $A(a,b)^\circ$ into a free $\mathcal{O}\langle S\rangle$-module of rank 2.

Remark A.3. In the rest of this section, we shall study finite étale coverings of $\mathbb{B}(a,b)$. We recall that there is a natural equivalence between the category of adic spaces which are finite and étale over X and the category of schemes which are finite and étale over $\mathrm{Spec}\,\mathcal{O}_X(X)$ (see [14, Example 1.6.6 ii)]).

In the rest of this subsection, we give two lemmas about the connected components of a finite étale covering of $\mathbb{B}(a,b)$.

Lemma A.4. *We assume that $a < b$. Let $f\colon X \to \mathbb{B}(a,b)$ be a finite étale morphism of adic spaces. For every $t \in |K^\times|$ with $a/b < t^2 < 1$, there exists an element $s_0 \in |K^\times|$ with $t < s_0 \leq 1$ such that every connected component of $f^{-1}(\mathbb{B}(a/s_0,s_0b))$ remains connected after restricting to $\mathbb{B}(a/s,sb)$ for every $s \in |K^\times|$ with $t < s \leq s_0$.*

Proof. The number of the connected components of $f^{-1}(\mathbb{B}(a/s,sb))$ increases with decreasing s, and it is bounded above by the degree of f (that is, the rank of $\mathcal{O}_X(X)$ as an $A(a,b)$-module) for every $s \in |K^\times|$ with $a/b < s^2 \leq 1$. The assertion follows from these properties. $\square$

Lemma A.5. *We assume that $a < b$. Let $f\colon X \to \mathbb{B}(a,b)$ be a finite étale morphism of adic spaces. We write $B := \mathcal{O}_X(X)$ and consider the composition*

$$\mathcal{O}\langle S\rangle \xrightarrow{\psi} A(a,b)^\circ \to B^\circ,$$

where ψ is the homomorphism defined in Example A.2 and the second homomorphism is the one induced by f. Let $\{\mathfrak{q}_1,\ldots,\mathfrak{q}_n\} \subset \mathrm{Spec}\,B^\circ$ be the set of the prime ideals of B° lying above the maximal ideal $\mathfrak{m}\mathcal{O}\langle S\rangle + S\mathcal{O}\langle S\rangle \subset \mathcal{O}\langle S\rangle$. Then, for every $t \in |K^\times|$ with $a/b < t^2 < 1$, the adic space $f^{-1}(\mathbb{B}(a/t,tb))$ has at least n connected components.

Proof. This lemma is proved in the proof of [27, Theorem 2.4.3]. We recall the arguments for the reader's convenience.

We define g as the composition

$$g\colon X \xrightarrow{f} \mathbb{B}(a,b) \xrightarrow{\Psi} \mathbb{B}(1) = \mathrm{Spa}(K\langle S\rangle).$$

Let $t \in |K^\times|$ be an element with $a/b < t^2 < 1$. We define $\mathbb{B}(t) := \{x \in \mathbb{B}(1) \,|\, |S(x)| \leq t\}$. Since $\Psi^{-1}(\mathbb{B}(t)) = \mathbb{B}(a/t,tb)$, it is enough to show that $g^{-1}(\mathbb{B}(t))$ has at least n connected

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components. The ring $\mathcal{O}[[S]]$ is a Henselian local ring. Since B° is a free $\mathcal{O}\langle S\rangle$-module of finite rank by [27, Proposition 2.3.5], we have a decomposition

$$B^\circ \otimes_{\mathcal{O}\langle S\rangle} \mathcal{O}[[S]] \cong R_1 \times \cdots \times R_n,$$

where $R_1,\ldots,R_n$ are local rings, which are free $\mathcal{O}[[S]]$-modules of finite rank. Since the natural homomorphism $\mathcal{O}\langle S\rangle = \mathcal{O}_{\mathbb{B}(1)}(\mathbb{B}(1))^\circ \to \mathcal{O}_{\mathbb{B}(t)}(\mathbb{B}(t))^\circ$ factors through $\mathcal{O}\langle S\rangle \to \mathcal{O}[[S]]$, we have

$$g^{-1}(\mathbb{B}(t)) \cong \mathrm{Spa}(B_1)\coprod\cdots\coprod\mathrm{Spa}(B_n),$$

where $B_i := R_i \otimes_{\mathcal{O}[[S]]} \mathcal{O}_{\mathbb{B}(t)}(\mathbb{B}(t))$. This proves our claim since $\mathrm{Spa}(B_i)$ is nonempty for every $1 \le i \le n$. □

A.2. Discriminant functions and finite étale coverings of open annuli

In this subsection, we give some properties of the discriminant function of a finite étale covering of $\mathbb{B}(a,b)$, and then we prove Theorem 6.2.

Let $a,b \in |K^\times|$ be elements with $a < b \le 1$. Let $f\colon X \to \mathbb{B}(a,b)$ be a finite étale morphism of adic spaces. Let us briefly recall the definition of the *discriminant function*

$$\delta_f\colon [-\log b, -\log a] \to \mathbb{R}_{\ge 0}$$

associated with f following [27].

Let $\mathcal{O}_X^+$ be the subsheaf of $\mathcal{O}_X$ defined by

$$\mathcal{O}_X^+(U) = \{g \in \mathcal{O}_X(U) \,|\, |g(x)| \le 1 \text{ for every } x \in U\}$$

for every open subset $U \subset X$. For an element $r \in |K^\times|$ with $a \le r \le b$, let

$$\mathscr{A}(r)^\flat := (f_*\mathcal{O}_X^+)_{\eta(r)^\flat}$$

be the stalk of $f_*\mathcal{O}_X^+$ at the point $\eta(r)^\flat$. The maximal ideal of the stalk $\mathcal{O}_{\mathbb{B}(a,b),\eta(r)^\flat}$ at $\eta(r)^\flat$ is zero, in other words, we have $\mathcal{O}_{\mathbb{B}(a,b),\eta(r)^\flat} = k(\eta(r)^\flat)$. Hence, there is a natural homomorphism $k(\eta(r)^\flat)^+ \to \mathscr{A}(r)^\flat$. By applying [27, Proposition 2.3.5] to the restriction $f^{-1}(\mathbb{B}(r,r)) \to \mathbb{B}(r,r)$ of f, we see that $\mathscr{A}(r)^\flat$ is a free $k(\eta(r)^\flat)^+$-module of finite rank. Then we can define the valuation

$$v_{\eta(r)^\flat}(\mathfrak{d}_f^\flat(r)) \in \mathbb{R}_{>0}$$

of the discriminant $\mathfrak{d}_f^\flat(r) \in k(\eta(r)^\flat)^+$ of $\mathscr{A}(r)^\flat$ over $k(\eta(r)^\flat)^+$ (in the sense of [27, Section 2.1]), and we define

$$\delta_f\colon [-\log b, -\log a] \cap -\log|K^\times| \to \mathbb{R}_{\ge 0}, \quad -\log r \mapsto -\log(v_{\eta(r)^\flat}(\mathfrak{d}_f^\flat(r))) \in \mathbb{R}_{\ge 0}$$

(see [27, 2.3.12] for details).

Theorem A.6 (Ramero [27, Theorem 2.3.25]). *The function δ_f extends uniquely to a continuous, piecewise linear and convex function*

$$\delta_f\colon [-\log b, -\log a] \to \mathbb{R}_{\ge 0}.$$

Moreover, the slopes of δ_f are integers.

Proof. See [27, Theorem 2.3.25]. □

Many basic properties of the discriminant function δ_f were studied in detail in [27]. Here, we are interested in the case where δ_f is linear.

Proposition A.7 (Ramero [27]). *Let $f\colon X = \mathrm{Spa}(B,B^\circ) \to \mathbb{B}(a,b)$ be a finite étale morphism of adic spaces with a complete affinoid ring (B,B°). Define g as the composition*

$$g\colon X \xrightarrow{f} \mathbb{B}(a,b) \xrightarrow{\Psi} \mathbb{B}(1) = \mathrm{Spa}(K\langle S\rangle),$$

where Ψ is the morphism defined in Example A.2. The map g induces a homomorphism $\mathcal{O}\langle S\rangle \to B^\circ$. Let us suppose that the following two conditions hold:

- *There is only one prime ideal $\mathfrak{q} \subset B^\circ$ lying above the maximal ideal $\mathfrak{m}\mathcal{O}\langle S\rangle + S\mathcal{O}\langle S\rangle \subset \mathcal{O}\langle S\rangle$.*
- *The discriminant function δ_f is linear.*

Then the inverse image $g^{-1}(\eta(1))$ consists of two points, or equivalently, the inverse images $f^{-1}(\eta(a)')$ and $f^{-1}(\eta(b))$ both consist of one point. Moreover, the closed point

$$x \in \mathrm{Spec}\, B^\sim = \mathrm{Spec}\, B^\circ/\mathfrak{m}B^\circ$$

corresponding to the prime ideal $\mathfrak{q}$ is an ordinary double point.

Proof. The first assertion is a consequence of [27, (2.4.4) in the proof of Theorem 2.4.3]. The assumption that the characteristic of the base field is zero in *loc. cit.* is not needed here. Moreover, the morphism f need not be Galois.

The second assertion is claimed in [27, Remark 2.4.8] (at least when K is of characteristic zero) without proof. Indeed, the hard parts of the proof were already done in [27]. We shall explain how to use the results in *loc. cit.* to deduce the second assertion.

Before giving the proof of the second assertion, let us prepare some notation. We write $A := K\langle S\rangle$. For the points $\eta(1),\eta(1)^\flat \in \mathbb{B}(1)$, we write

$$(k(1),k(1)^+) := (k(\eta(1)),k(\eta(1))^+) \quad \text{and} \quad (k(1^\flat),k(1^\flat)^+) := (k(\eta(1)^\flat),k(\eta(1)^\flat)^+).$$

Note that $\mathcal{O}_{\mathbb{B}(1),\eta(1)} = k(1)$ and $\mathcal{O}_{\mathbb{B}(1),\eta(1)^\flat} = k(1^\flat)$. We have a natural inclusion

$$k(1)^+ \to k(1^\flat)^+.$$

The residue field $k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+$ of $k(1^\flat)^+$ is naturally isomorphic to the field of fractions $\kappa(S)$ of $\kappa[S]$. The image of $k(1)^+$ in $\kappa(S)$ is the localization $\kappa[S]_{(S)}$ of $\kappa[S]$ at the maximal ideal $(S) \subset \kappa[S]$. More precisely, we have the following commutative diagram:

$$\begin{array}{ccccc} A^\sim = A^\circ/\mathfrak{m}A^\circ & \longrightarrow & k(1)^+/\mathfrak{m}k(1)^+ & \longrightarrow & k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+ \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \kappa[S] & \longrightarrow & \kappa[S]_{(S)} & \longrightarrow & \kappa(S). \end{array}$$

Let

$$\mathscr{B}(1)^+ := (g_*\mathcal{O}_X^+)_{\eta(1)}$$

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be the stalk of $g_*\mathcal{O}_X^+$ at the point $\eta(1) \in \mathbb{B}(1)$. We have a map

$$i\colon B^\circ \otimes_{A^\circ} k(1)^+ \to \mathscr{B}(1)^+.$$

The target and the source of i are both free $k(1)^+$-modules of finite rank by [27, Proposition 2.3.5 and Lemma 2.2.17], and clearly i becomes an isomorphism after tensoring with $k(1)$. In particular, the map i is injective. By [27, Proposition 2.3.5] again, it follows that i becomes an isomorphism after tensoring with $k(1^\flat)^+$. Consequently, we have inclusions

$$B^\circ \otimes_{A^\circ} k(1)^+ \hookrightarrow \mathscr{B}(1)^+ \hookrightarrow B^\circ \otimes_{A^\circ} k(1^\flat)^+,$$
$$B^\circ \otimes_{A^\circ} (k(1)^+/\mathfrak{m}k(1)^+) \hookrightarrow \mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+ \hookrightarrow B^\circ \otimes_{A^\circ} (k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+).$$

We shall now prove the second assertion. First, we prove that the ring $\mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+$ is integrally closed in $B^\circ \otimes_{A^\circ} (k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+)$. Let

$$(k(1)^\wedge, k(1)^{\wedge +}) \quad \text{and} \quad (k(1^\flat)^\wedge, k(1^\flat)^{\wedge +})$$

be the completions with respect to the valuation topologies. By [14, Lemma 1.1.10 iii)], we have $k(1)^\wedge \xrightarrow{\cong} k(1^\flat)^\wedge$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{B}(1)^+ \otimes_{k(1)^+} k(1)^{\wedge +} & \longrightarrow & B^\circ \otimes_{A^\circ} k(1^\flat)^{\wedge +} \\ \downarrow & & \downarrow \\ \mathscr{B}(1)^+ \otimes_{k(1)^+} k(1)^\wedge & \xrightarrow{\cong} & \mathscr{B}(1)^+ \otimes_{k(1)^+} k(1^\flat)^\wedge, \end{array}$$

where the vertical maps and the top horizontal map are injective. By using [27, Proposition 1.3.2 (iii)], we see that $\mathscr{B}(1)^+ \otimes_{k(1)^+} k(1)^{\wedge +}$ is integrally closed in $\mathscr{B}(1)^+ \otimes_{k(1)^+} k(1)^\wedge$, and, hence, integrally closed in $B^\circ \otimes_{A^\circ} k(1^\flat)^{\wedge +}$. This implies that $\mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+$ is integrally closed in $B^\circ \otimes_{A^\circ} (k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+)$.

By the assumptions, the ring

$$R := B^\circ \otimes_{A^\circ} (k(1)^+/\mathfrak{m}k(1)^+) \cong B^\sim \otimes_{\kappa[S]} \kappa[S]_{(S)}$$

is the local ring of $\operatorname{Spec} B^\sim$ at the closed point $x \in \operatorname{Spec} B^\sim$. Moreover, the ring

$$B^\circ \otimes_{A^\circ} (k(1^\flat)^+/\mathfrak{m}k(1^\flat)^+) \cong B^\sim \otimes_{\kappa[S]} \kappa(S)$$

is the total ring of fractions of R. Hence, the ring $\mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+$ is the normalization of R. By using [27, (2.4.4) in the proof of Theorem 2.4.3], one can show that there are exactly two maximal ideals of $\mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+$ and the length of $\mathscr{B}(1)^+/\mathfrak{m}\mathscr{B}(1)^+$ as an R-module is one. In other words, the closed point x is an ordinary double point.

The proof of Proposition A.7 is complete. □

We deduce the following result from Proposition A.7, which is used in the proof of Theorem 6.2.

Proposition A.8. *Let $f\colon X \to \mathbb{B}(a,b)$ be a finite étale morphism of adic spaces. We assume that the discriminant function δ_f is linear. Let $t \in |K^\times|$ be an element with $a/b <$*

$t^2 < 1$. *Then we have*

$$f^{-1}(\mathbb{B}(a/t,tb)) \cong \coprod_{i=1}^{n} \mathbb{B}(c_i,1)$$

for some elements $c_i \in |K^\times|$ *with* $c_i < 1$ $(1 \le i \le n)$.

Proof. By Lemma A.4, without loss of generality, we may assume that every connected component of X remains connected after restricting to $\mathbb{B}(a/s,sb)$ for every $s \in |K^\times|$ with $t < s \le 1$. Let $X_1,\dots,X_m$ be the connected components of X, and let $f_i\colon X_i \to \mathbb{B}(a,b)$ be the restriction of f. By Theorem A.6, each discriminant function δ_{f_i} associated with f_i is a continuous, piecewise linear and convex function. Since $\delta_f = \sum_{i=1}^{m} \delta_{f_i}$, it follows that δ_{f_i} is linear for every i. Thus, we may further assume that X is connected.

Let (B,B°) be a complete affinoid ring such that $X = \mathrm{Spa}(B,B^\circ)$. Define g as the composition

$$g\colon X \xrightarrow{f} \mathbb{B}(a,b) \xrightarrow{\Psi} \mathbb{B}(1) = \mathrm{Spa}(K\langle S\rangle).$$

By Lemma A.5, there is only one prime ideal $\mathfrak{q} \subset B^\circ$ lying above the maximal ideal $\mathfrak{m}\mathcal{O}\langle S\rangle + S\mathcal{O}\langle S\rangle \subset \mathcal{O}\langle S\rangle$. Let $x \in \mathrm{Spec}\, B^\sim$ be the closed point corresponding to the prime ideal $\mathfrak{q}$, which is an ordinary double point by Proposition A.7. Let

$$\lambda\colon X = d(\mathrm{Spf}(B^\circ)) \to \mathrm{Spf}(B^\circ)$$

be the specialization map associated with the formal scheme $\mathrm{Spf}(B^\circ)$ (see Section 5.2). By the proof of [4, Proposition 2.3], the interior $\lambda^{-1}(x)^\circ$ of the inverse image $\lambda^{-1}(x)$ in X is isomorphic to the interior $\mathbb{D}(d,1)^\circ$ of $\mathbb{D}(d,1)$ in $\mathbb{B}(1)$ for some element $d \in |K^\times|$ with $d < 1$ as an adic space over $\mathrm{Spa}(K)$, where

$$\mathbb{D}(d,1) := \{x \in \mathbb{B}(1) = \mathrm{Spa}(K\langle T\rangle) \mid d < |T(x)| < 1\} \subset \mathbb{B}(1).$$

We fix such an isomorphism. For every $s \in |K^\times|$ with $t \le s < 1$, we have

$$f^{-1}(\mathbb{B}(a/s,sb)) = g^{-1}(\mathbb{B}(s)) \subset g^{-1}(\mathbb{D}(1)^\circ) = \lambda^{-1}(x)^\circ \cong \mathbb{D}(d,1)^\circ \subset \mathbb{B}(1).$$

Thus, we may consider $f^{-1}(\mathbb{B}(a/s,sb))$ as a connected affinoid open subset of $\mathbb{B}(1)$.

We write $X_t := f^{-1}(\mathbb{B}(a/t,tb))$. Let X_t^c be the closure of X_t in $\mathbb{B}(1)$, which is contained in $g^{-1}(\mathbb{D}(1)^\circ)$. In view of Example A.1 (3), to prove the assertion, it suffices to prove that the complement $X_t^c \backslash X_t$ consists of exactly two points. The map f induces a map

$$f'\colon X_t^c \backslash X_t \to \mathbb{B}(a/t,tb)^c \backslash \mathbb{B}(a/t,tb) = \{\eta(a/t), \eta(tb)'\}.$$

We prove that f' is bijective. Since f is surjective and specialising by [14, Lemma 1.4.5 ii)], it follows that the map f' is surjective. To show that the map f' is injective, it suffices to prove the following claim:

Claim A.9. *The inverse images* $f^{-1}(\eta(a/t))$ *and* $f^{-1}(\eta(tb)')$ *both consist of one point.*

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Proof. Recall that we assume that X remains connected after restricting to $\mathbb{B}(a/s,sb)$ for every $s \in |K^\times|$ with $t < s \le 1$. Thus, by Lemma A.5 and Proposition A.7, the inverse images $f^{-1}(\eta(a/s)')$ and $f^{-1}(\eta(sb))$ both consist of one point for every $s \in |K^\times|$ with $t < s \le 1$. This fact implies that

$$f^{-1}(\mathbb{B}(s_1b,s_2b))$$

is connected for every $s_1,s_2 \in |K^\times|$ with $t \le s_1 < s_2 \le 1$ (indeed, if it is not connected, then there exist at least two points mapped to $\eta(s_2b)$). By applying Lemma A.5 and Proposition A.7 to

$$f^{-1}(\mathbb{B}(tb,b)) \to \mathbb{B}(tb,b),$$

we see that $f^{-1}(\eta(tb)')$ consists of one point. The same arguments show that $f^{-1}(\eta(a/t))$ consists of one point. □

The proof of Proposition A.8 is complete. □

Remark A.10. Here, we prove Proposition A.7 and Proposition A.8 in the context of adic spaces as in [27]. It is probably possible to prove these results (in a more geometric way) by using the methods of [22, 23].

We now give a proof of Theorem 6.2.

Proof of Theorem 6.2. Let $f\colon X \to \mathbb{B}(1)^*$ be a finite étale morphism. Clearly, the discriminant functions on open annuli constructed in Theorem A.6 can be glued to a continuous, piecewise linear and convex function

$$\delta_f\colon [0,\infty) \to \mathbb{R}_{\ge 0}.$$

Moreover, the slopes of δ_f are integers. By [27, Lemma 2.1.10], the function δ_f is bounded above by some positive real number (depending only on the degree of f). It follows that there exists an element $\epsilon_0 \in |K^\times|$ with $\epsilon_0 \le 1$ such that the restriction of δ_f to $[-\log \epsilon_0, \infty)$ is constant. Let $t \in |K^\times|$ be an element with $t < 1$. We put $\epsilon := t\epsilon_0$. Then, for elements $a,b \in |K^\times|$ with $a < b \le \epsilon$, we have

$$f^{-1}(\mathbb{B}(a,b)) \cong \coprod_{i=1}^{n} \mathbb{B}(c_i,d_i)$$

for some elements $c_i,d_i \in |K^\times|$ with $c_i < d_i \le 1$ $(1 \le i \le n)$ by Proposition A.8. If K is of characteristic zero, after replacing ϵ by a smaller one, we can easily show that the restriction $\mathbb{B}(c_i,d_i) \to \mathbb{B}(a,b)$ of f is a Kummer covering by using [27, Claim 2.4.5] (see also the proofs of [22, Theorem 2.2] and [27, Theorem 2.4.3]). □

A.3. Galois coverings and discriminant functions

In this subsection, we prove Theorem 6.3.

Let $a,b \in |K^\times|$ be elements with $a < b \le 1$. Let

$$f\colon X = \mathrm{Spa}(B,B^\circ) \to \mathbb{B}(a,b)$$

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be a finite étale morphism of adic spaces with a complete affinoid ring (B,B°). Let

$$G := \mathrm{Aut}(X/\mathbb{B}(a,b))^{\circ} \cong \mathrm{Aut}(B/A(a,b))$$

be the opposite of the group of $\mathbb{B}(a,b)$-automorphisms on X, or equivalently, the group of $A(a,b)$-automorphisms of B. We assume that f is Galois, that is, $A(a,b)$ coincides with the ring B^G of G-invariants (this is equivalent to saying that the finite étale morphism $\operatorname{Spec} B \to \operatorname{Spec} A(a,b)$ of schemes is Galois; see Remark A.3). In this case, we call G the Galois group of f.

We assume that X is connected. Let $r \in |K^{\times}|$ be an element with $a < r \le b$, and let $x \in f^{-1}(\eta(r))$ be an element. Let

$$\mathrm{Stab}_x := \{g \in G \,|\, g(x) = x\}$$

be the stabiliser of x. Let $k(x)^{\wedge h+}$ (resp. $k(r)^{\wedge h+}$) be the Henselization of the completion of the valuation ring $k(x)^{+}$ (resp. $k(\eta(r))^{+}$). Let $k(x)^{\wedge h}$ and $k(r)^{\wedge h}$ be the fields of fractions of $k(x)^{\wedge h+}$ and $k(r)^{\wedge h+}$, respectively. Then by [17, 5.5], the extension of fields

$$k(r)^{\wedge h} \to k(x)^{\wedge h}$$

is finite and Galois, and we have a natural isomorphism

$$\mathrm{Stab}_x \xrightarrow{\cong} \mathrm{Gal}(k(x)^{\wedge h}/k(r)^{\wedge h}).$$

In [17], Huber defined higher ramification subgroups and the Swan character of the Galois group $\mathrm{Gal}(k(x)^{\wedge h}/k(r)^{\wedge h})$. In [27], Ramero investigated the relation between the discriminant functions and the Swan characters. We are interested in the case where all higher ramification subgroups and the Swan character of $\mathrm{Gal}(k(x)^{\wedge h}/k(r)^{\wedge h})$ are trivial. All we need is the following lemma:

Lemma A.11 ([27, Lemma 3.3.10]). *Let $f\colon X \to \mathbb{B}(a,b)$ be a finite Galois étale morphism such that X is connected. We assume that $\sharp\mathrm{Stab}_x$ is invertible in $\mathcal{O}$ for every $r \in |K^{\times}|$ with $a < r \le b$ and every $x \in f^{-1}(\eta(r))$. Then the discriminant function*

$$\delta_f\colon [-\log b, -\log a] \to \mathbb{R}_{\ge 0}$$

associated with f is constant.

Proof. This follows from the second equality of [27, Lemma 3.3.10]. Indeed, under the assumption, we have $\mathrm{Sw}_x^{\natural} = 0$ for the Swan character $\mathrm{Sw}_x^{\natural}$ attached to $x \in f^{-1}(\eta(r))$ defined in [27, Section 3.3]. □

Finally, we prove Theorem 6.3.

Proof of Theorem 6.3. In fact, we will show that if a locally constant étale sheaf $\mathcal{F}$ with finite stalks on $\mathbb{B}(a,b)$ is tame at $\eta(r) \in \mathbb{B}(a,b)$ for every $r \in |K^{\times}|$ with $a < r \le b$, then, for every $t \in |K^{\times}|$ with $a/b < t^2 < 1$, there exists an integer m invertible in $\mathcal{O}$ such that the restriction $\mathcal{F}|_{\mathbb{B}(a/t,tb)}$ is trivialized by a Kummer covering φ_m.

There is a finite Galois étale morphism $f\colon X \to \mathbb{B}(a,b)$ such that X is connected and $f^{*}\mathcal{F}$ is a constant sheaf. Let G be the Galois group of f. By replacing X by a quotient

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of it by a subgroup of G (this makes sense by Remark A.3), we may assume that the induced homomorphism

$$\rho\colon G \to \mathrm{Aut}(\Gamma(X, f^*\mathcal{F}))$$

is injective. Let $t \in |K^\times|$ be an element with $a/b < t^2 < 1$. By Lemma A.4, we may assume that X remains connected after restricting to $\mathbb{B}(a/s, sb)$ for every $s \in |K^\times|$ with $t < s \leq 1$.

We claim that $\sharp \mathrm{Stab}_x$ is invertible in $\mathcal{O}$ for every $r \in |K^\times|$ with $a < r \leq b$ and every $x \in f^{-1}(\eta(r))$. Let $L(r)$ be a separable closure of $k(x)^{\wedge h}$. It induces a geometric point $\overline{x} \to X$ with support x. Let $\overline{r} \to \mathbb{B}(a,b)$ denote the composition $\overline{x} \to X \to \mathbb{B}(a,b)$. Since $f^*\mathcal{F}$ is a constant sheaf, we have the following identifications

$$\Gamma(X, f^*\mathcal{F}) \cong (f^*\mathcal{F})_{\overline{x}} \cong \mathcal{F}_{\overline{r}}.$$

Recall that we have $\mathrm{Stab}_x \cong \mathrm{Gal}(k(x)^{\wedge h}/k(r)^{\wedge h})$. Via these identifications, the action of $\mathrm{Stab}_x \subset G$ on $\Gamma(X, f^*\mathcal{F})$ is compatible with the action of $\mathrm{Gal}(L(r)/k(r)^{\wedge h})$ on $\mathcal{F}_{\overline{r}}$. Since $\mathcal{F}$ is tame at $\eta(r)$ and ρ is injective, it follows that $\sharp \mathrm{Gal}(k(x)^{\wedge h}/k(r)^{\wedge h})$ is invertible in $\mathcal{O}$. This proves our claim.

By Lemma A.11, it follows that the discriminant function δ_f is constant. By Lemma A.5 and Proposition A.7, there is exactly one point x in $f^{-1}(\eta(b))$. Therefore, the Galois group G is isomorphic to $\mathrm{Stab}_x \cong \mathrm{Gal}(k(x)^{\wedge h}/k(b)^{\wedge h})$.

Theorem 6.3 now follows from [22, Theorem 2.11]. Alternatively, we can argue as follows. By [17, Proposition 2.5, Corollary 2.7, and Corollary 5.4], we see that G is a cyclic group. Let us write $G \cong \mathbb{Z}/m\mathbb{Z}$. We consider $f\colon X \to \mathbb{B}(a,b)$ as a $\mathbb{Z}/m\mathbb{Z}$-torsor. As in the proof of [27, Theorem 2.4.3], since the Picard group of $\mathbb{B}(a,b)$ is trivial, the Kummer sequence gives an isomorphism

$$A(a,b)^\times/(A(a,b)^\times)^m \cong H^1(\mathbb{B}(a,b), \mathbb{Z}/m\mathbb{Z}).$$

Since m is invertible in $\mathcal{O}$, the left-hand side is a cyclic group of order m generated by the coordinate function $T \in A(a,b)^\times$. It follows that every $\mathbb{Z}/m\mathbb{Z}$-torsor over $\mathbb{B}(a,b)$ is the disjoint union of Kummer coverings. This completes the proof of Theorem 6.3. $\square$

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