

OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR NEUTRAL-TYPE DIFFERENCE EQUATIONS

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Abstract

The authors consider the higher-order nonlinear neutral delay difference equation

$$\Delta^m[y_{n-m+1} + p_{n-m+1}y_{n-m+1-k}] + \delta F(n, y_{n-l}) = 0$$

and obtain results on the asymptotic behavior of solutions when $\{p_n\}$ is allowed to oscillate about the bifurcation value -1 . We also consider the case where the sequence $\{p_n\}$ has arbitrarily large zeros. Examples illustrating the results are included, and suggestions for further research are indicated.

1. Introduction

Difference equations of the form

$$\Delta^m[y_{n-m+1} + py_{n-m+1-k}] + q_n y_{n-l} = 0 \quad (e)$$

are commonly called delay difference equations of neutral type, and have been studied by a number of different authors in the last few years (see, for example, [1–12] and the references contained therein). Here k and l are nonnegative integers, m is a positive integer, p is a constant, and $\{q_n\}$ is a sequence of real numbers. Equations of this type arise in a number of important applications including problems in population dynamics when maturation and gestation are included, in “cobweb” models in economics where demand depends on current price but supply depends on the price at an earlier time, and in electrical transmission in lossless transmission lines between circuits in high speed computers.

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From the study of such equations, it is known that the value $p = -1$ behaves as a bifurcation point for the behavior of nonoscillatory solutions of (e) (see, for example, [3] and [5 – 10]). That is, the nonoscillatory solutions behave in a considerably different fashion depending on whether $p > -1$ or $p < -1$. In particular, it is known that for $q_n \geq 0$ and m odd, if $p > -1$ all nonoscillatory solutions converge to zero as $n \rightarrow \infty$, while if $p < -1$ any nonoscillatory solution $\{y_n\}$ satisfies $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$. On the other hand, if m is even and again $q_n \geq 0$, for $p > -1$ all nonoscillatory solutions converge to zero, and for $p < -1$ bounded nonoscillatory solutions converge to zero as $n \rightarrow \infty$. Moreover, examples show that it is possible to have unbounded solutions in the latter case.

For the case of equation (e) with $q_n \leq 0$, nonoscillatory solutions satisfy $y_n \rightarrow 0$ or $|y_n| \rightarrow \infty$ if m is even and $-1 < p \leq 0$, while $|y_n| \rightarrow \infty$ if m is even and $p \leq -1$. With m odd either $y_n \rightarrow 0$ or $|y_n| \rightarrow \infty$ if $-1 < p \leq 0$ and bounded nonoscillatory solutions converge to zero if $p < -1$. It is possible to construct examples illustrating all these types of behaviors.

In this paper we examine the solutions of nonlinear equations of this type with p replaced by a sequence which is allowed to oscillate about the value -1 in some regular fashion. In another direction, the study of the asymptotic behavior of solutions of (e) often requires that the sequence $\{p_n\}$ satisfies either $p_n > 0$ or $p_n < 0$ (again see [1 – 12]). Here, we also examine the case where the sequence $\{p_n\}$ has arbitrarily large zeros. Our results appear to be new even in the case where F is linear. We illustrate our results with examples, and make some suggestions for further research.

2. Asymptotic behavior of solutions

We consider the nonlinear neutral delay difference equation

$$\Delta^m[y_{n-m+1} + p_{n-m+1}y_{n-m+1-k}] + \delta F(n, y_{n-l}) = 0, \tag{E}$$

where $m \geq 1$, $\delta = \pm 1$, Δ denotes the forward difference operator $\Delta y_n = y_{n+1} - y_n$, $\Delta^i y_n = \Delta(\Delta^{i-1}y_n)$, $1 \leq i \leq m$, $k, l \in \mathbb{N} = \{0, 1, 2, \dots\}$, $\{p_n\}$ is a sequence of real numbers, $F : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous with $uF(n, u) \geq 0$ for $u \neq 0$ and $n \geq N_0 \in \mathbb{N}$, and $F(n, u) \neq 0$ for $u \in \mathbb{R} \setminus \{0\}$ and $n \geq N_1$ for every $N_1 \geq N_0$. By a solution of (E), we mean a sequence $\{y_n\}$ of real numbers which is defined for $n \geq N_0 - M$, where $M = \max\{k, l\} + m - 1$ and which satisfies (E) for $n \geq N_0$. Such a solution $\{y_n\}$ of (E) is said to be *nonoscillatory* if the terms y_n are either eventually all positive or eventually all negative. Otherwise the solution is called *oscillatory*.

In many of our results we will require the condition that if $\{u_n\}$ is a sequence with

$u_n > 0$ ($u_n < 0$) and $\liminf_{n \rightarrow \infty} |u_n| > 0$, then

$$\sum_{i=N_0}^{\infty} F(i, u_i) = \infty \quad (-\infty). \quad (1)$$

For notational purposes, we shall let

$$z_n = y_n + p_n y_{n-k}.$$

The following lemma will be used in proving the main results in this paper.

LEMMA 1. *Suppose condition (I) holds, and $\{y_n\}$ is an eventually positive (negative) solution of (E) with $\delta = +1$ ($\delta = -1$). Then:*

(i) *If $z_n \rightarrow 0$ as $n \rightarrow \infty$, then $\{\Delta^i z_n\}$ is monotonic and*

$$\Delta^i z_n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ and } \Delta^i z_n \Delta^{i+1} z_n < 0 \quad (\text{A})$$

for $i = 0, 1, \dots, m - 1$.

(ii) *Let $z_n \rightarrow 0$ as $n \rightarrow \infty$. If m is even (odd), then $z_n < 0$ ($z_n > 0$). If m is odd (even), then $z_n > 0$ ($z_n < 0$).*

(iii) *If there exists a constant P_1 such that*

$$P_1 \leq p_n \leq 0, \quad (2)$$

then either $\{\Delta^i z_n\}$ is decreasing (increasing) $\{\text{increasing (decreasing)}\}$ with

$$\Delta^i z_n \rightarrow -\infty \quad (\infty) \quad \{\infty \quad (-\infty)\} \text{ as } n \rightarrow \infty \quad (\text{B})$$

for $i = 0, 1, \dots, m - 1$, or $\{\Delta^i z_n\}$ is monotonic and (A) holds.

(iv) *If (2) holds and m is even, then $z_n < 0$ ($z_n > 0$) $\{> 0$ (< 0)\}. If (A) holds and m is odd, then $z_n > 0$ ($z_n < 0$) $\{< 0$ (> 0)\}.*

REMARK. The parts of this lemma appear as parts (c) – (f) respectively of Lemma 1 in [10].

For our first result, we are interested in the situation where the sequence $\{p_n\}$ is allowed to oscillate about the value -1 . We ask that this behavior occur in a regular fashion, and in fact, we say that the sequence $\{p_n\}$ has *property (S)* if:

for every $n_1 \geq N_0$ there exists $N \geq n_1$ with the property that for each fixed n with $N \leq n \leq N + k$ there is a nonnegative integer M_n such that $p_{n+M_n k} = -Q_n > -1$. (S)

In all our results we give proofs only for the case when a nonoscillatory solution is eventually positive since the proofs for an eventually negative solution are similar.

THEOREM 2. *Suppose that conditions (1) – (2) hold and $\{p_n\}$ has property (S). If $\delta = +1$ and m is even, then any solution $\{y_n\}$ of (E) is either oscillatory or satisfies $y_n \rightarrow 0$ as $n \rightarrow \infty$, while if $\delta = -1$ and m is odd, then either $\{y_n\}$ is oscillatory, $y_n \rightarrow 0$, or $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$.*

PROOF. Let $\{y_n\}$ be an eventually positive solution of (E), say, $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \geq N_1$ for some $N_1 \geq N_0$. Lemma 1(iv) implies that $z_n = y_n + p_n y_{n-k} < 0$ for $n \geq N_2$ for some $N_2 \geq N_1$. Since (2) holds and the sequence $\{p_n\}$ has property (S), there exists $N \geq N_2$ with the property that for each fixed n with $N \leq n \leq N + k$ there is a nonnegative integer M_n such that $0 > p_{n+M_n k} = -Q_n > -1$. Hence, for each fixed n with $N \leq n \leq N + k$

$$y_{n+M_n k} + p_{n+M_n k} y_{n+(M_n-1)k} = z_{n+M_n k} < 0,$$

and so

$$y_{n+M_n k} < -p_{n+M_n k} y_{n+(M_n-1)k} = Q_n y_{n+(M_n-1)k}.$$

Iterating this inequality, we obtain

$$y_{n+(M_n+j)k} < (Q_n)^{j+1} y_{n+(M_n-1)k} \rightarrow 0$$

as $j \rightarrow \infty$. Since for each fixed n with $N \leq n \leq N + k$, we have $y_{n+ik} \rightarrow 0$ as $i \rightarrow \infty$, it follows that $y_n \rightarrow 0$ as $n \rightarrow \infty$.

Our next result is for the case where $\{p_n\}$ has arbitrarily large zeros.

THEOREM 3. *Suppose conditions (1) – (2) hold and the set of zeros of $\{p_n\}$ is unbounded from above. If $\delta = +1$ and m is even, then any solution $\{y_n\}$ of (E) is oscillatory, while if $\delta = -1$ and m is odd, then either $\{y_n\}$ is oscillatory or $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$.*

PROOF. Let $\{y_n\}$ be a nonoscillatory solution of (E), say, $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \geq N_1 \geq N_0$. Part (iii) of Lemma 1 implies that either (A) or (B) holds. If (A) holds, part (ii) of Lemma 1 implies there exists $N_2 \geq N_1$ such that $z_n < 0$ for $n \geq N_2$, so choose $N \geq N_2$ such that $p_N = 0$. Then $y_N = z_N < 0$, which is a contradiction. If (B) holds, then $z_n < y_n \rightarrow \infty$ as $n \rightarrow \infty$ if $\delta = -1$, and we again have $z_n < 0$ if $\delta = +1$.

In the results in the remainder of this paper, we shall make use of the following condition on the function F . Assume that there is a continuous increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $uf(u) > 0$ for $u \neq 0$ and constants $K_1, K_2 > 0$ such that

$$K_1 f(u) \leq F(n, u) \text{ if } u > 0 \text{ and } F(n, u) \leq K_2 f(u) \text{ if } u < 0 \tag{3}$$

for all $n \geq N_0$. The following lemma provides some additional information on the behavior of the nonoscillatory solutions of (E).

LEMMA 4. *Suppose that condition (3) holds and there is a positive constant P_2 such that*

$$|p_n| \leq P_2. \quad (4)$$

If $\{y_n\}$ is a nonoscillatory solution of (E), then either (A) or (B) holds.

PROOF. Suppose that $\{y_n\}$ is an eventually positive solution of (E), say $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \geq N_1$ for some $N_1 \geq N_0$. From (E), we have $\delta \Delta^m z_{n-m+1} = -F(n, y_{n-l}) \leq 0$, so $\{\Delta^i z_n\}$ is monotonic and $\Delta^{m-1} z_n \rightarrow \delta L < \infty$ as $n \rightarrow \infty$. If $\delta L = -\infty$, then clearly (B) holds. If $\delta L > -\infty$, then summing (E) from N_1 to n and then letting $n \rightarrow \infty$, we have

$$\delta \Delta^{m-1} z_{N_1-m+1} - \delta L = \sum_{i=N_1}^{\infty} F(i, y_{i-l}) \geq K_1 \sum_{i=N_1}^{\infty} f(y_{i-l}).$$

This implies that $f(y_n) \rightarrow 0$ as $n \rightarrow \infty$, and since f is increasing, we have $y_n \rightarrow 0$ as $n \rightarrow \infty$. Condition (4) then implies that $z_n \rightarrow 0$ as $n \rightarrow \infty$, and it follows immediately from Lemma 1(i) that (A) holds.

THEOREM 5. *Suppose conditions (3) – (4) hold and the sequence $\{p_n\}$ is not eventually negative. If $\delta = +1$ and m is even, then all solutions of (E) are oscillatory, while if $\delta = -1$ and m is odd, then all bounded solutions are oscillatory.*

PROOF. Let $\{y_n\}$ be a nonoscillatory solution of (E) with $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \geq N_1 \geq N_0$. Since $\{p_n\}$ is not eventually negative, (B) cannot hold if $\delta = +1$, so in this case (A) holds. Moreover, if m is even, Lemma 1(ii) again yields the contradiction that $z_n < 0$. For $\delta = -1$, (B) contradicts the boundedness of $\{y_n\}$, and Lemma 1(ii) yields a contradiction when m is odd.

The next theorem considers the complementary cases to those in Theorem 5.

THEOREM 6. *Suppose conditions (3) – (4) hold and $p_n \geq 0$. If $\delta = +1$ and m is odd, then any solution $\{y_n\}$ of (E) is either oscillatory or satisfies $y_n \rightarrow 0$ as $n \rightarrow \infty$. If $\delta = -1$ and m is even, then any bounded solution $\{y_n\}$ of (E) is either oscillatory or satisfies $y_n \rightarrow 0$ as $n \rightarrow \infty$.*

PROOF. Again let $\{y_n\}$ be a nonoscillatory solution of (E) with $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \geq N_1 \geq N_0$. By Lemma 4, either (A) or (B) holds. Now (B) contradicts $p_n \geq 0$ if $\delta = +1$ and contradicts the boundedness of $\{y_n\}$ if $\delta = -1$. Part (ii) of Lemma 1 yields the contradictions needed to complete the proof after it is observed that $z_n \rightarrow 0$ and $y_n \leq z_n$ implies $y_n \rightarrow 0$.

Our next theorem is just for the case $\delta = +1$; in it we allow $\{p_n\}$ to oscillate about 0.

THEOREM 7. *Suppose $\delta = +1$, condition (3) holds, and there exist P_3 and P_4 such that*

$$-1 \leq P_3 \leq p_n \leq P_4. \tag{5}$$

If m is odd, then every nonoscillatory solution $\{y_n\}$ of (E) satisfies $y_n \rightarrow 0$ as $n \rightarrow \infty$. If m is even, then any nonoscillatory solution is bounded; in addition, if $P_3 > -1$, $y_n \rightarrow 0$ as $n \rightarrow \infty$.

PROOF. Suppose $\{y_n\}$ is a nonoscillatory solution of (E) with $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \geq N \geq N_0$. First note that (5) implies (4), so by Lemma 4, either (A) or (B) holds. If (B) holds, then $-y_{n-k} \leq z_n \rightarrow -\infty$ as $n \rightarrow \infty$ and so $y_n \rightarrow \infty$ as $n \rightarrow \infty$. But $y_n = z_n - p_n y_{n-k} \leq -P_3 y_{n-k} \leq y_{n-k}$ which contradicts the fact that $y_n \rightarrow \infty$ as $n \rightarrow \infty$. Therefore (A) holds.

Suppose that m is odd. By Lemma 1(ii), $z_n > 0$, and since (A) holds, $\sum_{i=N}^{\infty} F(i, y_{i-l})$ converges. As in the proof of Lemma 4, this implies that $y_n \rightarrow 0$ as $n \rightarrow \infty$.

If m is even, Lemma 1(ii) implies that $z_n < 0$, and this in turn implies that p_n must eventually be negative. If $\limsup_{n \rightarrow \infty} y_n = \infty$, then there exists an increasing sequence $\{m_n\}$ with $m_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $p_n < 0$ for $n \geq m_1$, $y_{m_n} \rightarrow \infty$ as $n \rightarrow \infty$, and $y_{m_n} = \max\{y_i : N_1 \leq i \leq m_n\}$. Hence, $z_{m_n} = y_{m_n} + p_{m_n} y_{m_n-k} \geq y_{m_n} + p_{m_n} y_{m_n} = [1 + p_{m_n}] y_{m_n} \geq 0$. This contradicts $z_n < 0$ for sufficiently large n . Thus, $\limsup_{n \rightarrow \infty} y_n < \infty$, and so $\{y_n\}$ is bounded. Finally, if $P_3 > -1$, we have $y_n \leq -p_n y_{n-k} \leq -P_3 y_{n-k}$, and iterating, we obtain $y_n \rightarrow 0$ as $n \rightarrow \infty$.

Our final result is a consequence of Theorem 7 for the case when m is odd. As discussed in the introduction, solutions of the equation

$$\Delta^m [y_{n-m+1} + p y_{n-m+1-k}] + q_n y_{n-l} = 0 \tag{e}$$

behave in a considerably different fashion depending on whether $p > -1$ or $p < -1$. The following corollary completes the picture by considering the case $p \equiv -1$.

COROLLARY 8. *Suppose $\delta = +1$, m is odd, condition (3) holds, and $p_n \equiv -1$. Then every solution of (E) is oscillatory.*

PROOF. Suppose $\{y_n\}$ is a solution of (E). By Theorem 5 in [11], either $\{y_n\}$ is oscillatory or $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$. By Theorem 7 above, $\{y_n\}$ is bounded and so the conclusion follows.

REMARK. Corollary 8 includes Theorem 2.1 of Lalli et al. [5] on first-order linear equations as a special case.

3. Examples and suggestions for further research

The equation

$$\Delta^m \left\{ y_{n-m+1} - \frac{2 + (-1)^{n-m+1}}{2} y_{n-m+1-k} \right\} + 2^{m+1} y_{n-l}^\gamma = 0, \tag{E_1}$$

where k is an odd integer, γ is an odd positive integer, and l is an even integer satisfies the hypotheses of Theorem 2. The sequence $\{y_n\} = \{(-1)^n\}$ is an oscillatory solution of (E_1) . On the other hand, consider the equation

$$\Delta^m \left\{ y_{n-m+1} - \frac{2 + (-1)^{n-m+1}}{2} y_{n-m+1-k} \right\} - \left[\frac{(1 - e)^m (2 - 3e^k/2) + (-1)^{n+1} (1 + e)^m}{e^{\gamma l+1} [2 + (-1)^{n-l}]^\gamma} \right] e^{(\gamma-1)n} y_{n-l}^\gamma = 0, \tag{E_2}$$

where k is odd and γ is the ratio of odd positive integers with $\gamma \geq 1$. If k is chosen large enough so that $e^k > 2[2 + (1 + e)^m / (1 - e)^m] / 3$, then the hypotheses of Theorem 2 are satisfied ($\delta = +1$ if m is even and $\delta = -1$ if m is odd). Here $\{y_n\} = \{e^{-n} [2 + (-1)^n]\}$ is a nonoscillatory solution of (E_2) converging to zero as $n \rightarrow \infty$. Also, if γ is the ratio of odd positive integers with $0 < \gamma \leq 1$, k is even, and $e^k > 1 + (e + 1)^m / 2(e - 1)^m$, then the equation

$$\Delta^m \left\{ y_{n-m+1} - \frac{2 + (-1)^{n-m+1}}{2} y_{n-m+1-k} \right\} - e^{1-m} \left\{ \frac{(e - 1)^m (1 - e^{-k}) + (-1)^n (e + 1)^m / 2e^k}{[e^{n-l} + (-1)^{n-l}]^\gamma} \right\} e^n y_{n-l}^\gamma = 0 \tag{E_3}$$

satisfies the hypotheses of Theorem 2 for the case $\delta = -1$ and has the solution $\{y_n\} = \{e^n + (-1)^n\}$ such that $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$.

The equation

$$\Delta^m \{y_{n-m+1} + [(-1)^n - 1]y_{n-m}\} + (-1)^{\gamma l} 2^{m+1} y_{n-l}^\gamma = 0, \tag{E_4}$$

where $\gamma \geq 1$ is an odd positive integer, satisfies the hypotheses of Theorems 3 and 5. Here, $\delta = +1$ if l is even and $\delta = -1$ if l is odd. An oscillatory solution of this equation is $\{y_n\} = \{(-1)^n\}$.

If k is even, the equation

$$\Delta^m \{y_{n-m+1} + [(-1)^{n-m+1} - 1]y_{n-m+1-k}\} - e^{1-m} \left\{ \frac{(e - 1)^m (1 - e^{-k}) + (-1)^{n+1} (e + 1)^m e^{-k}}{e^{-l} + (-1)^{n-l} e^{-n}} \right\} y_{n-l} = 0, \tag{E_5}$$

will satisfy the hypotheses of Theorem 3 for the case $\delta = -1$ provided k is chosen large enough so that $e^k > (e + 1)^m / (e - 1)^m + 1$. The sequence $\{y_n\} = \{e^n + (-1)^n\}$ is a nonoscillatory solution with $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$.

The equation

$$\Delta^m \{y_{n-m+1} + [(-1)^{n-m+1} + 1]y_{n-m+1-k}\} + (-1)^{\gamma l} 2^{2m+1} y_{n-l}^\gamma = 0, \tag{E_6}$$

where k is even and $\gamma > 0$ is the ratio of odd positive integers, satisfies the hypotheses of Theorems 5 and 6. We have $\delta = +1$ if l is even, $\delta = -1$ if l is odd, and $\{y_n\} = \{(-1)^n\}$ is an oscillatory solution. The equation

$$\Delta^m [y_{n-m+1} + e^{-n+m-1} y_{n-m+1-k}] - e^{n(1-\gamma)} (e - 1)^m e^{\gamma l - m + 1} y_{n-l}^\gamma = 0, \tag{E_7}$$

where γ is the ratio of odd positive integers with $0 < \gamma \leq 1$, satisfies the hypotheses of Theorems 5 and 6 for $\delta = -1$. Here, $\{y_n\} = \{e^n\}$ is an unbounded solution which does not oscillate. In addition, the equation

$$\Delta^m \{y_{n-m+1} + [(-1)^{n-m+1} + 1]y_{n-m+1-k}\} - e^{n(1-\gamma)} e^{1-m} \left\{ \frac{(e + 1)^m (1 - e^{-k}) + (-1)^{n+1} (e - 1)^m e^{-k}}{[e^{-l} + e^{-n}]^\gamma} \right\} y_{n-l}^\gamma = 0, \tag{E_8}$$

where k is an odd positive integer and γ is the ratio of odd positive integers with $0 < \gamma \leq 1$, satisfies the hypotheses of Theorems 5 and 6 and has the unbounded oscillatory solution $\{y_n\} = \{(-1)^n e^n + (-1)^n\}$.

For the equation

$$\Delta^m [y_{n-m+1} + e^{-n+m-1} y_{n-m+1-k}] + \delta [2^m + e^{-n-1} (1 + e)^m] y_{n-l}^\gamma = 0, \tag{E_9}$$

where k is even and $\gamma > 0$ is the ratio of odd positive integers, the hypotheses of Theorems 5 and 6 are satisfied if (i) $\delta = +1$, m is odd, and l is even, or (ii) $\delta = -1$, m is even, and l is odd. We have that $\{y_n\} = \{(-1)^n\}$ is an oscillatory solution of (E₉). The hypotheses of Theorem 7 are also satisfied if (i) holds.

The equation

$$\Delta^m [y_{n-m+1} + e^{-n+m-1} y_{n-m+1-k}] - [(1 - e)^m + e^{-n} (1 - e^2)^m e^{k-1}] y_{n-l} / e^{l+1} = 0 \tag{E_{10}}$$

satisfies the hypotheses of Theorem 6, and if m is odd, it also satisfies the hypotheses of Theorem 7. Here, $\{y_n\} = \{e^{-n}\}$ is a nonoscillatory solution which converges to 0 as $n \rightarrow \infty$.

REMARK. As to some directions for further research, the extension of any of the results in this paper to the case where $F(n, u)$ can oscillate in sign for fixed u would be of

significant interest. Another suggestion is to study the effect of modifying property (S) on the sequence $\{p_n\}$ so that $p_{n+M_nk} = -Q_n < -1$ and examining the effect on the behavior of the nonoscillatory solutions especially in the cases where $\delta = +1$ and m is odd or $\delta = -1$ and m is even.

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