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CABLE KNOTS AND INFINITE NECKLACES OF KNOTS

SIEGFRIED MORAN

The group of an arbitrary companion knot is determined using the theory of braids. This seems to be a new result as far as the resulting group is concerned. The latter part of the paper considers infinite necklaces of knots, their groups and in special cases their Alexander power series.

The group of a cable knot (or more generally a companion knot) is determined in an algebraic manner using the theory of braids. The results as stated in this paper seem to be new. Other accounts of the theory can be found in Burau [2], Seifert [8], Burde and Zieschang [3]. Some examples can also be found in Rolfsen [7]. The appropriate theory will be preceded by an account of some relevant notation from the theory of braids.

The last part of this paper will be devoted to a consideration of infinite necklaces of a countable number of knots. They give rise to certain Alexander formal power series.

In this paper it is assumed that the reader is familar with the standard theory of braids and knots as given in Birman [1] and Moran [5]. The last part of this paper also assumes a knowledge of the results contained in Moran [6].

1. SOME NOTATION FROM BRAID THEORY

Suppose that i and j are positive integers with $i \leqslant j$. We shall be concerned with the usual generators

$$\sigma_1, \ldots, \sigma_i, \ldots, \sigma_j, \ldots$$

of a braid group. We denote

$$\sigma_i\sigma_{i+1}\ldots\sigma_j \quad \text{by} \quad {}_i\sigma_j$$

and

$${}_i\sigma_j . {}_i\sigma_{j-1} \ldots {}_i\sigma_{i+1} . \sigma_i \quad \text{by} \quad {}_i\Delta_j.$$

${}_1\Delta_j$ is abbreviated to Δ_j.

We study braids of braids which are generated by braids of the form

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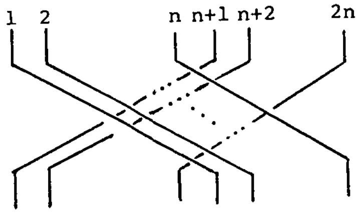


where n is a fixed positive integer. This braid is given by

$$ {}_{n}\sigma_{2n-1} \cdot {}_{n-1}\sigma_{2n-2} \cdot \cdots \cdot {}_{n-i}\sigma_{2n-i-1} \cdot \cdots \cdot {}_{1}\sigma_{n} = \sigma. $$

LEMMA 1.1. *The above given braid σ is equal to*

$$ \Delta_{n-1}^{-1} \cdot \Delta_{2n-1} \cdot \Delta_{n-1}^{-1}. $$

PROOF:

$$ \begin{aligned} \sigma &= {}_{n}\sigma_{2n-1} \cdot {}_{n-1}\sigma_{2n-2} \cdot \cdots \cdot {}_{n-i}\sigma_{2n-i-1} \cdot \cdots \cdot {}_{3}\sigma_{n+2} \cdot \sigma_1^{-1} \cdot {}_{1}\sigma_{n+1} \cdot {}_{1}\sigma_{n} \\ &= \sigma_1^{-1} \cdot {}_{n}\sigma_{2n-1} \cdot \cdots \cdot {}_{4}\sigma_{n+3} \cdot {}_{1}\sigma_2^{-1} \cdot {}_{1}\sigma_{n+2} \cdot {}_{1}\sigma_{n+1} \cdot {}_{1}\sigma_{n} \\ &= \sigma_{1 \cdot 1}^{-1}\sigma_2^{-1} \cdot {}_{n}\sigma_{2n-1} \cdot \cdots \cdot {}_{5}\sigma_{n+4} \cdot {}_{1}\sigma_3^{-1} \cdot {}_{1}\sigma_{n+3} \cdot {}_{1}\sigma_{n+2} \cdot {}_{1}\sigma_{n+1} \cdot {}_{1}\sigma_{n} \quad . \end{aligned} $$

Continuing to collect in this way one has that

$$ \begin{aligned} \sigma &= \left({}_{1}\sigma_{n-1} \cdot {}_{1}\sigma_{n-2} \cdot \cdots \cdot {}_{1}\sigma_2 \cdot {}_{1}\sigma_1\right)^{-1} \cdot \left({}_{1}\sigma_{2n-1} \cdot {}_{1}\sigma_{2n-2} \cdot \cdots \cdot \sigma_{n+1} \cdot {}_{1}\sigma_n\right) \\ &= \Delta_{n-1}^{-1} \cdot \Delta_{2n-1} \cdot \Delta_{n-1}^{-1}. \end{aligned} $$

∎

LEMMA 1.2.

$$ \left(\Delta_{n-1}^{-1} \cdot \Delta_{2n-1} \cdot \Delta_{n-1}^{-1}\right)^{k} = \Delta_{2n-1}^{k} \cdot \left(\Delta_{2n-1}^{-1} \cdot \Delta_{n-1} \cdot \Delta_{2n-1}\right)^{-k} \cdot \Delta_{n-1}^{-k} $$

for all integers k.

PROOF: We first note that Δ_{2n-1}^{2} commutes with all terms in the above expression and also that Δ_{n-1} and $\Delta_{2n-1}^{-1} \cdot \Delta_{n-1} \cdot \Delta_{2n-1}$ commute. Hence it is sufficient to consider the case when k is a positive integer. We proceed by induction on k.

$$ \begin{aligned} &\left(\Delta_{n-1}^{-1} \cdot \Delta_{2n-1} \cdot \Delta_{n-1}^{-1}\right)^{k+1} \\ &\quad = \Delta_{2n-1}^{k} \cdot \left(\Delta_{2n-1}^{-1} \cdot \Delta_{n-1} \cdot \Delta_{2n-1}\right)^{-k} \cdot \Delta_{n-1}^{-k} \cdot \Delta_{n-1}^{-1} \cdot \Delta_{2n-1} \cdot \Delta_{n-1}^{-1} \\ &\quad = \Delta_{2n-1}^{k+1} \cdot \Delta_{n-1}^{-k} \cdot \Delta_{2n-1}^{-1} \cdot \Delta_{n-1}^{-k-1} \cdot \Delta_{2n-1} \cdot \Delta_{n-1}^{-1} \\ &\quad = \Delta_{2n-1}^{k+1} \cdot \left(\Delta_{2n-1}^{-1} \cdot \Delta_{n-1} \cdot \Delta_{2n-1}\right)^{-k-1} \cdot \Delta_{n-1}^{-k-1}. \end{aligned} $$

Lemma 1.3.

$$\Delta_{2n-1}^{-1} \,.\, \Delta_{n-1} \,.\, \Delta_{2n-1} = {}_{n+1}\Delta_{2n-1}.$$

Proof:

$$\begin{aligned}
&\Delta_{2n-1} \cdot {}_{n+1}\Delta_{2n-1} \,.\, \Delta_{2n-1}^{-1} \\
&\quad = \Delta_{2n-1} \,.\, (\sigma_{n+1} \cdots \sigma_{2n-1}) \,.\, (\sigma_{n+1} \cdots \sigma_{2n-2}) \cdots (\sigma_{n+1}\sigma_{n+2})\sigma_{n+1} \,.\, \Delta_2^{-1} \\
&\quad = (\sigma_{n-1} \cdots \sigma_1) \,.\, (\sigma_{n-1} \cdots \sigma_2) \cdots (\sigma_{n-1}\sigma_{n-2})\sigma_{n-1} \\
&\quad = \Delta_{n-1}^{-1} \,.\, (\sigma_1 \cdots \sigma_{n-1}) \,.\, (\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)\sigma_1 \,.\, \Delta_{n-1} \\
&\quad = \Delta_{n-1}.
\end{aligned}$$

∎

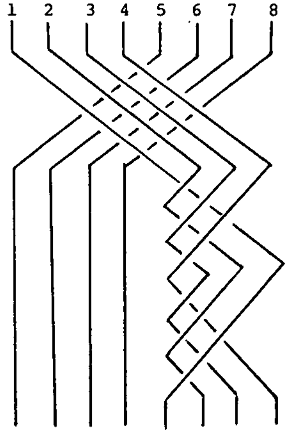
The counterpart to σ_1 in Braid Theory is played here by $\Lambda_1^{(n)}$ which we define to be

$$\Delta_{n-1}^{-1} \,.\, \Delta_{2n-1} \,.\, \Delta_{n-1}^{-1} \cdot {}_{n+1}\Delta_{2n-1}^{-2}.$$

We also have, by Lemma 1.3, that

$$\Lambda_1^{(n)} = \Delta_{2n-1} \,.\, \Delta_{n-1}^{-1} \cdot {}_{n+1}\Delta_{2n-1}^{-3}.$$

This has a picture of the following form in the particular case when $n = 4$:



It is a routine matter to show that

$$\left(\Lambda_1^{(n)}\right)^{-1} = \Delta_{2n-1}^{-1} \cdot {}_{n+1}\Delta_{2n-1} \,.\, \Delta_{n-1}^{3}.$$

∎

2. Free Automorphisms corresponding to some braids, longitudes

We determine the automorphism of the free group corresponding to $\Lambda_1^{(n)}$ and hence more generally that corresponding to $\Lambda_i^{(n)}$. It is well known that

$$x_j\Delta_{n-1} = (x_1x_2\cdots x_{n-j+1})\cdot x_{n-j+1}\cdot(x_1x_2\cdots x_{n-j+1})^{-1}$$

for $1\leqslant j\leqslant n$. Using this result, a tedious calculation shows that

$$x_j\Lambda_1^{(n)} = \begin{cases} [(x_1\cdots x_n)(x_{n+1}\cdots x_{2n})^{-1}]\cdot x_{n+j}\cdot[\cdots]^{-1} & \text{for } 1\leqslant j\leqslant n\\ x_{j-n} & \text{for } n<j\leqslant 2n.\end{cases}$$

We now define $\Lambda_i^{(n)}$ for general i.

$$\Lambda_i^{(n)} = {}_{n(i-1)+1}\Delta_{n(i+1)-1}\cdot{}_{n(i-1)+1}\Delta_{ni-1}^{-1}\cdot{}_{ni+1}\Delta_{n(i+1)-1}^{-3}$$

for $i=1,2,\ldots,m-1$. These braids all lie in the braid group B_{mn}. It follows from the above given automorphism corresponding to $\Lambda_1^{(n)}$ that the automorphism of the free group corresponding to $\Lambda_i^{(n)}$ is as follows:

$$x_j\Lambda_i^{(n)} = \begin{cases} x_j & \text{for } 1\leqslant j\leqslant n(i-1)\\ [(x_{n(i-1)+1}\cdots x_{ni})(x_{ni+1}\cdots x_{n(i+1)})^{-1}]\cdot x_{n+j}\cdot[\cdots]^{-1} & \\ & \text{for } n(i-1)<j\leqslant ni\\ x_{j-n} & \text{for } ni<j\leqslant n(i+1)\\ x_j & \text{for } j>n(i+1).\end{cases}$$

Hence one can deduce that the automorphism corresponding to its inverse is given by

$$x_j\left(\Lambda_i^{(n)}\right)^{-1} = \begin{cases} x_j & \text{for } 1\leqslant j\leqslant n(i-1)\\ x_{j+n} & \text{for } n(i-1)<j\leqslant ni\\ \left[(x_{ni+1}\cdots x_{n(i+1)})^{-1}(x_{n(i-1)+1}\cdots x_{ni})\right]\cdot x_{j-n}\cdot[\cdots]^{-1} & \\ & \text{for } ni<j\leqslant n(i+1)\\ x_j & \text{for } j>n(i+1).\end{cases}$$

Both these automorphisms are defined for $i=1,2,\ldots,m-1$.

The braid $\Lambda_i^{(n)}$ takes the ith strand (of n strings) over the $(i+1)$th strand (of n strings) for $i=1,2,\ldots,m-1$. Suppose that one has an arbitrary word in these operations which we denote by

$$w\left(\Lambda_1^{(n)},\cdots,\Lambda_{m-1}^{(n)}\right) = \Lambda^{(n)}$$

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for short. Then the tth strand will terminate at $t\overline{\pi}$, where $\overline{\pi}$ is a permutation of $1, 2, \ldots, m$. If one considers the jth string, then it lies in the tth strand, where t is determined by the equation

$$j = (t-1)n + r$$

with $1 \leqslant r \leqslant n$ and $1 \leqslant t \leqslant m$. The jth string will then terminate at $j\pi$, where

$$j\pi = (t\overline{\pi} - 1)n + r.$$

Using this notation we now determine the form of the automorphism of the free group corresponding to $\Lambda^{(n)}$.

Lemma 2.1. *Suppose that*

$$\Lambda^{(n)} = w\left(\Lambda_1^{(n)}, \ldots, \Lambda_{m-1}^{(n)}\right)$$

is a word in the generators $\Lambda_1^{(n)}, \ldots, \Lambda_{n-1}^{(n)}$. *Then the automorphism of the free group* $F(\{x_1, x_2, \ldots, x_{mn}\})$ *corresponding to* $\Lambda^{(n)}$ *is of the form*

$$x_j\Lambda^{(n)} = A_t^{(n)} \,.\, x_{j\pi} \,.\, \left(A_t^{(n)}\right)^{-1},$$

where the j*th string of* $\Lambda^{(n)}$ *belongs to the* t*th strand of* $\Lambda^{(n)}$ *for* $1 \leqslant j \leqslant mn$. *Further* $A_t^{(n)}$ *is a word in the elements* $X_1^{(n)}, X_2^{(n)}, \ldots, X_m^{(n)}$ *of the free group freely generated by them, where*

$$X_s^{(n)} = x_{(s-1)n+1} \cdot x_{(s-1)n+2} \cdot \cdots \cdot x_{sn}$$

for $1 \leqslant s, t \leqslant m$. *Also every* $A_t^{(n)}$ *has exponent sum equal to xero when it is considered as a word in the free group* $F\left(\{X_1^{(n)}, \ldots, X_m^{(n)}\}\right)$.

Proof: This proceeds by induction on the length of the word w. The result has already been proved true for a word of length 1 in the previous paragraphs. Now

$$\begin{aligned} &x_j\Lambda^{(n)}\left(\Lambda_i^{(n)}\right)^{\pm 1} \\ &\quad = \left[A_t^{(n)}\left(X_1^{(n)}\left(\Lambda_i^{(n)}\right)^{\pm 1}, \ldots, X_m^{(n)}\left(\Lambda_i^{(n)}\right)^{\pm 1}\right)\right] \,.\, x_{j\pi}\left(\Lambda_i^{(n)}\right)^{\pm 1} \,.\, [\cdots]^{-1}, \end{aligned}$$

which gives the required result. ∎

Lemma 2.2. *Using the notation of the previous Lemma one has that*

$$X_j^{(n)}\Lambda_i^{(n)} = \begin{cases} X_j^{(n)} & \text{for } j < i \\ X_i^{(n)} \,.\, X_{i+1}^{(n)} \,.\, \left(X_i^{(n)}\right)^{-1} & \text{for } j = i \\ X_i^{(n)} & \text{for } j = i+1 \\ X_j^{(n)} & \text{for } j \geqslant i+2 \end{cases}$$

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where $i = 1, 2, \ldots, m-1$. *Hence there is a natural one-to-one correspondence between the automorphisms of the free group* $F\left(\{X_1^{(n)}, \ldots, X_m^{(n)}\}\right)$ *induced by the braids* $w\left(\Lambda_1^{(n)}, \ldots, \Lambda_{m-1}^{(n)}\right)$ *and the automorphisms of the free group* $F(\{x_1, \ldots, x_m\})$ *induced by the braids* $w(\sigma_1, \ldots, \sigma_{m-1})$. *In fact one has that*

$$X_t^{(n)} w\left(\Lambda_1^{(n)}, \ldots, \Lambda_{m-1}^{(n)}\right) = A_t^{(n)}(X) \,.\, X_{t\overline{\pi}}^{(n)} \,.\, \left(A_t^{(n)}(X)\right)^{-1}$$

for $t = 1, 2, \ldots, m$ *and*

$$x_s w(\sigma_1, \ldots, \sigma_{m-1}) = a_s^{(n)}(x) \,.\, x_{s\overline{\pi}} \,.\, \left(a_s^{(n)}(x)\right)^{-1}$$

for $s = 1, 2, \ldots, m$. *The correspondence is obtained by mapping*

$$X_j^{(n)} \longrightarrow x_j \qquad \text{for } j = 1, 2, \ldots, m.$$

In particular one has that under this correspondence

$$A_t^{(n)}(X) \longrightarrow a_t^{(n)}(x) \qquad \text{for } t = 1, 2, \ldots, m.$$

We give a simple proof of the following well known result on the longitude of a knot. For a convenient reference concerning longitudes see Moran [**5**, Chapter 17]. For a different proof see [**3**, Proposition 3.12].

Lemma 2.3. *The longitude of a knot belongs to the second commutator subgroup of the group of the knot.*

Proof: Let G be the group of the knot K. Then, by definition, the longitude l belongs to the commutator subgroup G'. Now the required result is a consequence of the following three well known facts. Firstly $G/G' = \langle t; - \rangle$ acts by conjugation on the Abelian group G'/G''. Secondly the Alexander polynomial $f(t)$ of the knot K annihilates all elements of G'/G'' (see Moran [**5**, end of Chapter 11]). Thirdly the longitude l commutes with its meridan and $f(1) = \pm 1$. ∎

3. Cable and companion theorems

Suppose that the braid

$$\Lambda^{(n)} = w\left(\Lambda_1^{(n)}, \ldots, \Lambda_{m-1}^{(n)}\right)$$

is such that the permutation of its strands is an m-cycle. Then the associated link $L\left(\Lambda^{(n)}\right)$ is said to be a *thick* knot. The loops $X_1^{(n)}, X_2^{(n)}, \ldots, X_m^{(n)}$ are each called *thick* meridans and they have corresponding *thick* longitudes – the latter travel along the thick knot.

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Cable Theorem 3.1. *Suppose that the braid*

$$\Lambda^{(n)} = w\left(\Lambda_1^{(n)}, \ldots, \Lambda_{m-1}^{(n)}\right)$$

is such that the permutation $\overline{\pi}$ of its strands is an m-cycle. Then the permutation corresponding to the mn-braid

$$\Lambda^{(n)} \cdot {}_{(m-1)n+1}\sigma_{mn-1}$$

is an mn-cycle. So the link

$$L\left(\Lambda^{(n)} \cdot {}_{(m-1)n+1}\sigma_{mn-1}\right)$$

corresponding to this braid is a knot – a cable knot. The group

$$\mathcal{G}\left(L\left(\Lambda^{(n)} \cdot {}_{(m-1)n+1}\sigma_{mn-1}\right)\right)$$

of this cable knot has generators

$$X_1^{(n)}, X_2^{(n)}, \ldots, X_m^{(n)}, x_{(m-1)n+1}$$

with defining relations

$$(*)_t \qquad X_t^{(n)} = A_t^{(n)}(X) \,.\, X_{t\overline{\pi}}^{(n)} \,.\, \left(A_t^{(n)}(X)\right)^{-1}$$

for $t = 1, 2, \ldots, m$ and

$$(*) \qquad X_m^{(n)} = \left(x_{(m-1)n+1} \,.\, l_m\right)^n . \, l_m^{-n},$$

where the thick longitude l_m belongs to the second commutator subgroup. Further if the Alexander polynomial of the knot

$$L(w(\sigma_1, \sigma_2, \ldots, \sigma_{m-1}))$$

is $f(\tau)$, then the Alexander polynomial of the above given cable knot is $f(\tau^n)$.

Proof: According to the Theorem of Artin and Birman (see Moran [**5**, Chapter 6]) one has that the group of the cable knot has generators

$$x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}, \ldots, x_{(m-1)n+1}, \ldots, x_{mn}$$

and defining relations

$$x_j \Lambda^{(n)} = x_j \left({}_{(m-1)n+1}\sigma_{mn-1}\right)^{-1} \quad \text{for } 1 \leqslant j \leqslant mn.$$

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These relations are equivalent to the defining relations

$$(\dagger)\quad x_j\Lambda^{(n)} = \begin{cases} x_j & \text{for } 1 \leqslant j \leqslant (m-1)n \\ x_{j+1} & \text{for } (m-1)n < j < mn \\ \left(x_{(m-1)n+2}\cdots x_{mn}\right)^{-1}.x_{(m-1)n+1}.(\ldots) & \text{for } j = mn. \end{cases}$$

Here, by Lemmas 2.1 and 2.2, we have that

$$x_j\Lambda^{(n)} = A_t^{(n)}.x_{j\pi}.\left(A_t^{(n)}\right)^{-1}$$

where $j = (t-1)n + r$ with $1 \leqslant r \leqslant n$ and $1 \leqslant t \leqslant m$. We also use the generators $X_1^{(n)}, X_2^{(n)}, \ldots, X_m^{(n)}$ with defining relations

$$(\dagger)_s \qquad X_s^{(n)} = x_{(s-1)n+1}.x_{(s-1)n+2}\cdots x_{sn}$$

for $1 \leqslant s \leqslant m$. Consequences of these relations are the relations

$$(*)_t \qquad X_t^{(n)} = A_t^{(n)}(X).X_{t\pi}^{(n)}.\left(A_t^{(n)}(X)\right)^{-1}$$

for $t = 1, 2, \ldots, m$. The last relation of $(\dagger)$ is superfluous and hence can be ignored.

Suppose that we now were to eliminate the generators x_k for all $k \leqslant (m-1)n$ from the defining relations

$$A_t^{(n)}.x_{i\pi}.\left(A_t^{(n)}\right)^{-1} = x_{j+1} \quad \text{for } (m-1)n < j < mn$$

by using the defining relations

$$A_s^{(n)}.x_{k\pi}.\left(A_s^{(n)}\right)^{-1} = x_k \quad \text{for } 1 \leqslant k \leqslant (m-1)n$$

of $(\dagger)$. Then we obtain the relations

$$l_m.x_j.l_m^{-1} = x_{j+1} \quad \text{for } (m-1)n < j < mn,$$

where l_m denotes the thick longitude of the thick knot relative to the thick meridan $X_m^{(n)}$, by Lemma 2.1 (see also Moran [5, Chapter 17]). By Lemma 2.3, we know that l_m and $X_m^{(n)}$ commute and l_m belongs to the second commutator subgroup. So we obtain the relations

$$l_m^h.x_{(m-1)n+1}.l_m^{-h} = x_{(m-1)n+h+1}$$

for $h = 1, 2, \ldots, n-1$.

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We now do eliminate the generators x_k for all $k \leqslant (m-1)n$ from the defining relations. This gives that the group of the cable knot has generators

$$X_1^{(n)}, X_2^{(n)}, \ldots, X_m^{(n)}, x_{(m-1)n+1}, \ldots, x_{mn}$$

and defining relations:

$$\begin{aligned}
&(*)_t \quad \text{for } t = 1, 2, \ldots, m;\\
&(\dagger)_m;\\
&l_m^h \cdot x_{(m-1)n+1} \cdot l_m^{-h} = x_{(m-1)n+h+1} \text{ for } 1 \leqslant h < n.
\end{aligned}$$

Hence one obtains the required defining relations.

Finally let A denote the Alexander matrix of the relations $(*)_t$ with $t = 1, 2, \ldots, m$ on the generators $X_1^{(n)}, \ldots, X_m^{(n)}$ with the variable T being put equal to τ^n. Then the Alexander matrix of the cable knot is equal to

$$\left[\begin{array}{c:c}
A & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\
\hdashline
0 \cdots 0\, 1 & -\tau^{n-1} - \tau^{n-2} - \cdots - \tau - 1
\end{array}\right].$$

The elementary ideal of this matrix is the ideal generated by the determinant

$$\left|\begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_{m-1} \\ 0 \cdots 0\, 1 \end{array}\right|$$

where A_i denotes the ith row of the matrix A for $i = 1, 2, \ldots, m-1$. By Lemma 2.2, the above given determinant is equivalent to $f(T) = f(\tau^n)$. $\square$

NOTE 3.2. A similar Cable Theorem holds for the cable knot corresponding to the braid

$$\Lambda^{(n)} \cdot \left({}_{(m-1)n+1}\sigma_{mn-1}\right)^{-1}.$$

Some straight forward modifications of the proof of the Cable Theorem 3.1 give the following generalisation of this theorem.

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Companion Theorem 3.3. *Suppose that the braid*

$$\Lambda^{(n)} = w\left(\Lambda_1^{(n)}, \ldots, \Lambda_{m-1}^{(n)}\right)$$

is such that the permutation $\overline{\pi}$ of its strands is an m-cycle. Further suppose that σ is an mn-braid which acts nontrivially only on the last n strings in such a way that the corresponding permutation is an n-cycle. Then the permutation corresponding to the mn-braid

$$\Lambda^{(n)} . \sigma$$

is an mn-cycle. So the link

$$L\left(\Lambda^{(n)} . \sigma\right)$$

corresponding to this braid is a knot – a companion knot. The group

$$\mathcal{G}\left(L\left(\Lambda^{(n)} . \sigma\right)\right)$$

of this companion knot has generators

$$X_1^{(n)}, X_2^{(n)}, \ldots, X_m^{(n)}, x_{(m-1)n+1}, \ldots, x_{mn}$$

with defining relations

$$X_t^{(n)} = A_t^{(n)}(X) . X_{t\overline{\pi}}^{(n)} . \left(A_t^{(n)}(X)\right)^{-1}$$

for $t = 1, 2, \ldots, m$ and

$$X_m^{(n)} = x_{(m-1)n+1} \cdots x_{mn}$$

and

$$l_m . x_j . l_m^{-1} = x_j \sigma^{-1}$$

for $j = (m-1)n+1, \ldots, mn$. Here l_m denotes the thick longitude which is an element of the second commutator subgroup. If the Alexander polynomials of the knots

$$L(w(\sigma_1, \sigma_2, \ldots, \sigma_{m-1})) \quad \text{and} \quad L(\sigma)$$

are $f(\tau)$ and $g(\tau)$ respectively, where σ is considered to be an n-braid, then the Alexander polynomial of the above given companion knot is

$$f(\tau^n) . g(\tau).$$

Corollary 3.3.1. *The group*

$$\mathcal{G}\left(L\left(\Lambda^{(n)} . \sigma\right)\right)$$

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is isomorphic to the free product with amalgamation

$$\mathcal{G}\Big(L\big(\Lambda^{(n)}\big)\Big) * \mathcal{G}\big(L(\sigma^{-1}.\rho)\big)$$
$$l_m = x_{mn+1}$$
$$X_m^{(n)} = x_{(m-1)n+1} \cdots x_{mn}$$

which amalgamates a free Abelian subgroup of rank two. Here $L\big(\Lambda^{(n)}\big)$ *is considered to be a link on the strands of* $\Lambda^{(n)}$ *as strings and so* $\mathcal{G}\big(L\big(\Lambda^{(n)}\big)\big)$ *has generators*

$$X_1^{(n)}, X_2^{(n)}, \dots, X_m^{(n)}.$$

The braid ρ *is defined to be the* $(n+1)$*-braid*

$$\rho = \sigma_{mn+1}\,\sigma_{mn} \cdots \sigma_{(m-1)n+2}\,\sigma_{(m-1)n+1}\,\sigma_{(m-1)n+2} \cdots \sigma_{mn}\,\sigma_{mn+1}$$

and σ *will now be considered to be an* $(n+1)$*-braid.*

See Moran [**5**, Chapter 20] for the relevant facts concerning the braid ρ, whose corresponding automorphism of a free group behaves like an inner automorphism with respect to the generator x_{mn+1} in the group

$$\mathcal{G}\big(L(\sigma^{-1}.\rho)\big).$$

4. Tying braids

Suppose that σ and τ are finite braids on $s+t$ strings, where σ operates trivially on the last t strings while τ operates trivially on the first s strings. Then the following lemma is a variant of a well known result.

The Thai Lemma 4.1. *The group* $\mathcal{G}(L(\sigma\tau\sigma_s))$ *has the following decomposition into a free product with amalgamation*

$$\mathcal{G}(L(\sigma)) \underset{x_s = y_{s+1}}{*} \mathcal{G}\big(L(\tau^{-1})\big),$$

where the braids σ *and* τ^{-1} *are considered to operate only on the first* s *strings and the last* t *strings respectively.*

Proof: The group $\mathcal{G}(L(\sigma\tau\sigma_s))$ has generators $z_1, z_2, \dots, z_s, \dots, z_{s+t}$ and defining relations

$$z_i\sigma\tau = z_i\sigma_s^{-1} = \begin{cases} z_i & \text{if } i \neq s,\ s+1 \\ z_{s+1} & \text{if } i = s \\ z_{s+1}^{-1} z_s z_{s+1} & \text{if } i = s+1. \end{cases}$$

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The defining relations can also be written as

$$z_i\sigma = \begin{cases} z_i\tau^{-1} & \text{if } i \neq s,\ s+1 \\ z_{s+1}\tau^{-1} & \text{if } i = s \\ \left(z_{s+1}\tau^{-1}\right)^{-1} . z_s . \left(z_{s+1}\tau^{-1}\right) & \text{if } i = s+1. \end{cases}$$

Now we take

$$z_i = \begin{cases} x_i & \text{if } 1 \leqslant i \leqslant s \\ y_i & \text{if } 1 \leqslant i - s \leqslant t. \end{cases}$$

In this notation the defining relations become

$$\begin{aligned} x_i\sigma &= x_i && \text{for } 1 \leqslant i \leqslant s-1 \\ x_s\sigma &= y_{s+1}\tau^{-1} \\ y_{s+1} &= \left(y_{s+1}\tau^{-1}\right)^{-1} . x_s . \left(y_{s+1}\tau^{-1}\right) \\ y_i &= y_i\tau^{-1} && \text{for } s+2 \leqslant i \leqslant s+t. \end{aligned}$$

Now from the basic property of braids it follows that

$$\begin{aligned} (x_1x_2 \dots x_{s-1}x_s)\sigma &= (x_1\sigma).(x_2\sigma) \dots (x_{s-1}\sigma).(x_s\sigma) \\ &= x_1x_2 \dots x_{s-1}x_s \end{aligned}$$

and so $x_s\sigma = x_s$. For similar reasons one has that $y_{s+1} = y_{s+1}\tau^{-1}$. This gives the required decomposition into a free product with amalgamation. ☐

NOTE 4.2. It is not difficult to show that

$$\mathcal{G}\left(L\left(\tau^{-1}\right)\right) \cong \mathcal{G}(L(\tau)).$$

Further a similar result to that given in the above Lemma holds also for the group

$$\mathcal{G}\left(L\left(\sigma\tau\sigma_s^{-1}\right)\right).$$

We now consider an infinite version of the Thai Lemma. Suppose that

$$\sigma(1),\ \sigma(2), \dots,\ \sigma(r), \dots$$

is an infinite sequence of braids so that $\sigma(r)$ acts trivially on all those strings which do not have upper ends

$$n_1 + \dots + n_{r-1} + 1 \quad \text{to} \quad n_1 + \dots + n_{r-1} + n_r$$

with $n_r \geqslant 2$ for each positive integer r. Now the two infinite braids

$$\prod_{r\geqslant 1} \sigma(r) \quad \text{and} \quad \prod_{r\geqslant 1} \sigma_{n_1+n_2+\ldots+n_r}$$

is each a sane braid on positive strings in which the pushing down process is possible. The same is true of their inverses, since the terms in each case commute and thus one has

$$\left(\prod_{r\geqslant 1} \sigma(r)\right)^{-1} = \prod_{r\geqslant 1} (\sigma(r))^{-1} \qquad \text{and}$$

$$\left(\prod_{r\geqslant 1} \sigma_{n_1+n_2+\ldots+n_r}\right)^{-1} = \prod_{r\geqslant 1} \sigma^{-1}_{n_1+n_2+\ldots+n_r}.$$

Hence the above given proof of the Thai Lemma and the results of Moran [**6**] (see in particular Paragraphs 6, 7 and 9) give

THE INFINITE NECKLACE OF LINKS THEOREM 4.3. *The group*

$$\mathcal{G}\left(L\left(\left(\prod_{r\geqslant 1} \sigma(r)\right)\cdot\left(\prod_{r\geqslant 1} \sigma_{n_1+n_2+\ldots+n_r}\right)\right)\right)$$

is isomorphic to the Topologist's free product of the groups $\mathcal{G}(L(\sigma(r)))$, *where* $r \geqslant 1$, *with the topological amalgamations*

$$x_{n_1+n_2+\ldots+n_r} = x_{n_1+n_2+\ldots+n_r+1} \quad \textit{for each } r \geqslant 1.$$

In fact the results of Moran [**6**] gives us that the sought after group is isomorphic to the Topologist's free group on the countably infinite set of generators $x_1, x_2, \ldots, x_n, \ldots$ modulo the topological defining relations

$$x_i\sigma(1) = \begin{cases} x_i & \text{if } 1 \leqslant i < n_1 \\ x_{n_1+1} & \text{if } i = n_1 \end{cases}$$

and

$$x_i\sigma(r) = \begin{cases} x_i^{-1}x_{i-1}x_i & \text{if } i = n_1 + \ldots + n_{r-1} + 1 \\ x_i & \text{if } n_1 + \ldots + n_{r-1} + 1 < i < n_1 + \ldots + n_{r-1} + n_r \\ x_{i+1} & \text{if } i = n_1 + \ldots + n_{r-1} + n_r \end{cases}$$

for $r = 2, 3, \ldots$.

Griffiths [**4**] contains the definition of the Topologist's free product and some of its relevant properties can also be found there.

NOTE 4.4. The above given infinite necklace contains each of the links $L(\sigma(r))$ tied in and its group contains each group $\mathcal{G}L(\sigma(r))$ as a subgroup. In fact we have

$$L\Big(\Big(\prod_{r\geqslant 1}\sigma(r)\Big)\cdot\Big(\prod_{r\geqslant 1}\sigma_{n_1+\ldots+n_r}\Big)\Big)$$

is

$$\lim_{m\to\infty} L\Big(\Big(\prod_{r=1}^{m}\sigma(r)\Big)\cdot\Big(\prod_{r=1}^{m-1}\sigma_{n_1+\ldots+n_r}\Big)\Big)$$

and

$$\mathcal{G}L\Big(\Big(\prod_{r\geqslant 1}\sigma(r)\Big)\cdot\Big(\prod_{r\geqslant 1}\sigma_{n_1+\ldots+n_r}\Big)\Big)$$

is

$$\lim_{m\to\infty} \mathcal{G}L\Big(\Big(\prod_{r=1}^{m}\sigma(r)\Big)\cdot\Big(\prod_{r=1}^{m-1}\sigma_{n_1+\ldots+n_r}\Big)\Big).$$

COROLLARY 4.5. *Suppose that each $L(\sigma(r))$ is a knot whose Alexander polynomial is $f_r(t)$ with $f_r(t)$ being a polynomial in t^{k_r}, where k_r is a positive integer so that*

$$k_r\to\infty \quad \text{as} \quad r\to\infty.$$

Then the Alexander power series of the knot

$$L\Big(\Big(\prod_{r\geqslant 1}\sigma(r)\Big)\cdot\Big(\prod_{r\geqslant 1}\sigma_{n_1+\ldots+n_r}\Big)\Big)$$

is the formal power series

$$\prod_{r\geqslant 1} f_r(t).$$

This is so because we take the definition of the Alexander power series of such a knot to be

$$\lim_{m\to\infty}\ \text{Alexander polynomial}\ L\Big(\Big(\prod_{r=1}^{m}\sigma(r)\Big)\cdot\Big(\prod_{r=1}^{m-1}\sigma_{n_1+\ldots+n_r}\Big)\Big).$$

EXAMPLE 4.6. A rich source of knots whose Alexander polynomials satisfy the conditions of the above Corollary is given by cable knots. Further if one has two infinite necklaces of knots of the type given in the Corollary which have distinct Alexander power series, then these two infinite necklaces of knots are not equivalent.

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Institute of Mathematics
The University of Kent
Canterbury
Kent CT2 7NF
United Kingdom