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A TECHNIQUE TO GENERATE $\mathfrak{m}$-ARY FREE LATTICES FROM FINITARY ONES

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Introduction. Let $\mathfrak{m}$ be an infinite regular cardinal. A poset L is called an $\mathfrak{m}$-lattice if and only if for all $X \subseteq L$ satisfying $0 < |X| < m$, $\bigwedge X$ and $\bigvee X$ exist.

This paper is a part of a sequence of papers, [**5**], [**6**], [7], [**8**], developing the theory of $\mathfrak{m}$-lattices. For a survey of some of these results, see [**9**].

The $\mathfrak{m}$-lattice $D(\mathfrak{m})$ is described in [**6**]; γ denotes the zero and γ' the unit of $D(\mathfrak{m})$. In particular, formulas for $\mathfrak{m}$-joins and meets are given. (We repeat the essentials of this description in Section 4.)

In [**6**] we proved the theorem stated below. Our proof was based on characterization of $F_{\mathfrak{m}}(P)$ (the free $\mathfrak{m}$-lattice on P) due to [**1**]; as a result, our proof was very computational.

In this paper, we shall present a non-computational proof. This proof relies on the description of $D(\mathfrak{m})$ borrowed from [**6**], and on the finitary case: the description of the free lattice on H from [**10**]. (The proof in [**6**] does not rely on the finitary case.)

THEOREM. *The* $\mathfrak{m}$-*lattice* $D(\mathfrak{m}) - \{\gamma, \gamma'\}$ *is the free* $\mathfrak{m}$-*lattice on* H.

The universal algebraic background of the present proof is given in Section 1. Next, in Section 2, we generalize the concept of partial lattices to $\mathfrak{m}$-lattices. Some immediate applications of these results are presented in Section 3; these are applied in Section 5. $D(\mathfrak{m})$ is described in Section 4. The proof of the theorem is given in Section 5.

1. Some universal algebraic lemmas. We recall some concepts from [**3**]. Let **K** be a variety (equational class) of algebras of some finitary or infinitary type. For $\mathfrak{A} = \langle A; F \rangle \in \mathbf{K}$ and $H \subseteq A$, we define a *relative algebra* $\mathfrak{H} = \langle H; F \rangle$ of $\mathfrak{A}$ as follows: if $f \in F$, $a_0, a_1, \ldots \in H$ and $f(a_0, a_1, \ldots) = a \in H$ in $\mathfrak{A}$, then (and only then) $f(a_0, a_1, \ldots)$ is defined on H and equals a. A *partial* **K**-*algebra* is defined as a relative algebra of some $\mathfrak{A} \in \mathbf{K}$. Let $\mathbf{K}(\tau)$ be the class of all algebras of type τ. Then a partial algebra of type τ is a partial $\mathbf{K}(\tau)$-algebra, and vice versa.

If $\mathfrak{B} = \langle B; F \rangle$ is a partial algebra with the same type as that of **K**, then $F(\mathfrak{B})$ denotes the free **K**-algebra generated by $\mathfrak{B}$. The canonical map of $\mathfrak{B}$

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into $F(\mathfrak{B})$ is not necessarily one-to-one; if it is one-to-one, then it is an embedding of $\mathfrak{B}$ into $F(\mathfrak{B})$. It is an isomorphism if and only if $\mathfrak{B}$ is a partial $\mathbf{K}$-algebra; in this case, $\mathfrak{B}$ is isomorphic to the relative algebra of $F(\mathfrak{B})$ on the image of B. The following lemma is obvious.

LEMMA 1. *Let $\mathfrak{B}$ be a relative $\mathbf{K}$-algebra, $B \subseteq F(\mathfrak{B})$, and $B \subseteq C \subseteq F(\mathfrak{B})$. Let $\mathfrak{C}$ be the relative algebra of $F(\mathfrak{B})$ on C. If $\mathfrak{C}$ is generated by B, then $F(\mathfrak{B}) \cong F(\mathfrak{C})$ in the natural way.*

Let $\mathfrak{B} = \langle B; F\rangle$ be a partial algebra, $f \in F$, $a_0, \ldots \in B$ such that $f(a_0, \ldots)$ is not defined in $\mathfrak{B}$. We define a *one-point extension* $\mathfrak{B}^p$ of $\mathfrak{B}$ as follows: $B^p = B \cup \{p\}$; all partial operations are the same on $\mathfrak{B}$ and $\mathfrak{B}^p$ except that we add $\langle a_0, \ldots \rangle$ to the domain of f, and $f(a_0, \ldots) = p$.

The next lemma is again trivial.

LEMMA 2. *Let $\mathfrak{B}$ be a partial $\mathbf{K}$-algebra and let $\mathfrak{B}^p$ be a one-point extension of $\mathfrak{B}$. Then $F(\mathfrak{B}) \cong F(\mathfrak{B}^p)$ in the natural way.*

Note that, as a rule, $\mathfrak{B}^p$ is not a partial $\mathbf{K}$-algebra.

Generalizing this construction, we can define $\mathfrak{B}^P$ for a set of points P and for each $p \in P, f_p$, and $a_0^p, \ldots \in B$.

An immediate consequence of Lemma 2 is the following:

LEMMA 3. *Assume that there is an $\mathfrak{A} \in K$ and a homomorphism φ of $\mathfrak{B}^P$ into $\mathfrak{A}$ such that for all $a \in B$, $p_1, p_2 \in P$, $p_1 \neq p_2$, we have*

$$a\varphi \neq p_i\varphi,\ i = 1, 2 \quad \text{and} \quad p_1\varphi \neq p_2\varphi.$$

Then $F(\mathfrak{B}) \cong F(\mathfrak{B}^P)$ in the natural way.

Now let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be partial $\mathbf{K}$-algebras, $A_0 \cap A_1 = A_2$ such that $\mathfrak{A}_2$ as a relative algebra of $\mathfrak{A}_0$ is the same as $\mathfrak{A}_2$ as a relative algebra of $\mathfrak{A}_1$. We shall say that $\mathfrak{A}_0$ and $\mathfrak{A}_1$ *can be strongly amalgamated over* $\mathfrak{A}_2$, if there is an algebra $\mathfrak{A}_3 \in \mathbf{K}$ of which both $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are relative algebras and $A_0 \cap A_1 = A_2$ in $\mathfrak{A}_3$.

LEMMA 4. *Let $\mathfrak{A}$ be a partial $\mathbf{K}$-algebra, let $A' \subseteq A$, and let $\mathfrak{A}'$ be the corresponding relative algebra of $\mathfrak{A}$. If $\mathfrak{A}$ and $F(\mathfrak{A}')$ can be strongly amalgamated over $\mathfrak{A}'$, then the subalgebra $[A']$ of $F(\mathfrak{A})$ generated by A' is naturally isomorphic to $F(\mathfrak{A}')$.*

Proof. Let $\mathfrak{A}'' \in \mathbf{K}$ strongly amalgamate $\mathfrak{A}$ and $F(\mathfrak{A}')$. Let φ be the extension of the identity map on A to a homomorphism of $F(\mathfrak{A})$ into $\mathfrak{A}''$. Obviously, φ maps $[A']$ onto $F(\mathfrak{A}')$. We get an inverse map by the freeness of $F(\mathfrak{A}')$, and hence the isomorphism.

2. Partial $\mathfrak{m}$-lattices. It is clear that we can define a type of algebras such that $\mathfrak{m}$-lattices can be regarded as algebras of this type.

Let L be an $\mathfrak{m}$-lattice, $Q \subseteq L$, $Q \neq \emptyset$, and we restrict the $\vee$ and $\wedge$ of L to Q as follows: if $X \subseteq Q$, $0 < |X| < \mathfrak{m}$, and $x = \wedge X$ (formed in L) is in Q, then $\wedge X$ is defined in Q and $\wedge X = x$ in Q; otherwise, $\wedge X$ is not defined; $\vee X$ is defined similarly. Then Q with $\wedge$ and $\vee$ is called a *partial* $\mathfrak{m}$-*lattice*; Q is a *relative* $\mathfrak{m}$-*sublattice* of L. (For $\mathfrak{m} = \aleph_0$, see [**4**] for a detailed discussion of partial lattices.)

The partial $\mathfrak{m}$-lattice Q is an example of an $\mathfrak{m}$-structure defined as follows. Given a partially ordered set P, we can make P into an (infinitary) partial algebra of the type of partial $\mathfrak{m}$-lattices as follows: we designate two families of subsets of P:$\mathfrak{M}$ and $\mathfrak{J}$; if $X \in \mathfrak{M}$, then $0 < |X| < \mathfrak{m}$ and $\inf X$ exists in P; if $X \in \mathfrak{J}$, then $0 < |X| < \mathfrak{m}$ and $\sup X$ exists in P. We define $\vee$ and $\wedge$ on P as follows:

$$\wedge X = x \text{ if and only if } X \in \mathfrak{M} \text{ and } x = \inf X$$

$$\vee X = x \text{ if and only if } X \in \mathfrak{J} \text{ and } x = \sup X.$$

We denote this partial algebra by $\langle P, \mathfrak{M}, \mathfrak{J} \rangle$ and call it an $\mathfrak{m}$-*structure*. Note that for the same poset P, there are many $\mathfrak{m}$-structures on P.

Given an $\mathfrak{m}$-structure $\langle P, \mathfrak{M}, \mathfrak{J} \rangle$ and $I \subseteq P$, we call I an *ideal* if and only if $x, y \in P$, $x \leqq y$, and $y \in I$ imply that $x \in I$; and $X \in \mathfrak{J}$, $X \subseteq I$ imply that $\sup X \in I$. For $X \subseteq P$, let $(X]_{\mathfrak{J}}$ denote the ideal generated by X; if $X = \{x\}$ we write $(x]_{\mathfrak{J}}$ for $(\{x\}]_{\mathfrak{J}}$.

Observe that every partial $\mathfrak{m}$-lattice P is an $\mathfrak{m}$-structure, $\langle P, \mathfrak{M}, \mathfrak{J} \rangle$, in the natural way. The corresponding ideal concept is called $\mathfrak{m}$-*ideal*. The $\mathfrak{m}$-ideal generated by X will be denoted by $(X]_{\mathfrak{m}}$; if $X = \{x\}$, we write $(x]_{\mathfrak{m}}$ for $(\{x\}]_{\mathfrak{m}}$. If $|X| < \mathfrak{m}$, then the $\mathfrak{m}$-ideal $(X]_{\mathfrak{m}}$ is called $\mathfrak{m}$-*generated*.

LEMMA 5. *An* $\mathfrak{m}$-*structure* $\langle P, \mathfrak{M}, \mathfrak{J} \rangle$ *is a partial* $\mathfrak{m}$-*lattice if and only if the following conditions are satisfied*:

(i) *For every* $u, v \in P$, *if* $u \leqq v$, *then* $\{u, v\} \in \mathfrak{M}$ *and* $\{u, v\} \in \mathfrak{J}$;

(ii) *For* $X \subseteq P$, $0 < |X| < \mathfrak{m}$, *if* $(X]_{\mathfrak{J}} = (x]_{\mathfrak{J}}$, *then* $X \in \mathfrak{J}$; *and dually for* $\mathfrak{M}$.

The proof of this lemma is analogous to the proof in the finitary case due to N. Funayama [**2**], see also Theorem 1.5.20 in [**4**]. The present formulation seems to be new even in the finitary case.

LEMMA 6. *For any* $\mathfrak{m}$-*structure* $\langle P, \mathfrak{M}, \mathfrak{J} \rangle$, *there exists a smallest partial* $\mathfrak{m}$-*lattice* $\langle P; \wedge, \vee \rangle$ *containing* $\langle P, \mathfrak{M}, \mathfrak{J} \rangle$ *in the obvious sense.*

Proof. This is clear from Lemma 5; first, we add to $\mathfrak{M}$ and $\mathfrak{J}$ the singletons and doubletons needed in (i) containing $\mathfrak{M}_0$ and $\mathfrak{J}_0$. Then we add to $\mathfrak{M}_0$ and $\mathfrak{J}_0$ all subsets of P required by (ii), obtaining $\mathfrak{M}_1$, $\mathfrak{J}_1$. Now (ii) will have to be applied again to augment $\mathfrak{M}_1$, $\mathfrak{J}_1$. After at most $|P|^{\mathfrak{m}}$ steps we obtain $\overline{\mathfrak{M}}$, $\overline{\mathfrak{J}}$ satisfying (i) and (ii), hence $\langle P, \overline{\mathfrak{M}}, \overline{\mathfrak{J}} \rangle$ is the smallest partial $\mathfrak{m}$-lattice containing $\langle P, \mathfrak{M}, \mathfrak{J} \rangle$.

The next lemma follows from Lemmas 5 and 6.

LEMMA 7. *The free* $\mathfrak{m}$*-lattice generated by the* $\mathfrak{m}$*-structure* $\langle P, \mathfrak{M}, \mathfrak{J}\rangle$ *is isomorphic to the free* $\mathfrak{m}$*-lattice generated by the smallest partial* $\mathfrak{m}$*-lattice containing* $\langle P, \mathfrak{M}, \mathfrak{J}\rangle$.

Finally, we observe that when generating the free $\mathfrak{m}$-lattice, we can first generate the free lattice. Let $\mathfrak{A}$ be a partial lattice and $F(\mathfrak{A})$ the free lattice generated by $\mathfrak{A}$. We make $F(\mathfrak{A})$ into an $\mathfrak{m}$-structure $\langle F(\mathfrak{A}), \mathfrak{M}, \mathfrak{J}\rangle$ as follows: $\mathfrak{M}$ and $\mathfrak{J}$ both consist of the nonempty finite subsets of $F(\mathfrak{A})$.

LEMMA 8. *The free* $\mathfrak{m}$*-lattice generated by* $\mathfrak{A}$ *and by* $\langle F(\mathfrak{A}), \mathfrak{M}, \mathfrak{J}\rangle$ *are naturally isomorphic.*

In other words, we can form first finitary meets and joins freely, before we have to worry about infinitary meets and joins. The proof is obvious.

For a partial $\mathfrak{m}$-lattice $\mathfrak{A}$ or an $\mathfrak{m}$-structure $\mathfrak{A} = \langle P, \mathfrak{M}, \mathfrak{J}\rangle$, the free $\mathfrak{m}$-lattice on $\mathfrak{A}$ will be denoted by $F_{\mathfrak{m}}(\mathfrak{A})$. For a poset P, there is a smallest partial $\mathfrak{m}$-lattice $\mathfrak{P} = \langle P, \mathfrak{M}, \mathfrak{J}\rangle$; let $F_{\mathfrak{m}}(\mathfrak{P})$ denote the free $\mathfrak{m}$-lattice generated by it. Obviously, $F_{\mathfrak{m}}(\mathfrak{P})$ is the same as $F_{\mathfrak{m}}(P)$.

3. Chains and linear sums. Let Q be a chain. As the simplest application of the results of Sections 1 and 2, we determine the free $\mathfrak{m}$-lattice on Q. Observe that the finitary case is trivial.

Let $\tilde{Q} = Q \cup I \cup D$, where I is the set of nonprincipal $\mathfrak{m}$-generated ideals of Q ordered by $\subseteq$ D, is the set of nonprincipal $\mathfrak{m}$-generated dual ideals of Q ordered by $\supseteq$. We define the partial order on $\tilde{Q}$ in the obvious way:

let $a \in Q$ and $b \in I$, $a \leqq b$ means that $a \in b$, and $b \leqq a$ means that $b \subseteq (a]$;

let $a \in Q$ and $b \in D$; we use the dual definition;

let $a \in I$ and $b \in D$; $a < b$ if and only if $x < y$ in Q for all $x \in I$ and $y \in D$;

$b < a$ if and only if $a \cap b \neq \emptyset$.

LEMMA 9. $\tilde{Q}$ *is an* $\mathfrak{m}$*-chain.*

Proof. Let $X \subseteq \tilde{Q}$, $0 < |X| < \mathfrak{m}$. We show that $\bigvee X$ exists in $\tilde{Q}$. We can assume that $X \subseteq Q$, or $X \subseteq I$, or $X \subseteq D$. If $X \subseteq Q$, then let $a = (X]$. We show that $a = \bigvee X$ in $\tilde{Q}$. Indeed, if b is an upper bound of X in $\tilde{Q}$, and $b \in Q \cup I$, then $a \leqq b$ is obvious; if $b \in D$, $b = [Y)$, $0 < |Y| < \mathfrak{m}$, in Q, then $x < y$ for all $x \in X$ and $y \in Y$, hence, $a < y$ for all $y \in Y$, implying that $a < b$.

If $X \subseteq I$, then $a = \bigcup (x \mid x \in I)$ is an $\mathfrak{m}$-generated ideal by the regularity of $\mathfrak{m}$. If a is nonprincipal, then $a \in I$ and a is obviously the least upper bound of X. If a is principal, $a = (a_0]$, $a_0 \in Q$, and a_0 is the least upper bound of X.

If $X \subseteq D$, we can assume that X has no largest element and X is well-ordered, $X = \{d_i \mid i < \mathfrak{n}\}$, where $\mathfrak{n} < \mathfrak{m}$ and $d_i < d_j$ (i.e., $d_1 \supset d_j$) for $i < j$. For each $i < \mathfrak{n}$, choose $a_i \in d_i - d_{i+1}$. The ideal a of Q generated by the a_i, $i < \mathfrak{n}$, is $\mathfrak{m}$-generated, hence $a \in \widetilde{Q}$. It is easily seen that a is the least upper bound of X in $\widetilde{Q}$.

By duality, $\bigwedge X$ also exists, hence $\widetilde{Q}$ is an $\mathfrak{m}$-chain.

LEMMA 10. *$\widetilde{Q}$ is the free $\mathfrak{m}$-lattice on Q.*

Proof. Let us define an $\mathfrak{m}$-structure on $\widetilde{Q}$: let both $\mathfrak{J}$ and $\mathfrak{M}$ consist of all subsets X of $\widetilde{Q}$ with $0 < |X| < \mathfrak{m}$. This makes $\widetilde{Q}$ into an $\mathfrak{m}$-structure generated by Q as discussed in Lemma 2. The free $\mathfrak{m}$-lattice on Q is the same as the free $\mathfrak{m}$-lattice on this partial $\mathfrak{m}$-lattice on $\widetilde{Q}$. However, the computations of Lemma 9 show that the smallest partial $\mathfrak{m}$-lattice on this $\mathfrak{m}$-structure is the $\mathfrak{m}$-chain $\widetilde{Q}$. So we can apply Lemmas 7 and 8 to conclude that the $\mathfrak{m}$-chain $\widetilde{Q}$ is the free $\mathfrak{m}$-lattice on Q.

A similar application is to linear sums. Let Q be a chain and let P_i, $i \in Q$, be posets. Let $\widetilde{Q}$ denote the free $\mathfrak{m}$-lattice (chain) on Q. We now describe the free $\mathfrak{m}$-lattice on the linear sum P of the P_i, $i \in Q$.

LEMMA 11. *For $i \in \widetilde{Q}$, let us define the poset Q_i:*
$Q_i = F_{\mathfrak{m}}(P_i)$ for $i \in Q$;
Q_i is a singleton for $i \in \widetilde{Q} - Q$.
Then $F_{\mathfrak{m}}(P)$ is the linear sum of the Q_i, $i \in \widetilde{Q}$.

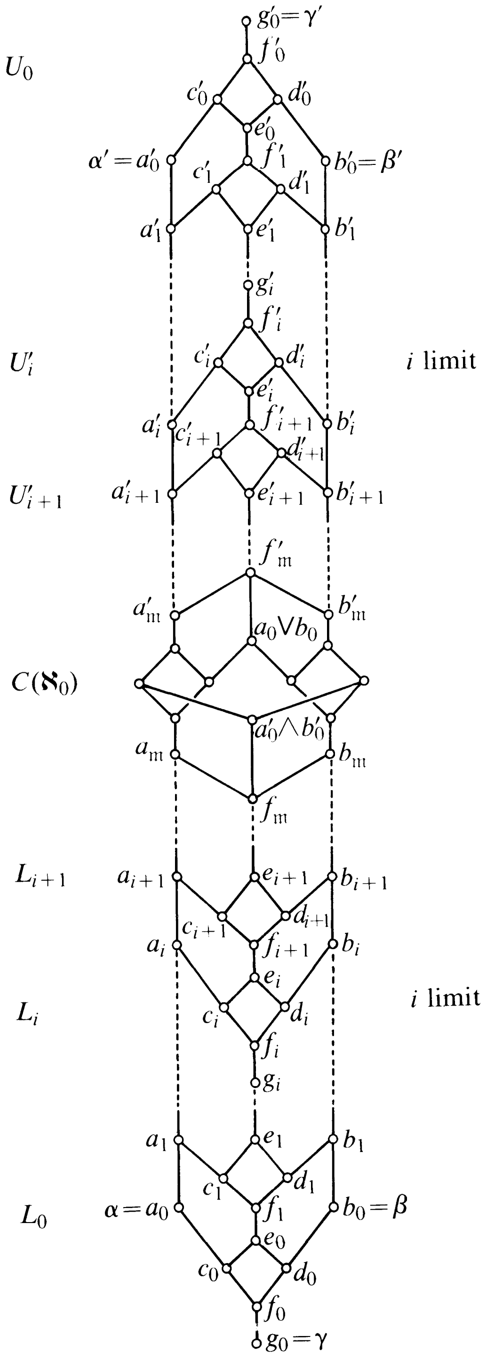
Proof. Let $\widetilde{P}$ stand for the linear sum of the Q_i, $i \in \widetilde{Q}$. Then $P \subseteq \widetilde{P}$. Let P^+ be the linear sum of the P_i for $i \in Q$ and the singleton Q_i for $i \in \widetilde{Q} - Q$. We can argue as in Lemmas 9 and 10 (the special case that all $|P_i| = 1$), that the free $\mathfrak{m}$-lattice on P and P^+ are the same.

For each $i \in I$, we can use Lemma 4 to show that, in P^+, we can replace P_i with $F_{\mathfrak{m}}(P_i)$. The resulting $\mathfrak{m}$-structure $\mathfrak{P}$ has $\widetilde{P}$ as the underlying poset; $\mathfrak{J}$ and $\mathfrak{M}$ consist of all subsets $X \subseteq \widetilde{P}$ satisfying $0 < |X| < \mathfrak{m}$, and $X \subseteq \widetilde{Q}$ or $X \subseteq F_{\mathfrak{m}}(P_i)$ for some i. However, the smallest partial $\mathfrak{m}$-lattice containing $\mathfrak{P}$ is the $\mathfrak{m}$-lattice $\widetilde{P}$. We apply again Lemmas 7 and 8 to conclude that $\widetilde{P} = F_{\mathfrak{m}}(P)$.

4. The $\mathfrak{m}$-lattice $D(\mathfrak{m})$. Let $\mathfrak{m}$ be a regular cardinal, $\mathfrak{m} > \aleph_0$. In this section, we sketch the definition of the complete lattice $D(\mathfrak{m})$. For a more detailed description, see **[6]**.

First, let $C(\mathfrak{m})$ be the lattice of Figure 1.

For every successor ordinal $j < \mathfrak{m}$, there is a lower j-th level of 6 elements $L_j = \{a_j, b_j, c_j, d_j\ e_j, f_j\}$, and for every limit ordinal $i < \mathfrak{m}$ (including $i = 0$), there is a lower i-th level of 7 elements $L_i = \{a_i, b_i, c_i, d_i, e_i, f_i, g_i\}$. These elements are ordered as shown in Figure 1. There is also an upper i-th level U_i for each $i < \mathfrak{m}$, defined dually and denoted by the same



The $\mathfrak{m}$-lattice $C(\mathfrak{m})$

Figure 1

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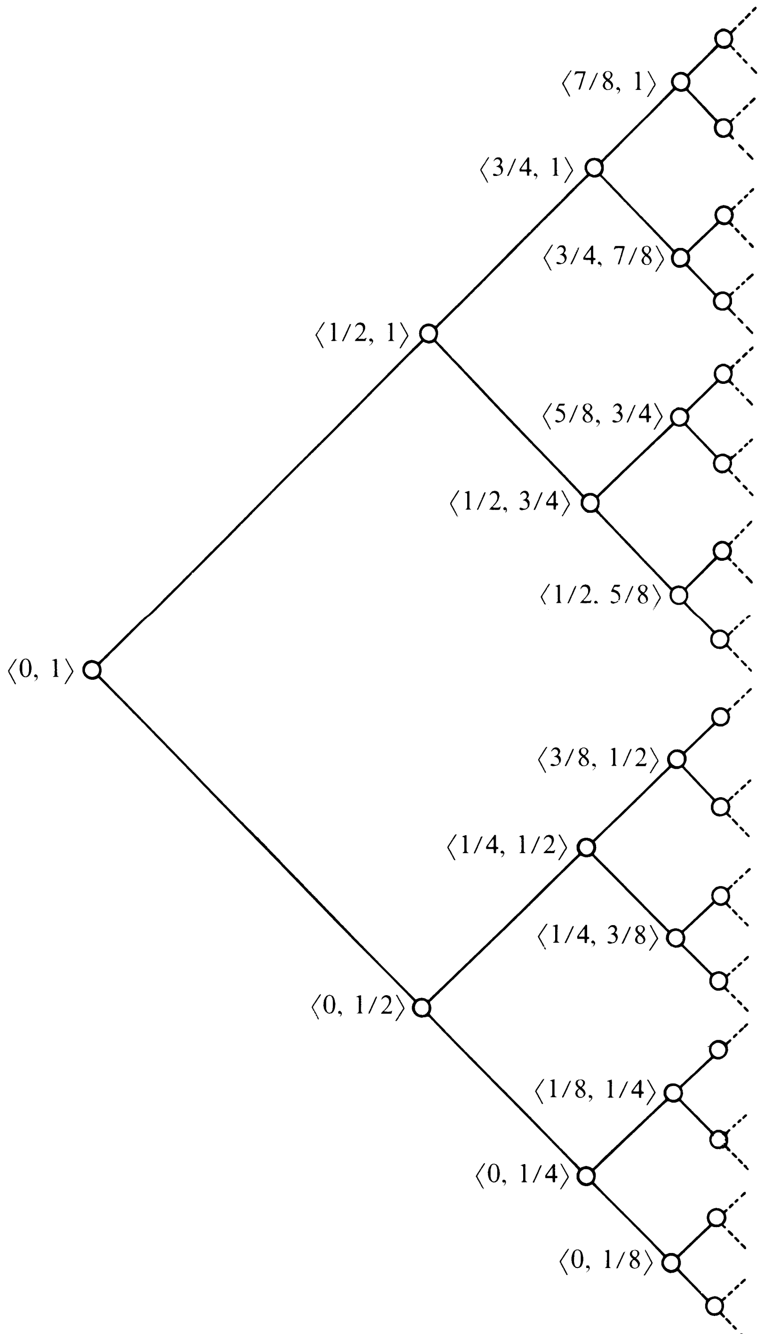

The lattice A
Figure 2

letters with primes. For convenience, we also label 6 elements of $C(\mathfrak{m})$ with Greek letters: $\alpha = a_0$, $\alpha' = a'_0$, $\beta = b_0$, $\beta' = b'_0$, $\gamma = g_0$, $\gamma' = g'_0$.

$C(\mathfrak{m}) - \{\gamma, \gamma'\}$ is $\mathfrak{m}$-generated by α, α', β, β'.

The second building block of $D(\mathfrak{m})$ is the lattice A of Figure 2, first described in [**10**]. Let J be the set of dyadic rationals r that satisfy $0 \leqq r \leqq 1$. Every $r \in J$, $r \neq 0$, has a unique representation, the *normal form*, $r = a \cdot 2^{-n}$, where a is an odd integer; n is the *order* or r; in notation, $n = \operatorname{ord}(r)$. By convention, $\operatorname{ord}(0) = 0$.

We define A as a subposet of J^2 with the product order:

$$A = \{ \langle r, s\rangle \mid r < s \text{ and } s - r = 2^n, n \geqq \max \{\operatorname{ord}(r), \operatorname{ord}(s) \}.$$

For $t \in J$, let us call the set of $a \in A$ of the form $\langle t, s\rangle$ the $x = t$ *line in* A, and define the $y = t$ *line* similarly. $\langle r, r + 2^{-\operatorname{ord}(r)}\rangle$ is the largest element on the $x = r$ line, and $\langle s - 2^{-\operatorname{ord}(s)}, s\rangle$ is the smallest element on the $y = s$ line.

Each $a \in A$ has a right upper cover a^*:

$$\langle r, s\rangle^* = \langle (r + s)/2, s\rangle.$$

Similarly, the left upper cover $^*\langle r, s\rangle$ exists and equals $\langle r, s + 2^{-\operatorname{ord}(s)}\rangle$ when $\operatorname{ord}(r) < \operatorname{ord}(s)$.

Let a and b be incomparable elements of A, with a to the left of b. The join of a and b is the least element on the y-line through a that is greater than b.

Finally, we define

$$B = \{ \langle r, s\rangle \mid \langle s, r\rangle \in A\},$$

a subposet of J^2. Clearly, B is a lattice and its diagram is obtained by reflecting Figure 2 about a vertical line.

Let I be the real interval $[0, 1]$, and recall that J denotes the subset of I consisting of dyadic rationals. For each $t \in J$, we take a copy C_t of $C(\mathfrak{m})$, with bounds γ_t and γ'_t, and generators α_t, α'_t, β_t, β'_t. For each $t \in I$ which is not a dyadic rational, $C_t = \{\gamma_t, \gamma'_t\}$ is the two-element chain with $\gamma_t < \gamma'_t$. We define C as the linear sum of the C_t, $t \in I$. Since I is complete and each C_t is complete, C is a complete lattice.

We define $D(\mathfrak{m}) = A \cup B \cup C$, partially ordered as follows (see Figures 3 and 4): Let

$$\langle r, s\rangle \in A, \langle t, u\rangle \in B, v \in I, p \in C_v;$$

$$\langle r, s\rangle < \langle t, u\rangle \text{ if and only if } s < u;$$

$$\langle r, s\rangle > \langle t, u\rangle \text{ if and only if } r > t;$$

$$\langle r, s\rangle < p \text{ if and only if } s < v \text{ holds, or } s = v \text{ and } \alpha_v \leqq p \text{ hold;}$$

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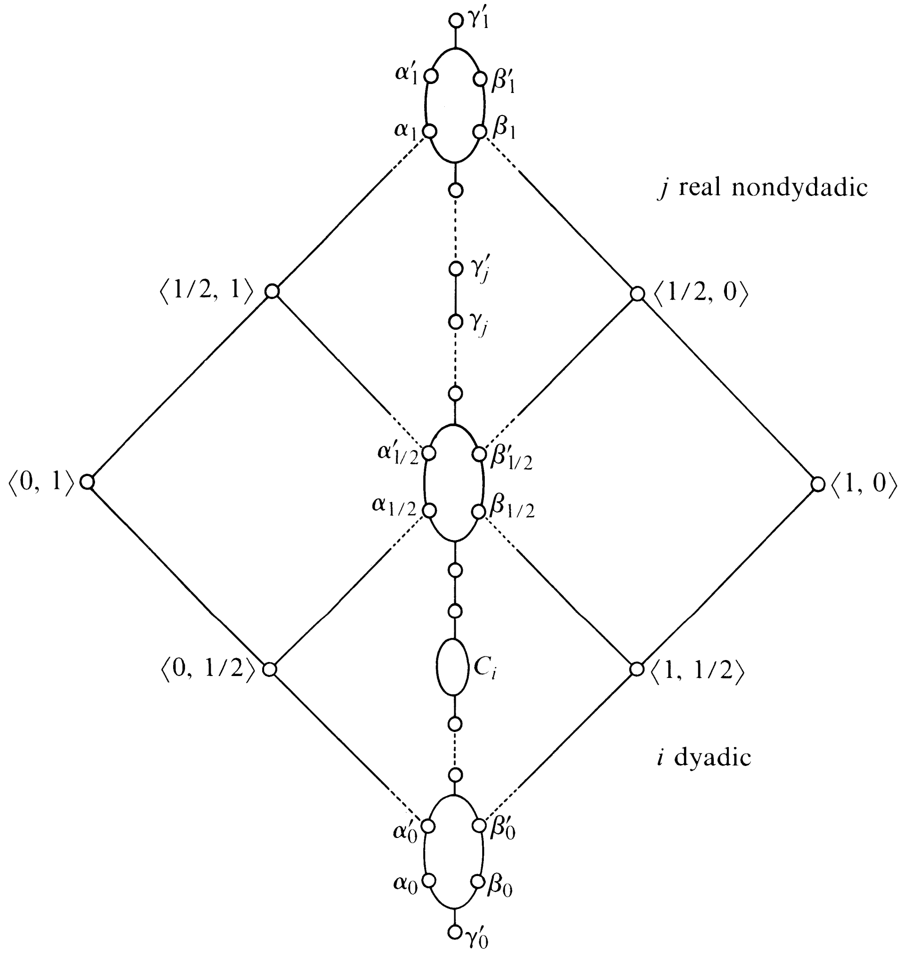


The $\mathfrak{m}$-lattice $D(\mathfrak{m})$

Figure 3

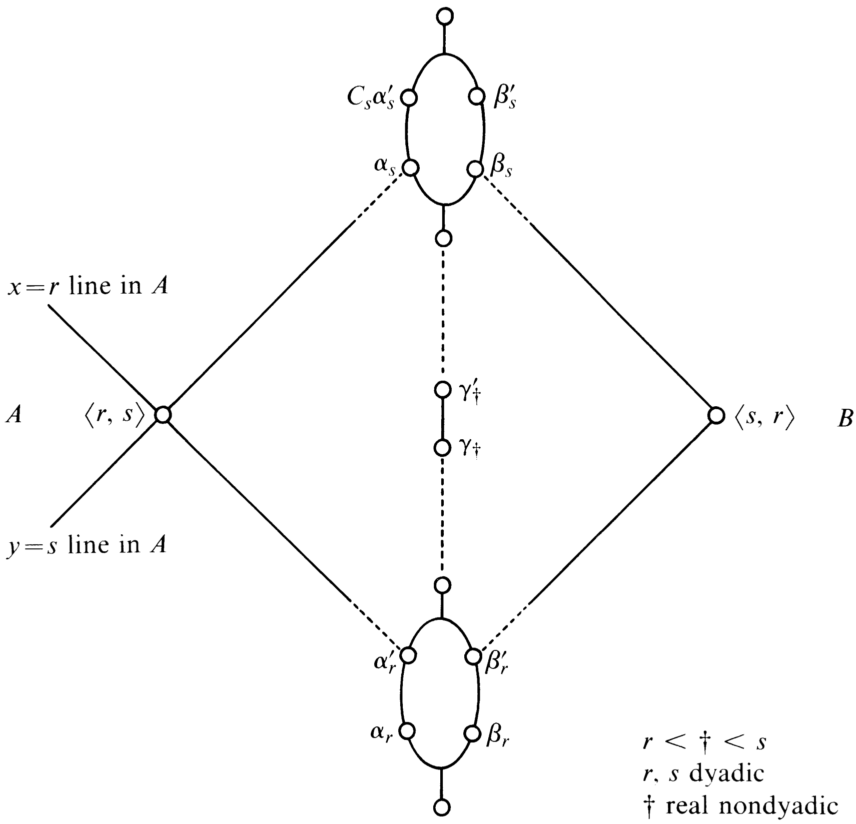
$$\langle r, s\rangle > p \text{ if and only if } r > v \text{ holds, or } r = v \text{ and } \alpha'_v \geqq p \text{ hold;}$$

$$\langle t, u\rangle < p \text{ if and only if } t < v \text{ holds, or } t = v \text{ and } \beta_v \leqq p \text{ hold;}$$

$$\langle t, u\rangle > p \text{ if and only if } u > v \text{ holds, or } u = v \text{ and } \beta'_v \geqq p \text{ hold.}$$

It is easily seen that $D(\mathfrak{m})$ is a poset.

It is not difficult to show that $D(\mathfrak{m})$ is a lattice, and that each of A, B, and C is a sublattice of $D(\mathfrak{m})$. For $\langle r, s\rangle \in A$, $\langle t, u\rangle \in B$, $v \in I$, and $p \in C_v$, we give the formulas for joining pairs:



Details of $D(m)$

Figure 4

(a) $\langle r, s\rangle \vee p$ is

(i) $\alpha_s \vee p \in C$, where the join is formed in C, if $s \leqq v$;

(ii) $\langle r, s\rangle$, if $r > v$, or $r = v$ and $p \leqq \alpha_r'$ in C_v;

(iii) the least $\langle w, s\rangle$ such that $w > v$, if $r \leqq v < s$ and $p \nleqq \alpha_v'$ in C_v;

(iv) the least $\langle w, s\rangle$ such that $w \geqq v$, if $r \leqq v < s$ and $p \leqq \alpha_v'$ in C_v;

(b) $\langle r, s\rangle \vee \langle t, u\rangle$ is

(i) $\langle t, u\rangle$, if $s < u$;

(ii) $\langle r, s\rangle$, if $t < r$;

(iii) the least $\langle w, s\rangle$ on the $y = s$ line in A such that $w > t$, if $s > t$;

(iv) the least $\langle t, w\rangle$ on the $x = t$ line in B such that $w > s$, if $s > t$;

(v) $\alpha_s \vee \beta_s$, if $s = t$, where the join is formed in C_s.

To show that $D(\mathfrak{m})$ is a complete lattice, it suffices to find $\bigvee X$ for a nonempty subset X of A. (The formula is similar for B and we already know that C is complete.) Let X_1 and X_2 be the first and the second projections of X, and form $u = \bigvee X_1$ and $v = \bigvee X_2$ in I.

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If $u < v$, then $v \in J$, and $\bigvee X$ is the least element of A on the $y = v$ line whose first coordinate is $\geqq u$.

If $u = v$, then

$$\bigvee X = \begin{cases} \gamma_u \text{ if } u = v \text{ and } u \notin X_2; \\ \alpha_u \text{ if } u = v \text{ and } u \in X_2. \end{cases}$$

$D(\mathfrak{m}) - \{\gamma_0, \gamma_1'\}$ is $\mathfrak{m}$-generated by $\alpha_0, \beta_0, \langle 0, 1\rangle, \langle 1, 0\rangle, \alpha_1', \beta_1'$.

5. $D(\mathfrak{m})$ as an $\mathfrak{m}$-structure. Let $P = D(\mathfrak{m}) - \{\gamma_0, \gamma_1'\}$ be the partially ordered set underlying $D(\mathfrak{m}) - \{\gamma_0, \gamma_1'\}$.

For a dyadic rational i, $0 \leqq i \leqq 1$, let C_i^{fin} be the 16 element sublattice $C(\aleph_0)$ of $C(\mathfrak{m})$. Let

$$P_0 = A \cup B \cup C^{\text{fin}},$$

where C^{fin} is the union of all C_i^{fin} where i is a dyadic rational, $0 \leqq i \leqq 1$. We know that P_0 is a sublattice of $D(\mathfrak{m})$. By **[10]**, P_0 is the free lattice generated by

$$H = \{\alpha_0, \beta_0, \alpha_1', \beta_1', \langle 1, 0\rangle, \langle 0, 1\rangle\}.$$

By Lemma 7, $FL_{\mathfrak{m}}(H)$ is isomorphic to the free $\mathfrak{m}$-lattice generated by $\langle P_0, \underline{\text{Fin}}, \underline{\text{Fin}}\rangle$, where $\underline{\text{Fin}}$ is the family of finite nonempty subsets of P_0.

Let P_1 be an extension of P_0 in the style of Lemma 3: We add to P_0 all $\alpha_i, \beta_i, \alpha_i', \beta_i', i \in J$; we define α_i as the $\mathfrak{m}$-join of the $y = i$ line in A; $\alpha_i', \beta_i, \beta_i'$ are defined analogously. To apply Lemma 3 we have to find an $\mathfrak{m}$-lattice where all these elements are distinct; of course, $D(\mathfrak{m})$ does the trick.

Now we apply Lemma 4 to P_1 and C. By Lemma 4, $P = P_1 \cup C$ as an $\mathfrak{m}$-structure $\langle P, \mathfrak{M}, \mathfrak{J}\rangle$ generates the same free $\mathfrak{m}$-lattice as H. $\langle P, \mathfrak{M}, \mathfrak{J}\rangle$ is defined as follows:

1. All finite nonempty subsets of P_0 are in $\mathfrak{M}$ and $\mathfrak{J}$.
2. The $y = i$ line in A is in $\mathfrak{J}$ (and analogously for $\mathfrak{M}$).
3. All subsets X of C are in $\mathfrak{J}$ and $\mathfrak{M}$ provided that $0 < |X| < \mathfrak{m}$.

Now the crucial statement is:

LEMMA 12. *The smallest partial* $\mathfrak{m}$-*lattice containing* $\langle P, \mathfrak{M}, \mathfrak{J}\rangle$ *is the* $\mathfrak{m}$-*lattice:* $D(\mathfrak{m}) - \{\gamma_0, \gamma_1'\}$.

It is clear, by Lemma 8, that Lemma 12 implies the theorem since the free $\mathfrak{m}$-lattice generated by an $\mathfrak{m}$-lattice is an $\mathfrak{m}$-lattice.

Proof of Lemma 12. By duality and Lemma 5, it is sufficient to prove the following statement:

For every subset X of P with $0 < |X| < \mathfrak{m}$ and $a = \sup X$, we have $(X]_I = (a]$.

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A, B, $A \cup B$, and C are sublattices of P since all finite sets are in $\mathfrak{M}$ and $\mathfrak{J}$. Thus, it is sufficient to verify the above statement in the following cases:

1. $X = \{x_1, x_2\}$, and x_1, x_2 are incomparable:
(a) $x_1 \in A$, $x_2 \in C$;
(b) $x_1 \in B$, $x_2 \in C$.
2. X is an infinite chain:
(a) $X \subseteq A$;
(b) $X \subseteq B$;
(c) $X \subseteq C$.

By the symmetry of $D(\mathfrak{m})$, it is enough to consider (1a), (2a), and (2c). Of these, (2c) is trivial, since all such X are in $\mathfrak{M}$ and $\mathfrak{J}$.

Case (1a). Let $x_1 = \langle r, s\rangle$ and $x_2 = p$ be given as in Section 4 in the description of the join in $D(\mathfrak{m})$. We proceed by subcases (i)-(iv) corresponding to part (a) of the join definition.

(i) In this case, $s \leqq v$. Let

$$\langle r, s\rangle = \langle r_1, s\rangle < \langle r_2, s\rangle < \dots$$

be the $y = s$ line in A. We prove by induction that

$$\langle r_i, s\rangle \in (\{\langle r, s\rangle, p\}]_J = I.$$

This holds for $i = 1$ by definition. For $i = 2$, observe that $r_1 < r_2 < s$, hence $\alpha_{r_2} < p$, and all $\langle x, r_2\rangle < \alpha_{r_2}$; thus all $\langle x, r_2\rangle \in I$. Choose x so that $r_1 < x < r_2$ and $\langle x, r_2\rangle \in A$. Then

$$\langle r_1, s\rangle \vee \langle x, r_2\rangle = \langle r_2, s\rangle \quad \text{and}$$

$$\{\langle r_1, s\rangle, \langle x, r_2\rangle\} \in \mathfrak{J},$$

hence, $\langle r_2, s\rangle \in I$. The induction step is similar. By the definition of $\mathfrak{J}$ (clause 2), and since $\{\langle r_i, s\rangle \mid i = 1, 2, \dots\}$ is cofinite with the $y = s$ line,

$$\bigvee(\langle r_i, s\rangle \mid i = 1, 2, \dots) = \alpha_s,$$

hence

$$I = (\alpha_s \vee p]_{\mathfrak{J}},$$

as required.

(ii) does not define incomparable pairs of elements.

(iii) and (iv) are similar to (i) except we prove $\langle r_i, s\rangle \in I$ only up to the first i such that $r_i > v$, while in (iv) up to the first i with $r_i \geqq v$.

Case (2a). Since A is a countable sublattice, we can assume that X is an ω chain:

$$\langle r_0, s_0\rangle < \langle r_1, s_1\rangle < \dots < \langle r_n, s_n\rangle < \dots$$

If there is an n, such that $s = s_n = s_{n+1} = \ldots$, then obviously,

$$(X] = (\alpha_s]_{\mathfrak{I}}.$$

If there is no such n, then set $s = \bigvee s_i$. For every $u < s$, u dyadic, there is an i such that $u < r_i$, hence $\gamma_u \in (X]$. By the definition of $\mathfrak{I}$ (clause 3),

$$\bigvee (\gamma_u \mid u < s) \in (X],$$

hence $\gamma_s \in (X]$. This proves that $(X] = (\gamma_s]$.

This completes the proof of Lemma 12.

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