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WEAKLY REGULAR RINGS

BY

V. S. RAMAMURTHI[1]

This paper attempts to generalize a property of regular rings, namely, $I^2=I$ for every right (left) ideal. Rings with this property are called right (left) weakly regular. A ring which is both left and right weakly regular is called weakly regular. Kovacs in [6] proved that, for commutative rings, weak regularity and regularity are equivalent conditions and left open the question whether for arbitrary rings the two conditions are equivalent. We show in §1 that, in general weak regularity does not imply regularity. In fact, the class of weakly regular rings strictly contains the class of regular rings as well as the class of biregular rings. Various characterisations of weak regularity are also given. Weakly regular rings with chain conditions are discussed in §2. For right Artinian rings, right weak regularity is equivalent to regularity and biregularity. Weakly regular Noetherian rings are also completely characterised. More generally it is shown that any finite dimensional ring is weakly regular if and only if it is a finite direct sum of simple D-regular rings. In passing we note also the following results: Any (right) weakly regular ring can be embedded as an ideal in a (right) weakly regular ring with identity. Every ring R has a maximal right weakly regular ideal W such that R/W has no non-zero right weakly regular ideals.

1. **Characterisation and properties.** A ring R is *right* (*left*) *weakly regular* if $I^2=I$ for each right (left) ideal of R. R is *weakly regular* if it is both left and right weakly regular. A ring R is *right* (*left*) *D-regular* if $x \in xR$ ($x \in Rx$) for each $x \in R$. A ring R is *D-regular* if it is both left and right D-regular. A ring R is (von Neumann) *regular* if for each $x \in R$ there exists $y \in R$ such that $x=xyx$. We begin with the following characterisation of right weakly regular rings.

1. Proposition. *The following conditions are equivalent for any ring R.*

(a) *R is right weakly regular.*

(b) *For every $r \in R$, $r \in (rR)^2$.*

(c) *For any 2 right ideals $K_1 \subseteq K_2$, of R, $K_1K_2=K_1$.*

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Proof. (a)$\Rightarrow$(b). Let $r \in R$ and K be the right ideal generated by r. Then $r \in K = K^2 \subseteq rR$ and thus $K = rR$. Therefore $r \in (rR)^2$.

(b)$\Rightarrow$(c). Taking $r \in K_1$ we have $r \in (rR)^2 \subseteq K_1K_2$. Hence $K_1 \subseteq K_1K_2$. The opposite inclusion is trivial.

(c)$\Rightarrow$(a). Trivial.

2. Remark. (b) is obviously equivalent to the statement that for every $r \in R$, $r \in (rR)^t$ holds for some $t \geq 2$. Also notice that if R is right weakly regular, then $r \in rR$ for all $r \in R$.

3. Remark. If R is a ring with identity it is easily verified that the following conditions are equivalent for R:

(a) R is right weakly regular.

(b) If I is a right ideal and K is a two sided ideal of R then
$$I \cap K = IK.$$

(c) If I and J are two right ideals of R then $I \cap RJ = IJ$.

(d) Every two-sided ideal of R is right D-regular.

4. Remark. It is immediate from (b) of Proposition 1, that any regular ring is weakly regular. So also is any biregular ring [5] and in particular any simple ring with identity. But there are weakly regular rings which are neither regular nor biregular (see Remark (6)).

It is clear from the definition that if R is right weakly regular then every quotient ring of R is right weakly regular. Also if I is a 2-sided ideal of R and $r \in I$, then $r \in (rR)^4 \subseteq (rI)^2$ so that I is also right weakly regular. On the other hand, suppose that a ring R has a 2-sided ideal I which is a right weakly regular ring and further that the quotient ring R/I is also right weakly regular. Then, if $r \in R$, there is an element $s \in (rR)^2$ such that $(r-s) \in I$ so that $(r-s) \in ((r-s)I)^2 \subseteq (rR)^2$. Thus $r \in (rR)^2$ and so R is right weakly regular. We collect these facts in the following:

5. Proposition. *Every 2-sided ideal and every quotient ring of a right weakly regular ring, is right weakly regular. On the other hand, if a ring R has a 2-sided ideal I such that the rings I and R/I and both right weakly regular, then R is right weakly regular.*

6. Remark. Let R be a regular ring which is not biregular and S be a biregular ring which is not regular [5]. Then $R \oplus S$ is weakly regular (by the above Proposition) but is neither regular nor biregular.

In the next two Propositions we note some properties of ideals of a right weakly regular ring.

7. LEMMA. *If K is a right ideal of a ring R, then any idempotent right ideal I of K is also a right ideal of R.*

Proof. $IK \subseteq I=I^2 \subseteq IK$. Thus $IK=I$. Therefore, $I=IK \supseteq IKR$ as required.

8. PROPOSITION. *If R is a right weakly regular ring and A is a 2-sided ideal of R, then any right ideal of A is also a right ideal of R.*

Proof. Follows from Proposition 5 and Lemma 7.

9. PROPOSITION. *If A is a proper two sided ideal of a right weakly regular ring R, then each element of A is a left zero divisor.*

Proof. If the element x of A is not a left zero divisor, then the equation $xR=xRxR$ yields $R=RxR \subseteq A$.

10. PROPOSITION. *If a right weakly regular ring R has more than one element and has no left zero divisor, then it is a simple ring with identity.*

Proof. If x is a nonzero element of R, then since $x=xy$ for some $y \in R$, $xy=xy^2$. Therefore, if t is any element of R, then $yt=y^2t$ which yields $y(t-yt)=0$ or $t=yt$. Using this we have $(yt-ty)^2=0$ or $yt=ty$. Thus y is an identity for R. By Proposition 9, R can have no proper ideals.

11. COROLLARY. *A right weakly regular ring has no nonzero nilpotent elements if and only if it is a subdirect sum of simple integral domains with identity. In this case it is also left weakly regular.*

Proof. The first statement follows from the above Proposition and the fact that any ring without nonzero nilpotent elements contains completely prime ideals the intersection of all of which is zero. To prove the second statement note that if $x=xy$ for two elements x, y of a ring, then $(x-yx)^2=0$.

12. PROPOSITION. *The centre of any right weakly regular ring is regular.*

Proof. If $x \in C$, the centre of the right weakly regular ring R, then $x \in (xR)^2 \subseteq x^2R$ implies immediately that $x=x(x^ky)x$ with $y \in R$ and $k \geq 1$. Moreover, for every $z \in R$, $z(x^ky)=x^{k-1}(xz)y=x^{k-1}(x^{k+1}yx)zy=x^{k-1}yz(x^{k+1}yx)=x^{k-1}y(zx)=(x^ky)z$ i.e. $x^ky \in C$ as required.

13. COROLLARY. *For any commutative ring the following conditions are equivalent: (1) R is right weakly regular, (2) R is regular.*

14. Proposition. (1) *The Jacobson radical of any right (left) weakly regular ring R is zero.* (2) *The left (right) singular ideal of any right (left) weakly regular ring R is zero.*

Proof. (1) If $x \in J$, the Jacobson radical of R then $x=xy$ with $y \in J$. But for any $r \in R$ there is an $s \in R$ such that $ys-s=r$. Consequently $xr=xys-xs=0$ and thus $xR=0$ implying that $x=0$.

(2) If $x \in S$, the left singular ideal of R, then $x=xy$ with $y \in S$. Now, the left annihilator $l(y)$ of y satisfies obviously $l(y) \cap Rx=0$ and since it is essential in R, necessarily $Rx=0$. Therefore $x=0$ as required.

15. Remark. Proposition 14(1) shows that if, in a ring R, each two-sided ideal is idempotent, it need not be right weakly regular; because any simple radical ring has the former property.

2. **Weakly regular rings with chain conditions.** From Proposition 14(1), it follows that, for right Artinian rings, right weak regularity is equivalent to regularity and biregularity. The main result of this section characterises right weakly regular rings among noetherian rings. A ring R will be called a Goldie ring if (1) there exists no infinite direct sum of right ideals of R and (2) R has ascending chain condition on annihilator right ideals. A ring that satisfies condition (1) alone will be called finite dimensional. If I is a right ideal then $l(I)$ and $r(I)$ will denote the left and right annihilating ideals of I.

16. Lemma. *In any finite dimensional right weakly regular ring with zero right singular ideal, each two sided ideal is a direct summand.*

Proof. If A is a two sided ideal of the right weakly regular ring R then as R is semi prime, $l(A+r(A))=r(A+r(A))=0$. Hence $A+r(A)$ is an essential right ideal of R and hence contains an element which is not a zero divisor [4, Theorem 3.9]. Lemma follows by Proposition 9.

17. Theorem. *Let R be a finite dimensional ring with zero right singular ideal. R is right weakly regular if and only if it is a direct sum of a finite number of simple right D-regular rings.*

Proof. Let R be right weakly regular. Then R is semiprime and so, by [4] R is Goldie, which implies that R has descending chain condition on annihilating left ideals. Now, by Lemma 16, each two sided ideal I of R is (a direct summand of R and hence) a left annihilator so that I contains minimal two-sided ideals of R. If $\{K_s\}$, $s \in S$, is a maximal independent family of minimal two-sided ideals of R, then, by finite dimensionality of R, S is finite and by Lemma 16, it is easy to see that $R=\Sigma \oplus K_s$ where each K_s is clearly simple and right D-regular. The converse follows by Proposition 5 and the fact that each simple right D-regular ring is right weakly regular.

18. Corollary. *A finite dimensional ring is weakly regular if and only if it is a direct sum of a finite number of simple D-regular rings.*

19. Corollary. *Any prime right weakly regular Goldie ring is simple.*

3. **Some additional remarks.** (I) Adapting the ideas of Fuchs and Halperin [3] we can show:

20. Proposition. *Every (right) weakly regular ring can be embedded as a two sided ideal in a (right) weakly regular ring with identity.*

(II) Right weak regularity is a radical property. Indeed, using the results of section 1, one can show that every ring R has a 2-sided ideal $W(R)$ which is right weakly regular, contains all the right weakly regular ideals of R and such that $R/W(R)$ has no nonzero right weakly regular ideals. Moreover, if R is right Artinian, then $W(R)$ coincides with the maximal regular ideal $M(R)$ [1].

We conclude by pointing out the following extension of Theorem 6 of [1], which can be proved by similar arguments.

21. Proposition. *If J is the Jacobson radical of R and R/J is right weakly regular, then $W(R)=0$ if and only if $r(J) \subseteq J$.*

At the time of revision, our attention was drawn to the fact that radical properties of right weak regularity have been briefly mentioned in [2].

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De la Salle College,
Karumathur 626514, Tamilnadu, India