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RESEARCH ARTICLE

Urod algebras and Translation of W-algebras

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Abstract
In this work, we introduce Urod algebras associated to simply laced Lie algebras as well as the concept of translation of W-algebras.

Both results are achieved by showing that the quantum Hamiltonian reduction commutes with tensoring with integrable representations; that is, for V and L an affine vertex algebra and an integrable affine vertex algebra associated with $\mathfrak{g}$, we have the vertex algebra isomorphism $H^0_{DS,f}(V \otimes L) \cong H^0_{DS,f}(V) \otimes L$, where in the left-hand-side the Drinfeld–Sokolov reduction is taken with respect to the diagonal action of $\widehat{\mathfrak{g}}$ on $V \otimes L$.

The proof is based on some new construction of automorphisms of vertex algebras, which may be of independent interest. As corollaries, we get fusion categories of modules of many exceptional W-algebras, and we can construct various corner vertex algebras.

A major motivation for this work is that Urod algebras of type A provide a representation theoretic interpretation of the celebrated Nakajima–Yoshioka blowup equations for the moduli space of framed torsion free sheaves on $\mathbb{CP}^2$ of an arbitrary rank.

1. Introduction

In [BFL16] Bershtein, Litvinov and the third named author introduced the *Urod algebra*, which gives a representation theoretic interpretation of the celebrated *Nakajima–Yoshioka blowup equations* [NY05] for the moduli space of framed torsion free sheaves on $\mathbb{CP}^2$ of rank two via the Alday–Gaiotto–Tachikawa (AGT) correspondence [AGT10]. One of the aims of the present paper is to introduce the *higher-rank Urod algebras*, which generalizes the result of [BFL16] to the case of sheaves at arbitrary rank.

In fact, it turned out in recent works (see, e.g., [FG18, CG17]) that Urod algebras appear not only in the AGT correspondence but also in various theories of vertex algebras in connection with higher-dimensional quantum field theories. This work provides the first systematic study of Urod algebras appearing in various contexts.

Another aim of this paper is to introduce the *translation for (affine) W-algebras*. Namely, we show that any integrable highest representation L of level $\ell \in \mathbb{Z}_{>0}$ gives rise to an exact functor

$$T_L : \mathscr{W}^k(\mathfrak{g}, f)\text{-Mod} \to \mathscr{W}^{k+\ell}(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto M \otimes L, \tag{1}$$

where $\mathscr{W}^k(\mathfrak{g}, f)$ is the W-algebra associated with $\mathfrak{g}$ and its nilpotent element f at level $k \in \mathbb{C}$ ([FF90a, KRW03]).

We establish these results by showing that the quantized Drinfeld–Sokolov reduction commutes with tensoring with integrable representations.

Let us describe our results in more details.

1.1. Main Theorem

Let $\mathfrak{g}$ be a simple Lie algebra, $\widehat{\mathfrak{g}} = \mathfrak{g}((t)) \oplus \mathbb{C}K$ be the affine Kac–Moody Lie algebra associated with $\mathfrak{g}$ defined by the commutation relation

$$[xf, yg] = [x, y]fg + (x, y)\,\mathrm{Res}_{t=0}(gdf)K,$$

$[K, \widehat{\mathfrak{g}}] = 0$, where $(\ |\ )$ is the normalized inner product of $\mathfrak{g}$ (it is $1/2h^\vee$ times the Killing form of $\mathfrak{g}$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$). For $k \in \mathbb{C}$, let $V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]] \otimes \mathbb{C}K)} \mathbb{C}$, where $\mathbb{C}$ is regarded as a one-dimensional representation of $\mathfrak{g}[[t]] \otimes \mathbb{C}K$ on which $\mathfrak{g}[[t]]$ acts trivially and K acts via multiplication by k. $V^k(\mathfrak{g})$ is naturally a vertex algebra and is called the *universal affine vertex algebra* associated with $\mathfrak{g}$ at level k.

Any (graded) quotient V of $V^k(\mathfrak{g})$ inherits the vertex algebra structure from $V^k(\mathfrak{g})$. Let $\mathbb{L}_k(\mathfrak{g})$ be the unique simple (graded) quotient of $V^k(\mathfrak{g})$. The vertex algebra $\mathbb{L}_k(\mathfrak{g})$ is integrable as an $\widehat{\mathfrak{g}}$-module if and only if $k \in \mathbb{Z}_{\geqslant 0}$.

For a nilpotent element f of $\mathfrak{g}$, let $H^\bullet_{DS,f}(M)$ be the BRST cohomology of the quantized Drinfeld–Sokolov reduction associated with $(\mathfrak{g}, f)$ with coefficients in a $\widehat{\mathfrak{g}}$-module M ([FF90a, KRW03]). The W-algebra associated with $(\mathfrak{g}, f)$ at level k is by definition the vertex algebra

$$\mathscr{W}^k(\mathfrak{g}, f) = H^0_{DS,f}(V^k(\mathfrak{g})).$$

For any smooth $\widehat{\mathfrak{g}}$-module M of level k, $H^i_{DS,f}(M)$, $i \in \mathbb{Z}$, is a module over $\mathscr{W}^k(\mathfrak{g}, f)$. More generally, for any vertex algebra V equipped with a vertex algebra homomorphism $V^k(\mathfrak{g}) \to V$ and a V-module M, $H^i_{DS,f}(M)$, $i \in \mathbb{Z}$, is a module over the vertex algebra $H^0_{DS,f}(V)$.

Theorem 1. *Let V be a quotient of the universal affine vertex algebra $V^k(\mathfrak{g})$ and $\ell \in \mathbb{Z}_{\geqslant 0}$. We have a vertex algebra isomorphism*

$$H^0_{DS,f}(V \otimes \mathbb{L}_\ell(\mathfrak{g})) \cong H^0_{DS,f}(V) \otimes \mathbb{L}_\ell(\mathfrak{g}), \tag{2}$$

where in the left-hand-side the Drinfeld–Sokolov reduction is taken with respect to the diagonal action of $\widehat{\mathfrak{g}}$ on $V \otimes \mathbb{L}_\ell(\mathfrak{g})$. More generally, let V be a vertex algebra equipped with a vertex algebra homomorphism $V^k(\mathfrak{g}) \to V$. Then we have an isomorphism

$$H^\bullet_{DS,f}(V \otimes \mathbb{L}_\ell(\mathfrak{g})) \cong H^\bullet_{DS,f}(V) \otimes \mathbb{L}_\ell(\mathfrak{g}), \tag{3}$$

of graded vertex algebras, and for any V-module M, $\mathbb{L}_\ell(\mathfrak{g})$-module N, there is an isomorphism

$$H^\bullet_{DS,f}(M \otimes N) \cong H^\bullet_{DS,f}(M) \otimes N.$$

as modules over $H^\bullet_{DS,f}(V \otimes \mathbb{L}_\ell(\mathfrak{g})) \cong H^\bullet_{DS,f}(V) \otimes \mathbb{L}_\ell(\mathfrak{g})$.

In the case that $\mathfrak{g} = \mathfrak{sl}_2$ and $\ell = 1$, the isomorphism (30) was established in [BFL16]. Our argument only requires certain properties of integrable representations and especially also works for superalgebras as well; see Section 5.

We note that the existence of the isomorphism (30) as vector spaces is not difficult to see. However, it is not a priori clear at all why there should exist an isomorphism of vertex algebras. We also note that equation (30) is not compatible with the standard conformal gradings of both sides. To remedy this, we need to change the conformal vector of $\mathbb{L}_\ell(\mathfrak{g})$ on the right-hand side to a new conformal vector ω_{Urod}, which we call the *Urod conformal vector* (see Section 6).

The proof of Theorem 1 is based on some new construction of automorphisms of vertex algebras, which may be of independent interest; see Section 2 for the details.

1.2. Translation for W-algebras

By applying Theorem 1 to $V = V^k(\mathfrak{g})$, $k \in \mathbb{C}$, we obtain the vertex algebra isomorphism

$$H^0_{DS,f}(V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g})) \cong \mathscr{W}^k(\mathfrak{g},f)\otimes\mathbb{L}_\ell(\mathfrak{g}).$$

Consequently, the natural vertex algebra homomorphism $V^{k+\ell}(\mathfrak{g}) \to V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g})$ induces a vertex algebra homomorphism

$$\mathscr{W}^{k+\ell}(\mathfrak{g},f) \to H^0_{DS}(V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g})) \xrightarrow{\sim} \mathscr{W}^k(\mathfrak{g},f)\otimes\mathbb{L}_\ell(\mathfrak{g}).$$

Therefore, for any $\mathscr{W}^k(\mathfrak{g},f)$-module M and any integrable representation L of $\widehat{\mathfrak{g}}$ of level ℓ, $M\otimes L$ is has the structure of an $\mathscr{W}^{k+\ell}(\mathfrak{g},f)$-module. As a consequence, we obtain *the translation by* L, that is, the exact functor (1) as we wished.

Recall that the Zhu algebra of $\mathscr{W}^k(\mathfrak{g},f)$ is isomorphic to the finite W-algebra [Pre02] $U(\mathfrak{g},f)$ associated with $(\mathfrak{g},f)$ ([Ara07, DSK06]). Also, the Zhu algebra of $\mathbb{L}_\ell(\mathfrak{g})$ is isomorphic to the quotient $U_\ell(\mathfrak{g})$ of $U(\mathfrak{g})$ by the two-sided ideal generated by $e_\theta^{\ell+1}$ ([FZ92]), and so is that of $\mathbb{L}_\ell(\mathfrak{g})$ with the Urod conformal structure ([Ara15b]). Therefore, by taking the Zhu algebras, we obtain from equation (1) an algebra homomorphism

$$U(\mathfrak{g},f) \to U(\mathfrak{g},f)\otimes U_\ell(\mathfrak{g}). \tag{4}$$

Since any finite-dimensional $\mathfrak{g}$-module is an $U_\ell(\mathfrak{g})$-module for a sufficiently large ℓ, equation (4) gives $M\otimes E$ a structure of $U(\mathfrak{g},f)$-module for any $U(\mathfrak{g},f)$-module M and a finite-dimensional $\mathfrak{g}$-module E. We expect that this $U(\mathfrak{g},f)$-module structure of $M\otimes E$ does not depend on the choice of a sufficiently large ℓ, and coincides with the one obtained by Goodwin [Goo11].

1.3. Higher-rank Urod algebras

We denote also by $\mathscr{W}^k(\mathfrak{g})$ the W-algebra $\mathscr{W}^k(\mathfrak{g},f_{prin})$ associated with a principal nilpotent element f_{prin} of $\mathfrak{g}$. Let $\mathfrak{g}$ be simply laced, and suppose that $k+h^\vee-1 \notin \mathbb{Q}_{\leqslant 0}$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. By the coset construction [ACL19] of the principal W-algebra $\mathscr{W}^k(\mathfrak{g})$, we have a conformal vertex algebra embedding

$$V^k(\mathfrak{g})\otimes\mathscr{W}^\ell(\mathfrak{g}) \hookrightarrow V^{k-1}(\mathfrak{g})\otimes\mathbb{L}_1(\mathfrak{g}), \tag{5}$$

where ℓ is the number defined by the formula

$$\frac{1}{k+h^\vee} + \frac{1}{\ell+h^\vee} = 1. \tag{6}$$

By taking the Drinfeld–Sokolov reduction with respect to the level k action of $\widehat{\mathfrak{g}}$, equation (5) gives rise to the full vertex algebra embedding

$$\mathscr{W}^k(\mathfrak{g},f)\otimes\mathscr{W}^\ell(\mathfrak{g}) \hookrightarrow H_{DS,f}(V^{k-1}(\mathfrak{g})\otimes\mathbb{L}_1(\mathfrak{g})) \cong \mathscr{W}^{k-1}(\mathfrak{g},f)\otimes\mathcal{U}(\mathfrak{g}), \tag{7}$$

where the last isomorphism follows from Theorem 1 and $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g},f)$ is the vertex algebra $\mathbb{L}_1(\mathfrak{g})$ equipped with the Urod conformal vector ω_{Urod}. We call the vertex operator algebra $\mathcal{U}(\mathfrak{g})$ the *Urod* *algebra*. In the case that $\mathfrak{g} = \mathfrak{sl}_2$ and $f = f_{prin}$, $\mathcal{U}(\mathfrak{g})$ is exactly the Urod algebra introduced in [BFL16].

Let L be a level one integrable representation of $\widehat{\mathfrak{g}}$, which is naturally a module over the Urod algebra $\mathcal{U}(\mathfrak{g})$. By equation (43), for any $\mathscr{W}^k(\mathfrak{g}, f)$-module M, the tensor product $M \otimes L$ has the structure of a $\mathscr{W}^k(\mathfrak{g}, f) \otimes \mathscr{W}^\ell(\mathfrak{g})$-module. We are able to describe the decomposition of $M \otimes L$ as $\mathscr{W}^k(\mathfrak{g}, f) \otimes \mathscr{W}^\ell(\mathfrak{g})$-modules for various M and L (Theorems 8.1, 8.2, 8.4, 8.7 and Corollaries 8.4, 8.8).

In particular, Corollary 8.4 states that, when M is a generic Verma module of $\mathscr{W}^{k-1}(\mathfrak{g})$, $M \otimes L$ decomposes into a direct sum of tensor products of Verma modules of $\mathscr{W}^k(\mathfrak{g})$ and $\mathscr{W}^\ell(\mathfrak{g})$. In the case that $\mathfrak{g} = \mathfrak{sl}_2$ with an appropriate choice of L, this provides the decomposition that was used in [BFL16] to give a representation theoretic interpretation of the Nakajima–Yoshioka blowup equations for the moduli space of rank two framed torsion-free sheaves on $\mathbb{CP}^2$. In a forthcoming paper, we show how the decomposition for $\mathfrak{g} = \mathfrak{sl}_n$ stated in Corollary 8.4 can be used to give a representation theoretic interpretation of Nakajima–Yoshioka blowup equations for the moduli space of framed rank n sheaves on $\mathbb{CP}^2$ via the AGT conjecture established by Schiffmann and Vasserot [SV13].

1.4. Higher-rank Urod algebras and VOA$[M_4]$

The Urod algebra is proposed to be important for general smooth 4-manifolds. This appeared in the recent work [FG18] of Gukov and the third named author. For a compact simply laced Lie group G, one can conjecturally [FG18] associate a vertex operator algebra $\mathrm{VOA}[M] = \mathrm{VOA}[M, G]$ to every smooth 4-manifold M and an category of modules for $\mathrm{VOA}[M]$ to every boundary component of M. The vertex algebra $\mathrm{VOA}[M]$ should then act on the cohomology of the moduli space of G-instantons on M. Moreover, the invariant $\mathrm{VOA}[M]$ should have the following property: Glueing 4-manifolds along a common boundary amounts to extending the tensor product of the two associated vertex algebras along the categories of modules; see [CKM19] for the theory of these vertex algebra extensions.

When $G = SU(2)$ we have [FG18] that

$$\mathrm{VOA}[M_4 \# \overline{\mathbb{CP}^2}] = \mathcal{U}(\mathfrak{sl}_2) \otimes \mathrm{VOA}[M_4],$$

where $M_4 \# \overline{\mathbb{CP}^2}$ is the connected sum of M_4 and $\overline{\mathbb{CP}^2}$. One expects the same type of formula for any simply laced G with $\mathcal{U}(\mathfrak{sl}_2)$ replaced by the corresponding higher-rank Urod algebra.

1.5. Higher-rank Urod algebras and vertex algebras for S-duality

The present work was originally motivated by a conjecture that appeared in the context of vertex algebras for S-duality [CG17] .

The problem lives inside four-dimensional supersymmetric GL-twisted gauge theories and vertex algebras appear on the intersection of three-dimensional topological boundary conditions. Such vertex algebras are typically constructed out of W-algebras and affine vertex algebras associated to the Lie algebra $\mathfrak{g}$ of the gauge group G, and the coupling constant Ψ relates to the level shifted by the dual Coxeter number $h^\vee$. Different boundary conditions can be concanated to yield other boundary conditions and corresponding vertex algebras are related via vertex algebra extensions. Most of [CG17] is dealing with simply laced $\mathfrak{g}$, and the discussion at the bottom of page 22 of [CG17] is concerned with the concatenation of boundary conditions called $B_{1,0}, B_{-1,1}$ and $B_{0,1}$. The main expectation is that the resulting corner vertex algebra coincides with the corner vertex algebra between the boundary conditions $B_{1,0}$ and $B_{0,1}$ dressed by extra decoupled degrees of freedom corresponding to $\mathbb{L}_1(\mathfrak{g})$. In vertex algebra language, this expectation is precisely the statement of Theorem 8.1 with $\mu = 0 = \nu$.

There are further conjectures around vertex algebras and S-duality. These are mainly the construction of junction or corner vertex algebras which are typically large extensions of products of two vertex algebras associated to $\mathfrak{g}$. Theorem 9.1, which gives a lattice type construction of vertex operator algebras of CFT type using W-algebras in place of Heisenberg algebras, proves them in a series of cases.

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1.6. Rigidity of vertex tensor categories

One of the most difficult problems of the subject of vertex algebras is the understanding of tensor categories of modules of a given vertex algebra. The theory of tensor categories of modules of vertex algebras has been developed in the series of papers [HL94, HL95]. In particular, Yi-Zhi Huang has shown the existence of vertex tensor categories for lisse vertex algebras without the rationality assumption [Hua09].

The most challenging technical problem for vertex tensor categories is to prove the rigidity of objects. This is crucial as rigidity gives the categories substantial structure, and many useful theorems only hold for rigid tensor categories. While this problem was settled by Huang for rational, lisse vertex algebras [Hua08], it is wide open for nonrational lisse vertex algebras.

In fact, it is expected that the tensor categories of modules should exist for much more general vertex algebras. Strong evidence was given in [CHY18] that showed the category of ordinary modules over an admissible affine vertex algebra associated with a simply laced Lie algebra has the structure of a vertex tensor category.

We conjecture that the category of ordinary modules over a quasi-lisse vertex algebra [AK18] has the structure of a vertex tensor category (Conjecture 1). Note that an admissible affine vertex algebra is quasi-lisse.

Assuming this conjecture and using an idea of [Cre19], we use the decomposition stated in Theorem 8.1 to prove that certain categories of quasi-lisse W-algebras at admissible level are fusion (Theorem 10.4). Since the conjecture is valid for lisse W-algebras, this gives strong evidence for the rationality conjecture [KW08, Ara15a] of lisse W-algebras at admissible levels, which has been settled only in some special cases [Ara15b, AvE].

It seems that the translation functor is a good tool to prove rationality of vertex operator algebras in suitable cases. We will explain in forthcoming work how to employ the translation functor in order to get new rational $\mathscr{W}$-algebras at nonadmissible levels associated to Deligne's exceptional series [ACK].

2. A construction of automorphisms of vertex algebras

Let V be a vertex algebra. As usual, we set $a_{(n)} = \mathrm{Res}_{z=0}\, z^n a(z)$ for $a \in V$, where $a(z)$ is the quantum field corresponding to a. We have

$$[a_{(m)}, b_{(n)}] = \sum_{j \geqslant 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)} \tag{8}$$

for $a, b \in V$, $m, n \in \mathbb{Z}$. In particular, $[a_{(0)}, b_{(n)}] = (a_{(0)}b)_{(n)}$.

Let A be an element of V such that its zero mode $A_{(0)}$ acts semisimply on V so that $V = \bigoplus_{\lambda\in\mathbb{C}} V[\lambda]$, where $V[\lambda] = \{v \in V \mid A_{(0)}v = \lambda v\}$. We assume that $A \in V[0]$. Set $V[\geqslant \lambda] = \bigoplus_{\mu\in\lambda+\mathbb{R}_{\geqslant 0}} V[\mu] \supset V[> \lambda] = \bigoplus_{\mu>\lambda} V[\mu]$.

Suppose that there exists another element $\hat{A} \in V[\geqslant 0]$ such that $\hat{A} \equiv A (\mathrm{mod}\ V[> 0])$ and $\hat{A}_{(0)}$ acts locally finitely on V. Then, $\hat{A}_{(0)}$ acts semisimply on V, and for $v \in V[\lambda]$, there exists a unique eigenvector $\tilde{v}$ of $\hat{A}_{(0)}$ eigenvalue λ such that $\tilde{v} \equiv v (\mathrm{mod}\ V[> \lambda])$. By extending this correspondence linearly, we obtain a linear map

$$V \to V, \quad v \mapsto \tilde{v}. \tag{9}$$

Lemma 2.1. *The map (9) is an automorphism of V.*

Proof. Clearly, equation (9) is a linear isomorphism. We wish to show that equation (9) is a homomorphism of vertex algebras. It is clear that $\widetilde{|0\rangle} = |0\rangle$. Let $v \in V[\lambda]$, $w \in V[\mu]$, $n \in \mathbb{Z}$. By equation (8), $\tilde{v}_{(n)}\tilde{w}$ is an eigenvector of $\tilde{A}_{(0)}$ of eigenvalue $\lambda + \mu$ and $\tilde{v}_{(n)}\tilde{w} \equiv v_{(n)}w (\mathrm{mod}\ V[> \lambda + \mu])$. Therefore,

$\tilde{v}_{(n)}\tilde{w} = \widetilde{v_{(n)}w}$. Similarly, $T\tilde{v}$ is an eigenvector of $\tilde{A}_{(0)}$ of eigenvalue λ such that $T\tilde{v} \equiv Tv (\mathrm{mod}\ V[>\lambda])$. Hence, $T\tilde{v} = \widetilde{Tv}$. This completes the proof. □

We also have the following.

Lemma 2.2. *The action of* $\hat{A}_{(0)}$ *on* V *coincides with that of* $\tilde{A}_{(0)}$.

Proof. For an eigenvector $v \in V[\geqslant \lambda]$ of $\hat{A}_{(0)}$ of eigenvalue λ, $(\hat{A}_{(0)} - \tilde{A}_{(0)})v$ is also an eigenvector of $\hat{A}_{(0)}$ of eigenvalue λ. On the other hand, $(\hat{A}_{(0)} - \tilde{A}_{(0)})v$ belongs to $V[>\lambda]$. Since all eigenvalues of $\hat{A}_{(0)}$ on $V[>\lambda]$ are greater than λ, the vector $(\hat{A}_{(0)} - \tilde{A}_{(0)})v$ must be zero. This completes the proof. □

Let M be a V-module on which both $A_{(0)}$ and $\hat{A}_{(0)}$ act semisimply. Set $M[\lambda] = \{m \in M \mid A_{(0)}m = \lambda m\}$, $M[>\lambda] = \sum_{\mu>\lambda} M[\mu]$. We can define a linear isomorphism

$$M \xrightarrow{\sim} M, \quad m \mapsto \tilde{m} \tag{10}$$

that sends $m \in M[\lambda]$ to a unique eigenvector $\tilde{m}$ of $\hat{A}_{(0)}$ of eigenvalue λ such that $\tilde{m} \equiv m (\mathrm{mod}\ M[>\lambda])$. The following assertion can be shown in the same manner as Lemma 2.1.

Lemma 2.3. *We have* $\widetilde{a_{(n)}m} = \tilde{a}_{(n)}\tilde{m}$ *for* $a \in V$, $m \in M$, $n \in \mathbb{Z}$, *that is, (10) is an isomorphism of* V*-modules.*

3. Preliminaries on Drinfeld–Sokolov reduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let $V^k(\mathfrak{g})$ be the universal affine vertex algebra associated with $\mathfrak{g}$ at level k as in the introduction. A $V^k(\mathfrak{g})$-module is the same as a smooth module M of level k over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$. Here, a $\widehat{\mathfrak{g}}$-module M is called smooth if $x(z) = \sum_{n\in\mathbb{Z}}(xt^n)z^{-n-1}$ is a (quantum) field on M for all $x \in \mathfrak{g}$, that is, $(xt^n)m = 0$ for a sufficiently large n for any $m \in M$.

Let f be a nilpotent element of $\mathfrak{g}$. Let

$$\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j \tag{11}$$

be a good grading ([KRW03]) of $\mathfrak{g}$ for f, that is, $f \in \mathfrak{g}_{-1}$, $\mathrm{ad} f : \mathfrak{g}_j \to \mathfrak{g}_{j-1}$ is injective for $j \geqslant 1/2$ and surjective for $j \leqslant 1/2$. Denote by x_0 the semisimple element of $\mathfrak{g}$ that defines the grading, i.e.,

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [x_0, x] = jx\}. \tag{12}$$

We write $\deg x = d$ if $x \in \mathfrak{g}_d$.

Let $\{e, h, f\}$ be an $\mathfrak{sl}_2$-triple associated with f in $\mathfrak{g}$. Then the grading defined by $x_0 = 1/2h$ is good and is called a *Dynkin grading*.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is contained in the Lie subalgebra $\mathfrak{g}_0$. Let Δ be the set of roots of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ the root space decomposition. Set $\Delta_j = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_j\}$ so that $\Delta = \bigsqcup_{j\in\frac{1}{2}\mathbb{Z}} \Delta_j$. Put $\Delta_{>0} = \bigsqcup_{j>0} \Delta_j$. Let $I = \{1, 2, \dots, \mathrm{rk}\,\mathfrak{g}\}$, and let $\{x_a \mid a \in I \sqcup \Delta\}$ be a basis of $\mathfrak{g}$ such that $x_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta$, and $x_i \in \mathfrak{h}$, $i \in I$. Denote by c_{ab}^d the corresponding structure constant.

Set $\mathfrak{g}_{\geqslant 1} = \bigoplus_{j\geqslant 1} \mathfrak{g}_j$, $\mathfrak{g}_{>0} = \bigoplus_{j\geqslant 1/2} \mathfrak{g}_j$, and let $\chi : \mathfrak{g}_{\geqslant 1} \to \mathbb{C}$ be the character defined by $\chi(x) = (f|x)$. We extend χ to the character $\hat{\chi}$ of $\mathfrak{g}_{\geqslant 1}[t, t^{-1}]$ by setting $\hat{\chi}(xt^n) = \delta_{n,-1}\chi(x)$. Define

$$F_\chi = U(\mathfrak{g}_{>0}[t, t^{-1}]) \otimes_{U(\mathfrak{g}_{>0}[t]+\mathfrak{g}_{\geqslant 1}[t,t^{-1}])} \mathbb{C}_{\hat{\chi}},$$

where $\mathbb{C}_{\hat{\chi}}$ is the one-dimensional representation of $\mathfrak{g}_{>0}[t] + \mathfrak{g}_{\geqslant 1}[t, t^{-1}]$ on which $\mathfrak{g}_{\geqslant 1}[t, t^{-1}]$ acts by the character $\hat{\chi}$ and $\mathfrak{g}_{>0}[t]$ acts trivially. Since it is a smooth $\mathfrak{g}_{>0}[t, t^{-1}]$-module, the space F_χ is a module

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over the vertex subalgebra $V(\mathfrak{g}_{>0}) \subset V^k(\mathfrak{g})$ generated by $x_\alpha(z)$ with $\alpha \in \Delta_{>0}$. For $\alpha \in \Delta_{>0}$, let $\Phi_\alpha(z)$ denote the image of $x_\alpha(z)$ in $(\operatorname{End} F_\chi)[[z, z^{-1}]]$. Then

$$\Phi_\alpha(z) = \chi(x_\alpha) \quad \text{for } \alpha \in \Delta_{\geqslant 1}$$

and

$$\Phi_\alpha(z)\Phi_\beta(w) \sim \frac{\chi([x_\alpha, x_\beta])}{z-w}.$$

There is a unique vertex algebra structure on F_χ such that $|0\rangle = 1 \otimes 1$ is the vacuum vector and

$$Y((\Phi_\alpha)_{(-1)}|0\rangle, z) = \Phi_\alpha(z)$$

for $\alpha \in \Delta_{>0}$. (Note that $(\Phi_\alpha)_{(-1)}|0\rangle = \chi(x_\alpha)|0\rangle$ for $\alpha \in \Delta_{\geqslant 1}$.) In other words, F_χ has the structure of the $\beta\gamma$-system associated with the symplectic vector space $\mathfrak{g}_{1/2}$ with the symplectic form

$$\mathfrak{g}_{1/2} \times \mathfrak{g}_{1/2} \to \mathbb{C}, \quad (x, y) \mapsto \chi([x, y]). \tag{13}$$

Next, let $\bigwedge^{\infty/2+\bullet}(\mathfrak{g}_{>0})$ be the vertex superalgebra generated by odd fields $\psi_\alpha(z)$, $\psi^*_\alpha(z)$, $\alpha \in \Delta_{>0}$, with the OPEs

$$\psi_\alpha(z)\psi^*_\beta(z) \sim \frac{\delta_{\alpha,\beta}}{z-w}, \quad \psi_\alpha(z)\psi_\beta(z) \sim \psi^*_\alpha(z)\psi^*_\beta(z) \sim 0$$

Let $\bigwedge^{\infty/2+\bullet}(\mathfrak{g}_{>0}) = \bigoplus_{n\in\mathbb{Z}} \bigwedge^{\infty/2+n}(\mathfrak{g}_{>0})$ be the $\mathbb{Z}$-gradation defined by $\deg|0\rangle = 0$, $\deg(\psi_\alpha)_{(n)} = -1$, $\deg(\psi^*_\alpha)_{(n)} = 1$.

For a smooth $\widehat{\mathfrak{g}}$-module M, set

$$C(M) := M \otimes F_\chi \otimes \bigwedge\nolimits^{\infty/2+\bullet}(\mathfrak{g}_{>0}). \tag{14}$$

Then $C(M) = \bigoplus_{i\in\mathbb{Z}} C^i(M)$, $C^i(M) = M \otimes F_\chi \otimes \bigwedge^{\infty/2+i}(\mathfrak{g}_{>0})$. Note that $C(V^k(\mathfrak{g}))$ is naturally a vertex superalgebra, and $C(M)$ is a module over the vertex superalgebra $C(V^k(\mathfrak{g}))$ for any smooth $\widehat{\mathfrak{g}}$-module M. Define

$$Q(z) = \sum_{\alpha\in\Delta_{>0}} x_\alpha(z)\psi^*_\alpha(z) + \sum_{\alpha\in\Delta_{>0}} \Phi_\alpha(z)\psi^*_\alpha(z) - \frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta_{>0}} c^\gamma_{\alpha,\beta}\psi^*_\alpha(z)\psi^*_\beta(z)\psi_\gamma(z),$$

where we have omitted the tensor product symbol. Then $Q(z)Q(w) \sim 0$, and we have $Q^2_{(0)} = 0$ on any $C(V^k(\mathfrak{g}))$-module. The cohomology

$$H^\bullet_{DS,f}(M) := H^\bullet(C(M), Q_{(0)})$$

is called the BRST cohomologyof the Drinfeld–Sokolov reduction associated with f with coefficients in M ([FF90a, KRW03]; see also [Ara05]). By definition [Feĭ84], $H^\bullet_{DS,f}(M)$ is the semi-infinite $\mathfrak{g}_{>0}[t, t^{-1}]$-cohomology $H^{\frac{\infty}{2}+\bullet}(\mathfrak{g}_{>0}[t, t^{-1}], M \otimes F_\chi)$ with coefficients in the diagonal $\mathfrak{g}_{>0}[t, t^{-1}]$-module $M \otimes F_\chi$.

The vertex algebra

$$\mathscr{W}^k(\mathfrak{g}, f) := H^0_{DS,f}(V^k(\mathfrak{g}))$$

is called the W-algebra associated with $(\mathfrak{g}, f)$ at level k, which is conformal provided that $k \neq -h^\vee$. The vertex algebra structure of $\mathscr{W}^k(\mathfrak{g}, f)$ does not depend on the choice of a good grading ([AKM15]);

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however, its conformal structure does. The central charge of $\mathscr{W}^k(\mathfrak{g}, f)$ is given by

$$\dim \mathfrak{g} - \frac{1}{2}\dim \mathfrak{g}_{1/2} - 12\left|\frac{\rho}{\sqrt{k+h^\vee}} - \sqrt{k+h^\vee}x_0\right|^2, \tag{15}$$

where ρ is the half sum of positive roots of $\mathfrak{g}$.

Let $\mathscr{W}_k(\mathfrak{g}, f)$ be the unique simple graded quotient of $\mathscr{W}^k(\mathfrak{g}, f)$.

Let KL be the full subcategory of $\widehat{\mathfrak{g}}$-modules consisting of objects on which $\mathfrak{g}$ acts semisimply and $t\mathfrak{g}[t]$ acts locally nilpotently, and let KL_k be the full subcategory of KL consisting of modules of level k.

Let Q be the root lattice of $\mathfrak{g}$, $\check{Q}$ the coroot lattice, P the weight lattice, $\check{P}$ the coweight lattice, P_+ the set of dominant integral weights and $\check{P}_+$ the set of dominant integral coweights of $\mathfrak{g}$. For $\lambda \in P_+$, set

$$\mathbb{V}_\lambda^k := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} E_\lambda \in \mathrm{KL}_k,$$

where E_λ is the irreducible finite-dimensional $\mathfrak{g}$-module of highest weight λ lifted to a $\mathfrak{g}[t]$-module by letting $\mathfrak{g}[t]t$ act trivially and K by multiplication with the scalar k. Note that $V^k(\mathfrak{g}) = \mathbb{V}_0^k$ as a $\widehat{\mathfrak{g}}$-module. We denote by $\mathbb{L}_\lambda^k$ the unique simple graded quotient of $\mathbb{V}_\lambda^k$. More generally, for any weight λ of $\mathfrak{g}$, we denote by $\mathbb{L}_\lambda^k$ the irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight λ and level k.

Theorem 3.1 ([FG10, Ara15a]). *We have $H^i_{DS,f}(M) = 0$ for $i \neq 0$ and $M \in \mathrm{KL}_k$. In particular, the functor*

$$\mathrm{KL}_k \to \mathscr{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto H^0_{DS,f}(M)$$

is exact.

Note that for $M \in \mathrm{KL}_k, N \in \mathrm{KL}_\ell$, we have $M \otimes N \in \mathrm{KL}_{k+\ell}$. Therefore, $H^i_{DS,f}(M\otimes N) = 0$ for $i \neq 0$. In particular, $H^i_{DS,f}(M\otimes L) = 0$ for $i \neq 0$ if $M \in \mathrm{KL}$ and L is an integrable representation of $\widehat{\mathfrak{g}}$.

4. Proof of Theorem 1

Let $k \in \mathbb{C}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, and set

$$C := C(V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g})) = V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g})\otimes F_{\chi\otimes} \bigwedge\nolimits^{\infty/2+\bullet}(\mathfrak{g}_{>0}).$$

For $t \in \mathbb{C}$, define the element $Q_t \in C$ by

$$\begin{aligned} Q_t(z) = &\sum_{\alpha\in\Delta_{>0}} (\pi_1(x_\alpha)(z) + t^{2\alpha(x_0)}\pi_2(x_\alpha)(z))\psi^*_\alpha(z) \\ &+ \sum_{\alpha\in\Delta_{>0}} \Phi_\alpha(z)\psi^*_\alpha(z) - \frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta_{>0}} c^\gamma_{\alpha,\beta}\psi^*_\alpha(z)\psi^*_\beta(z)\psi_\gamma(z), \end{aligned} \tag{16}$$

where $\pi_1(x_a)(z)$ (resp. $\pi_2(x_a)(z)$) denotes the action of $x_a(z)$, $a \in I \sqcup \Delta$, on $V^k(\mathfrak{g})$ (resp. on $\mathbb{L}_\ell(\mathfrak{g})$). Then $Q_t(z)Q_t(w) \sim 0$, and therefore, $(Q_t)^2_{(0)} = 0$. It follows that $(C, (Q_t)_{(0)})$ is a differential graded vertex algebra, and the corresponding cohomology $H^\bullet(C, (Q_t)_{(0)})$ is naturally a vertex algebra. Clearly,

$$H^i(C, (Q_{t=0})_{(0)}) \cong H^i_{DS,f}(V^k(\mathfrak{g}))\otimes\mathbb{L}_\ell(\mathfrak{g}) = \delta_{i,0}\mathscr{W}^k(\mathfrak{g}, f)\otimes\mathbb{L}_\ell(\mathfrak{g}), \tag{17}$$

$$H^i(C, (Q_{t=1})_{(0)}) \cong H^i_{DS,f}(V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g})) = \delta_{i,0}H^0_{DS,f}(V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g})); \tag{18}$$

see Theorem 3.1.

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By [Ara05, 3.7], the differential $(Q_t)_{(0)}$ decomposes as

$$(Q_t)_{(0)} = d_t^{st} + d^{\chi}, \quad (d_t^{st})^2 = (d^{\chi})^2 = \{d_t^{st}, d^{\chi}\} = 0, \tag{19}$$

where

$$\begin{aligned} d_t^{st} = &\sum_{\alpha\in\Delta_{>0}}\sum_{n\in\mathbb{Z}}(\pi_1(x_\alpha)_{(n)} + t^{2\alpha(x_0)}\pi_2(x_\alpha)_{(n)})\psi^*_{\alpha,(-n)} \\ &+ \sum_{\alpha\in\Delta_{1/2}}\sum_{n<0}\Phi_{\alpha,(n)}\psi^*_{\alpha,(-n)} \\ &- \frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta_{>0}} c^{\gamma}_{\alpha,\beta}\sum_{m,n\in\mathbb{Z}}\psi^*_{\alpha,(m)}\psi^*_{\beta,(n)}\psi_{\gamma,(-m-n)}, \end{aligned} \tag{20}$$

$$d^{\chi} = \sum_{\alpha\in\Delta_{1/2}}\sum_{n\geqslant 0}\Phi_{\alpha,(n)}\psi^*_{\alpha,(-n)} + \sum_{\alpha\in\Delta_1}\chi(x_\alpha)\psi^*_{\alpha,(0)}, \tag{21}$$

and $\{x, y\} = xy + yx$.

Define the Hamiltonian H on C by

$$H = H^{V^k(\mathfrak{g})}_{standard} + (\omega_{\mathbb{L}_\ell(\mathfrak{g})})_{(1)} + (\omega_{F_\chi})_{(1)} + (\omega_{\bigwedge^{\infty/2+\bullet}(\mathfrak{g}_{>0})})_{(1)} - \pi_1(x_0)_{(0)} - \pi_2(x_0)_{(0)},$$

where $H^{V^k(\mathfrak{g})}_{standard}$ is the standard Hamiltonian of $V^k(\mathfrak{g})$ that gives $\pi_1(x)$, $x \in \mathfrak{g}$, conformal weight one and $\omega_{\mathbb{L}_\ell(\mathfrak{g})}$ is the Sugawara conformal vector of $\mathbb{L}_\ell(\mathfrak{g})$,

$$\begin{aligned} \omega_{F_\chi}(z) &= \frac{1}{2}\sum_{\alpha\in\Delta_{1/2}} :\partial_z\Phi^{\alpha}(z)\Phi_{\alpha}(z):, \\ \omega_{\bigwedge^{\infty/2+\bullet}(\mathfrak{g}_{>0})}(z) &= \sum_{j>0}\sum_{\alpha\in\Delta_j} j:\psi^*_\alpha(z)\partial\psi_\alpha(z): + \sum_{j>0}\sum_{\alpha\in\Delta_j}(1-j):(\partial\psi^*_\alpha(z))\psi_\alpha(z):. \end{aligned}$$

Here, $\{\Phi^{\alpha}\}$ is a dual basis to $\{\Phi_\alpha\}$, that is, $\Phi^{\alpha}(z)\Phi_\beta(w) \sim \delta_{\alpha\beta}/(z-w)$. Then $[H, d_t^{st}] = 0 = [H, d_t^{\chi}]$, and thus, H defines a Hamiltonian on the vertex algebra $H^{\bullet}(C, (Q_t)_{(0)})$.

Following [FBZ04, KRW03], define

$$J^a(z) = \pi_1(x_a)(z) + \sum_{\beta,\gamma\in\Delta_+} c^{\gamma}_{a,\beta} :\psi_\gamma(z)\psi^*_\beta(z):$$

for $a \in I \sqcup \Delta$. We also denote by J^x the linear combination of J^a, $a \in I \sqcup \Delta_{\leqslant 0}$, corresponding to $x \in \mathfrak{g}_{\leqslant 0} := \bigoplus_{j\leqslant 0}\mathfrak{g}_j$. Let $C_{\leqslant 0}$ be the vertex subalgebra of C generated by $J^x(z)$ ($x \in \mathfrak{g}_{\leqslant 0}$), $\pi_2(x)(z)$ ($x \in \mathfrak{g}$), $\psi^*_\alpha(z)$ ($\alpha \in \Delta_{>0}$) and $\Phi_\alpha(z)$ ($\alpha \in \Delta_{1/2}$), and let $C_{>0}$ be the vertex subalgebra of C generated by $\psi_\alpha(z)$ and

$$((Q_t)_{(0)}\psi_\alpha)(z) = J^{\alpha}(z) + t^{\alpha(h)}\pi_2(x_\alpha)(z) + \Phi_\alpha(z)$$

($\alpha \in \Delta_{>0}$).

As in [FBZ04, KRW03], we find that both $C_{\leqslant 0}$ and $C_{>0}$ are subcomplexes of $(C, (Q_t)_{(0)})$ and that $C \cong C_{\leqslant 0}\otimes C_{>0}$ as complexes. Moreover, we have

$$H^i(C_{>0}, (Q_t)_{(0)}) = \begin{cases} \mathbb{C} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0, \end{cases}$$

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and therefore,

$$H^\bullet(C,(Q_t)_{(0)}) \cong H^\bullet(C_{\leqslant 0},(Q_t)_{(0)}) \tag{22}$$

as vertex algebras. Since the cohomological gradation takes only nonnegative values on $C_{\leqslant 0}$, it follows that $H^0(C,(Q_t)_{(0)}) = H^0(C_{\leqslant 0},(Q_t)_{(0)})$ is a vertex subalgebra of $C_{\leqslant 0}$.

Note that the vertex algebra $C_{\leqslant 0}$ does not depend on the parameter $t \in \mathbb{C}$. Also, $C_{\leqslant 0}$ is preserved by the action of both d_t^{st} and d^χ.

Let $C_{\leqslant 0,\Delta} = C_{\leqslant 0} \cap C_\Delta$ so that $C_{\leqslant 0} = \bigoplus_\Delta C_{\leqslant 0,\Delta}$.

Lemma 4.1. *For each Δ, $C_{\leqslant 0,\Delta}$ is a finite-dimensional subcomplex of $C_{\leqslant 0}$.*

Proof. The generators $J^x(z)$ $(x \in \mathfrak{g}_{\leqslant 0})$, $\psi_\alpha^*(z)$ $(\alpha \in \Delta_{>0})$ and $\Phi_\alpha(z)$ $(\alpha \in \Delta_{1/2})$ have positive conformal weights with respect to the Hamiltonian H. Therefore, it is sufficient to show that the vertex subalgebra $\mathbb{L}_\ell(\mathfrak{g})$ that is generated by $\pi_2(x)(z), x \in \mathfrak{g}$, has finite-dimensional weight spaces $\mathbb{L}_\ell(\mathfrak{g})_\Delta := \mathbb{L}_\ell(\mathfrak{g}) \cap C_{\leqslant 0,\Delta}$ and the conformal weights of $L_\ell(\mathfrak{g})$ is bounded from the above. On the other hand, the action of H on $\mathbb{L}_\ell(\mathfrak{g})$ is the same as the twisted action of $(\omega_{\mathbb{L}_\ell(\mathfrak{g})})_{(1)}$ corresponding to Li's delta operator associated with $-x_0$. Since $\mathbb{L}_\ell(\mathfrak{g})$ is rational and lisse, the conformal weights of this twisted representation are bounded from above and the weight spaces are finite-dimensional. This completes the proof. □

Since both d_t^{st} and d^χ preserve $C_{\leqslant 0,\Delta}$, we can consider the spectral sequence $E_r \Rightarrow H^\bullet(C_{\leqslant 0})$ such that $d_0 = d^\chi$ and $d_1 = d_t^{st}$, which is converging since each $C_{\leqslant 0,\Delta}$ is finite dimensional. As in [FBZ04, KW04, KW05], we find that

$$E_1^{\bullet,q} = H^q(C_{\leqslant 0}, d^\chi) = \delta_{q,0} V^{k^\natural}(\mathfrak{g}^f) \otimes \mathbb{L}_\ell(\mathfrak{g}), \tag{23}$$

where $V^{k^\natural}(\mathfrak{g}^f)$ is the vertex subalgebra of $C_{\leqslant 0}$ generated by $J^x(z), x \in \mathfrak{g}^f$. Thus, the spectral sequence collapses at $E_1 = E_\infty$, and we get the vertex algebra isomorphism

$$\operatorname{gr} H^q(C_{t,\leqslant 0}) \cong \delta_{q,0} V^{k^\natural}(\mathfrak{g}^f) \otimes \mathbb{L}_\ell(\mathfrak{g}). \tag{24}$$

Here, $\operatorname{gr} H^q(C_{t,\leqslant 0})$ is the associated graded vertex algebra with respect to the filtration that defines the spectral sequence. By equation (24), for each $v \in V^{k^\natural}(\mathfrak{g}^f) \otimes \mathbb{L}_\ell(\mathfrak{g})$, there exists a cocycle

$$\hat{v} = v_0 + v_1 + v_2 + \ldots \quad \text{(a finite sum)}$$

such that

$$v_0 = v, \quad d^{st} v_i = -d^\chi v_{i+1}.$$

Set

$$A := \pi_2(x_0) \in \mathbb{L}_\ell(\mathfrak{g}), \tag{25}$$

where we recall that x_0 is defined by equation (12). Then $C_{\leqslant 0} = \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} C_{\leqslant 0}[\lambda]$, $C_{\leqslant 0}[\lambda] = \{c \in C_{\leqslant 0} \mid A_{(0)}c = \lambda c\}$, and $A \in C_{\leqslant 0}[0]$. Consider the corresponding cocycle $\hat{A}$ that has cohomolological degree zero. Since $d_t^{st} A \subset C_{\leqslant 0}[>0]$ and $d^\chi C_{\leqslant 0}[\lambda] \subset C_{\leqslant 0}[\lambda]$, we may assume that $\hat{A} \equiv A (\operatorname{mod} C_{\leqslant 0}[>0])$. Moreover, we may assume that $\hat{A}$ is homogenous with respect to the Hamiltonian H so that $\hat{A}$ preserves each finite-dimensional subcomplex $C_{\leqslant 0,\Delta}$. In particular, $\hat{A}$ is locally finite on $C_{\leqslant 0}$. Therefore, we can apply the construction of Section 2 to obtain the automorphism

$$\varphi_t \colon C_{\leqslant 0} \xrightarrow{\sim} C_{\leqslant 0}, \quad c \mapsto \tilde{c} \tag{26}$$

of vertex algebras.

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Note that $\tilde{A} \equiv A(\mathrm{mod}\ C[>0])$ and the operator $A_{(0)}$ acts semisimply on the whole complex C so that $C = \bigoplus_{\lambda\in\frac{1}{2}\mathbb{Z}} C[\lambda]$, $C[\lambda] = \{c \in C \mid A_{(0)}c = \lambda c\}$. Since φ_t is an automorphism of the vertex algebra, $\tilde{A} \in \varphi_t(\mathbb{L}_\ell(\mathfrak{g}))$ generates a Heisenberg vertex subalgebra. In particular, $\tilde{A}_{(0)}$ acts semisimply on C. Therefore, by replacing $\hat{A}$ by $\tilde{A}$ (see Lemma 2.2), we can extend the automorphism (26) to the automorphism

$$C \xrightarrow{\sim} C \quad c \mapsto \tilde{c}, \tag{27}$$

which is also denoted by φ_t.

Proposition 4.2. *We have $\varphi_t(Q_{t=0})_{(0)} = (Q_t)_{(0)}$ on C. In particular, φ_t defines an isomorphism*

$$(C, (Q_{t=0})_{(0)}) \cong (C, (Q_t)_{(0)})$$

of differential graded vertex algebras.

Proof. We first show that $\varphi_t(Q_{t=0})_{(0)} = (Q_t)_{(0)}$ on $C_{\leqslant 0}$. Since $\varphi_t(Q_{t=0})_{(0)}\tilde{A} = \varphi_t(Q_{t=0})_{(0)}\varphi_t(A) = \varphi_t((Q_{t=0})_{(0)}A) = 0$, we have $[\varphi_t(Q_{t=0})_{(0)}, \tilde{A}_{(0)}] = 0$. Also, $[(Q_t)_{(0)}, \tilde{A}_{(0)}] = [(Q_t)_{(0)}, \hat{A}_{(0)}] = 0$ on $C_{\leqslant 0}$ by Lemma 2.2. Let $c \in C_{\leqslant 0}[\geqslant \lambda]$ be an eigenvector of $\tilde{A}_{(0)}$ of eigenvalue λ. Then the vector $(\varphi_t(Q_{t=0})_{(0)} - (Q_t)_{(0)})c$ is also an eigenvector of $\tilde{A}_{(0)}$ of eigenvalue λ. On the other hand, note that $\varphi_t(Q_{t=0}) \equiv Q_{t=0} \equiv Q_{t=1}(\mathrm{mod}\ C_{\leqslant 0}[\geqslant \lambda])$. Hence, $(\varphi_t(Q_{t=0})_{(0)} - (Q_t)_{(0)})c \in C_{\leqslant 0}[> \lambda]$. Since all the eigenvalues of $\tilde{A}_{(0)}$ on $C_{\leqslant 0}[> \lambda]$ are greater than λ, we get that $(\varphi_t(Q_{t=0})_{(0)} - (Q_t)_{(0)})c = 0$. Hence, $\varphi_t(Q_{t=0})_{(0)} = (Q_t)_{(0)}$ on $C_{\leqslant 0}$.

Next, since $(Q_t)_{(0)}\tilde{A} = 0$, we have $[(Q_t)_{(0)}, \tilde{A}_{(0)}] = 0$ on the whole space C. Therefore, we can repeat the same argument for an eigenvector $c \in C[\geqslant \lambda]$ of $\tilde{A}_{(0)}$ of eigenvalue λ to obtain that $(\varphi_t(Q_{t=0})_{(0)} - (Q_t)_{(0)})c = 0$ for all $c \in C$.

The last assertion follows since φ_t preserves the cohomological gradation as $\hat{A}$ has cohomological degree zero. □

By Proposition 4.2, $\varphi := \varphi_{t=1}$ defines an isomorphism

$$(C, (Q_{t=0})_{(0)}) \cong (C, (Q_{t=1})_{(0)}) \tag{28}$$

of differential graded vertex algebras. We have shown the following assertion.

Theorem 4.3. *The automorphism φ induces an isomorphism $H^\bullet(C, (Q_{t=0})_{(0)}) \cong H^\bullet(C, (Q_{t=1})_{(0)})$. In particular,*

$$\mathscr{W}^k(\mathfrak{g}, f)\otimes\mathbb{L}_\ell(\mathfrak{g}) \cong H^0_{DS,f}(V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g}))$$

as vertex algebras.

Proof of Theorem 1 . Let V be a vertex algebra equipped with a vertex algebra homomorphism $V^k(\mathfrak{g}) \to V$. Since $\mathbb{L}_\ell(\mathfrak{g})$ is rational, both $A_{(0)}$ and $\tilde{A}_{(0)}$ acts semisimply on $C(V\otimes\mathbb{L}_\ell(\mathfrak{g}))$. Hence, we can apply Lemma 2.1 to extend φ to the isomorphism $C(V\otimes\mathbb{L}_\ell(\mathfrak{g}), (Q_{t=0})_{(0)}) \cong C(V\otimes\mathbb{L}_\ell(\mathfrak{g}), (Q_{t=t})_{(0)})$ of differential graded vertex algebras, which gives the the isomorphism $H^\bullet_{DS,f}(V\otimes\mathbb{L}_\ell(\mathfrak{g})) \cong H^\bullet_{DS,f}(V)\otimes\mathbb{L}_\ell(\mathfrak{g})$ of vertex algebras. Similarly, if M is a V-module and N is an $\mathbb{L}_\ell(\mathfrak{g})$-module, it follows from Lemma 2.3 that we have an isomorphism $H^\bullet_{DS,f}(M\otimes N) \cong H^\bullet_{DS,f}(M)\otimes N$ as modules over $H^\bullet_{DS,f}(V\otimes\mathbb{L}_\ell(\mathfrak{g})) \cong H^\bullet_{DS,f}(V)\otimes\mathbb{L}_\ell(\mathfrak{g})$. □

Example 4.4. Let $\mathfrak{g} = \mathfrak{sl}_2 = \mathrm{span}_{\mathbb{C}}\{e, h, f\}$ and $\ell = 1$, and consider the Drinfeld–Sokolov reduction associated with f. Then $C = V^k(\mathfrak{g})\otimes\mathbb{L}_1(\mathfrak{g})\otimes \bigwedge^{\infty/2+\bullet}(\mathfrak{g}_{>0})$, and $\bigwedge^{\infty/2+\bullet}(\mathfrak{g}_{>0})$ is generated by odd fields $\psi(z) = \psi_\alpha(z)$, $\psi^*(z) = \psi^*_\alpha(z)$. We have

$$Q_t(z) = (e_1(z) + t^2 e_2(z))\psi^*(z) + \psi^*(z),$$

where we have set $x_i = \pi_i(x)$ for $x \in \mathfrak{g}, i = 1, 2$. The vertex subalgebra $C_{\leqslant 0}$ is generated by $J^{f_1}(z) = f_1(z)$, $J^{h_1}(z) = h_1(z) + 2 :\psi(z)\psi^*(z):$, $e_2(z)$, $h_2(z)$, $f_2(z)$ and $\psi^*(z)$. We have $A = h_2/2$, and

$$\hat{A}(z) = h_2(z) + t^2 J^{h_1}(z) e_2(z).$$

We find that the isomorphism

$$\varphi_t : V^k(\mathfrak{g}) \otimes \mathbb{L}_1(\mathfrak{g}) \otimes \bigwedge^{\infty/2+\bullet}(\mathfrak{g}_{>0}) \xrightarrow{\sim} V^k(\mathfrak{g}) \otimes \mathbb{L}_1(\mathfrak{g}) \otimes \bigwedge^{\infty/2+\bullet}(\mathfrak{g}_{>0})$$

is given by

$$\begin{aligned}
&e_1(z) \mapsto e_1(z)(1 - t^2 e_2(z)),\\
&h_1(z) \mapsto h_1(z) - t^2 k \partial_z e_2(z),\\
&f_1(z) \mapsto f_1(z)(1 + t^2 e_2(z)),\\
&e_2(z) \mapsto e_2(z),\\
&h_2(z) \mapsto h_2(z) + t^2 J^{h_1}(z) e_2(z),\\
&f_2(z) \mapsto f_2(z) - \frac{t^2}{2} J^{h_1}(z) h_2(z) + \partial_z J^{h_1}(z) - \frac{t^4}{4} : e_2(z) J^{h_1}(z)^2 :,\\
&\psi(z) \mapsto \psi(z)(1 - t^2 e_2(z)),\\
&\psi^*(z) \mapsto \psi^*(z)(1 + t^2 e_2(z)).
\end{aligned}$$

Note that we have $(1 - t^2 e_2(z))(1 + t^2 e_2(z)) = 1$ on $\mathbb{L}_1(\mathfrak{g})$.

5. Remarks on superalgebras

We restricted our attention to vertex algebras, and here we remark that the results also hold for vertex superalgebras, i.e.,

Remark 5.1. All results of section 2 also hold if we allow V to be a vertex superalgebra such that A and $\hat{A}$ are even elements of V.

Remark 5.2. In the proofs of section 4, we only used the following properties of $\mathbb{L}_\ell(\mathfrak{g})$:

1. finite dimensionality and boundedness from above of conformal weight spaces of $\mathbb{L}_\ell(\mathfrak{g})$ with respect to the twisted action of $(\omega_{\mathbb{L}_\ell(\mathfrak{g})})_{(1)}$ corresponding to Li's delta operator associated with $-x_0$ (see the proof of Lemma 4.1);
2. semisimple action of A_0, where $A := \pi_2(x_0) \in \mathbb{L}_\ell(\mathfrak{g})$, on $\mathbb{L}_\ell(\mathfrak{g})$-modules.

Thus, the results are also true by replacing $\mathbb{L}_\ell(\mathfrak{g})$ by any vertex algebra that carries an action of $V^\ell(\mathfrak{g})$ and satisfies these two properties. We can also allow $\mathfrak{g}$ to be a Lie superalgebra and f an even nilpotent element such that $\mathbb{L}_\ell(\mathfrak{g})$ satisfies the above two properties.

First, if we take $\mathfrak{g} = \mathfrak{osp}_{1|2n}$ and a positive integer ℓ, then $\mathbb{L}_\ell(\mathfrak{g})$ is rational [CL21, Thm. 7.1]. Hence, Theorem 1 holds for $\mathfrak{g} = \mathfrak{osp}_{1|2n}$.

Unfortunately, except for the $\mathfrak{g} = \mathfrak{osp}_{1|2n}$ cases, there are only few integrable representations of $\widehat{\mathfrak{g}}$ [KW94, Theorem 8.1]. For this reason, the notion of principal integrable representations and subprincipal integrable representations of $\widehat{\mathfrak{g}}$ was introduced in [KW01], which are integrable modules with respect to a certain affine vertex subalgebra, call it $\mathbb{L}_\ell(\mathfrak{a})$, with $\mathfrak{a} \subset \mathfrak{g}_{\bar{0}}$, where $\mathfrak{g}_{\bar{0}}$ is the even part of $\mathfrak{g}$.

Theorem 5.3. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra with $\mathfrak{g}_{\bar{0}} = \mathfrak{a} \oplus \mathfrak{a}'$, $\mathfrak{a}$ semisimple, $\mathfrak{a}'$ reductive. Suppose that $\mathbb{L}_\ell(\mathfrak{g})$ is integrable as a representation of $\widehat{\mathfrak{a}} \subset \widehat{\mathfrak{g}}$, and f is a nilpotent element in $\mathfrak{a}$. For a*

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quotient V of $V^k(\mathfrak{g})$, we have a vertex algebra isomorphism

$$H^0_{DS,f}(V\otimes\mathbb{L}_\ell(\mathfrak{g}))\cong H^0_{DS,f}(V)\otimes\mathbb{L}_\ell(\mathfrak{g}), \tag{29}$$

where in the left-hand side the Drinfeld–Sokolov reduction is taken with respect to the diagonal action of $\widehat{\mathfrak{g}}$ on $V\otimes\mathbb{L}_\ell(\mathfrak{g})$. More generally, let V be a vertex algebra equipped with a vertex algebra homomorphism $V^k(\mathfrak{g})\to V$. Then we have an isomorphism

$$H^\bullet_{DS,f}(V\otimes\mathbb{L}_\ell(\mathfrak{g}))\cong H^\bullet_{DS,f}(V)\otimes\mathbb{L}_\ell(\mathfrak{g}) \tag{30}$$

of graded vertex algebras, and for any V-module M, $\mathbb{L}_\ell(\mathfrak{g})$-module N, there is an isomorphism

$$H^\bullet_{DS,f}(M\otimes N)\cong H^\bullet_{DS,f}(M)\otimes N,$$

as modules over $H^\bullet_{DS,f}(V\otimes\mathbb{L}_\ell(\mathfrak{g}))\cong H^\bullet_{DS,f}(V)\otimes\mathbb{L}_\ell(\mathfrak{g})$.

Proof. Since $f\in\mathfrak{a}$, there is an $\mathfrak{sl}_2$-triple $\{e,h,f\}$ in $\mathfrak{a}$, and we can assume that $x=\frac{1}{2}h$.

Let $V_\ell(\mathfrak{g}_{\bar{0}})$ be the vertex subalgebra of $\mathbb{L}_\ell(\mathfrak{g})$ generated by $y\in\mathfrak{g}_{\bar{0}}$. We have $V_\ell(\mathfrak{g}_{\bar{0}})\cong V_\ell(\mathfrak{a})\otimes V_\ell(\mathfrak{a}')$, where $V_\ell(\mathfrak{a})$ and $V_\ell(\mathfrak{a}')$ are vertex subalgebras of $V_\ell(\mathfrak{g}_{\bar{0}})$ generated by $y\in\mathfrak{a}$ and $y\in\mathfrak{a}'$, respectively. By the assumption, $V_\ell(\mathfrak{a})$ is integrable, and hence is rational. It follows that any $\mathbb{L}_\ell(\mathfrak{g})$-module is a direct sum of integrable $\widehat{\mathfrak{a}}$-modules. Thus, the condition (2) is satisfied.

To see that the condition (1) is satisfied, let us consider the twisted $\mathbb{L}_\ell(\mathfrak{g})$ -module $\sigma^*_{-x_0}\mathbb{L}_\ell(\mathfrak{g})$ corresponding to Li's delta operator associated with $-x_0$. Since it is irreducible, it is sufficient to show that $\sigma^*_{-x_0}\mathbb{L}_\ell(\mathfrak{g})$ is a highest weight representation of $\mathbb{L}_\ell(\mathfrak{g})$. Clearly, any root vector corresponding to an odd root acts locally nilpotently on $\sigma^*_{-x_0}\mathbb{L}_\ell(\mathfrak{g})$. Hence, it suffices to show that any root vector corresponding to a even positive root acts locally nilpotently on $\sigma^*_{-x_0}\mathbb{L}_\ell(\mathfrak{g})$. This is clear for a root corresponding to $\widehat{\mathfrak{a}'}$ as elements of $\mathfrak{a}'$ are orthogonal to $x\in\mathfrak{a}$. On the other hand, since $V_\ell(\mathfrak{a})$ is integrable, $\sigma^*_{-x_0}\mathbb{L}_\ell(\mathfrak{g})$ is integrable over $V_\ell(\mathfrak{a})$. Hence, any root vector corresponding to a even positive root acts locally nilpotently on $\sigma^*_{-x_0}\mathbb{L}_\ell(\mathfrak{g})$. This completes the proof. □

6. Urod conformal vector

In this section, we assume that k is not critical, and discuss how the conformal structure match under the isomorphism

$$\varphi\colon \mathscr{W}^k(\mathfrak{g},f)\otimes\mathbb{L}_\ell(\mathfrak{g})\cong H^0_{DS,f}(V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g}))$$

of vertex algebras.

On the right-hand side, the conformal vector of $H^0_{DS,f}(V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g}))$ is given by

$$\omega_{total}=\omega_{V^k(\mathfrak{g})}+\omega_{\mathbb{L}_\ell(\mathfrak{g})}+\omega_{F_\chi}+\omega_{\bigwedge^{\infty/2+\bullet}(\mathfrak{g}_{>0})}+T\pi_1(x_0)+T\pi_2(x_0),$$

where $\omega_{V^k(\mathfrak{g})}$ is the Sugawara conformal vector of $V^k(\mathfrak{g})$. It has the central charge

$$\begin{aligned}&\frac{k\dim\mathfrak{g}}{k+h^\vee}+\frac{\ell\dim\mathfrak{g}}{\ell+h^\vee}-\frac{(k+\ell)\dim\mathfrak{g}}{k+\ell+h^\vee}\\&\qquad+\dim\mathfrak{g}_0-\frac{1}{2}\dim\mathfrak{g}_{1/2}-12\Big|\frac{\rho}{\sqrt{k+\ell+h^\vee}}-\sqrt{k+\ell+h^\vee}x_0\Big|^2.\end{aligned} \tag{31}$$

Clearly, $\varphi^{-1}(\omega_{total})$ is a conformal vector of $\mathscr{W}^k(\mathfrak{g},f)\otimes\mathbb{L}_\ell(\mathfrak{g})$. Set

$$\omega_{Urod}=\varphi^{-1}(\omega^{total})-\omega_{\mathscr{W}^k(\mathfrak{g},f)}, \tag{32}$$

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where $\omega_{\mathscr{W}^k(\mathfrak{g},f)}$ is the conformal vector of $\mathscr{W}^k(\mathfrak{g},f)$. Since it commutes with $\omega_{\mathscr{W}^k(\mathfrak{g},f)}$, ω_{Urod} defines a conformal vector of $\mathbb{L}_\ell(\mathfrak{g})$ ([LL04, 3.11]), which is called *the Urod conformal vector* of $\mathbb{L}_\ell(\mathfrak{g})$. It depends on the choice of x_0 and k and has the central charge

$$\frac{k\dim\mathfrak{g}}{k+h^\vee}+\frac{\ell\dim\mathfrak{g}}{\ell+h^\vee}-\frac{(k+\ell)\dim\mathfrak{g}}{k+\ell+h^\vee}\\ +12\left(|\frac{\rho}{\sqrt{k+h^\vee}}-\sqrt{k+h^\vee}x_0|^2-|\frac{\rho}{\sqrt{k+\ell+h^\vee}}-\sqrt{k+\ell+h^\vee}x_0|^2\right). \tag{33}$$

In the case that $\mathfrak{g}$ is simply laced and $f=f_{prin}$, the central charge (33) becomes

$$-\frac{\ell(\ell h+h^2-1)\dim\mathfrak{g}}{\ell+h}, \tag{34}$$

where h is the Coxeter number, and we have used the strange formula $|\rho|^2/2h^\vee=\dim\mathfrak{g}/24$, and so it does not depend on the parameter k.

By definition, the conformal vertex algebra $\mathscr{W}^k(\mathfrak{g},f)\otimes\mathbb{L}_\ell(\mathfrak{g})$ with the conformal vector $\omega_{\mathscr{W}^k(\mathfrak{g},f)}+\omega_{Urod}$ is isomorphic to $H^0_{DS,f}(V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g}))$ as conformal vertex algebras.

Lemma 6.1. *The Hamiltonian of $\mathbb{L}_\ell(\mathfrak{g})$ defined by ω_{Urod} coincides with*

$$H_{Urod}:=(\omega_{\mathbb{L}_\ell(\mathfrak{g})})_{(1)}-(x_0)_{(0)}.$$

Proof. For a homogenous element $x\in\mathfrak{g}$, $\pi_2(x)\in C$ has the conformal weight $1-\deg x$. Hence, $\tilde{x}$ has the the conformal weight $1-\deg x$ as well. □

Example 6.2 (Continued from Example 4.4). The Urod conformal vector of $\mathbb{L}_1(\mathfrak{g})$ is given by

$$\omega_{Urod}=\omega_{\mathbb{L}_1(\mathfrak{g})}+Th/2-(k+1)T^2e/2,$$

which agrees with [BFL16].

7. Compatibility with twists

In this section, we show that the various *twisted* Drinfeld–Sokolov reduction functors commute with tensoring integrable representations as well.

For $\check{\mu}\in\check{P}$, we define a character $\hat{\chi}_{\check{\mu}}$ of $\mathfrak{g}_{\geqslant 1}[t,t^{-1}]$ by the formula

$$\hat{\chi}_{\check{\mu}}(x_\alpha f(t))=\chi(x_\alpha)\cdot\mathrm{Res}_{t=0}\,f(t)t^{\langle\check{\mu},\alpha\rangle}dt,\qquad f(t)\in\mathbb{C}[t,t^{-1}]. \tag{35}$$

Define

$$F_{\chi,\check{\mu}}=U(\mathfrak{g}_{>0}[t,t^{-1}])\otimes_{U(\mathfrak{g}_{>0}[t]+\mathfrak{g}_{\geqslant 1}[t,t^{-1}])}\mathbb{C}_{\hat{\chi}_{\check{\mu}}},$$

where $\mathbb{C}_{\hat{\chi}_{\check{\mu}}}$ is the one-dimensional representation of $\mathfrak{g}_{>0}[t]+\mathfrak{g}_{\geqslant 1}[t,t^{-1}]$ on which $\mathfrak{g}_{\geqslant 1}[t,t^{-1}]$ acts by the character $\hat{\chi}_{\check{\mu}}$ and $\mathfrak{g}_{>0}[t]$ acts trivially. Then $F_{\chi,\check{\mu}}$ is naturally a $V(\mathfrak{g}_{>0})$-module, and we denote by $\Phi^{\check{\mu}}_\alpha(z)$ the image of $x_\alpha(z)$ in $(\mathrm{End}\,F_{\chi,\check{\mu}}))[z,z^{-1}]$. We have

$$\Phi^{\check{\mu}}_\alpha(z)=z^{\langle\check{\mu},\alpha\rangle}\chi(x_\alpha)\quad\text{for }\alpha\in\Delta_{>0},$$
$$\Phi^{\check{\mu}}_\alpha(z)\Phi^{\check{\mu}}_\beta(w)\sim\frac{w^{\langle\check{\mu},\alpha+\beta\rangle}\chi([x_\alpha,x_\beta])}{z-w}.$$

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For a smooth $\widehat{\mathfrak{g}}$-module M of level k, we define

$$H^{\bullet}_{DS,f,\check{\mu}}(M) = H^{\infty/2+\bullet}(\mathfrak{g}_{>0}[t,t^{-1}], M\otimes F_{\chi,\check{\mu}}), \tag{36}$$

where $\mathfrak{g}_{>0}[t,t^{-1}]$ acts diagonally on $M\otimes F_{\chi,\check{\mu}}$. By definition, $H^{\bullet}_{DS,f,\check{\mu}}(M)$ is the cohomology of the complex $(C_{\check{\mu}}(M), (Q_{\check{\mu}})_{(0)})$, where $C_{\check{\mu}}(M) = M\otimes F_{\chi,\check{\mu}}\otimes \bigwedge^{\infty/2+\bullet}(\mathfrak{g}_{>0})$ and $(Q_{\check{\mu}})_{(0)}$ is the zero mode of the field

$$Q_{\check{\mu}}(z) = \sum_{\alpha\in\Delta_{>0}} x_{\alpha}(z)\psi^*_{\alpha}(z) + \sum_{\alpha\in\Delta_{>0}} \Phi^{\check{\mu}}_{\alpha}(z)\psi^*_{\alpha}(z) - \frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta_{>0}} c^{\gamma}_{\alpha,\beta}\psi^*_{\alpha}(z)\psi^*_{\beta}(z)\psi_{\gamma}(z),$$

on $C_{\check{\mu}}(M)$.

We define the structure of a $\mathscr{W}^k(\mathfrak{g},f)$-module on $H^i_{DS,f,\check{\mu}}(M), i\in\mathbb{Z}$, as follows. Let $\Delta(J^{\{\check{\mu}\}},z)$ be Li's delta operator corresponding to the field

$$J^{\{\check{\mu}\}}(z) := \check{\mu}(z) + \sum_{\alpha\in\Delta_{>0}}\langle\alpha,\check{\mu}\rangle :\psi_{\alpha}(z)\psi^*_{\alpha}(z): -\frac{1}{2}\sum_{\alpha\in\Delta_{1/2}}\langle\alpha,\check{\mu}\rangle :\Phi_{\alpha}(z)\Phi^{\alpha}(z): \tag{37}$$

in $C(V^k(\mathfrak{g}))$, where $\Phi^{\alpha}(z)$ is a linear sum of $\Phi^{\beta}(z)$ corresponding to the vector x^{α} dual to x_{α} with respect to the symplectic form (13), that is, $(f|[x_{\alpha},x^{\beta}]) = \delta_{\alpha,\beta}$. We have

$$J^{\{\check{\mu}\}}(z)x_{\alpha}(w) \sim \frac{\langle\alpha,\check{\mu}\rangle}{z-w}x_{\alpha}(w), \quad J^{\{\check{\mu}\}}(z)\Phi_{\alpha}(w) \sim \frac{\langle\alpha,\check{\mu}\rangle}{z-w}\Phi_{\alpha}(w),$$
$$J^{\{\check{\mu}\}}(z)\psi_{\alpha}(w) \sim \frac{\langle\alpha,\check{\mu}\rangle}{z-w}\psi_{\alpha}(w), \quad J^{\{\check{\mu}\}}(w)\psi^*_{\alpha}(z) \sim -\frac{\langle\alpha,\check{\mu}\rangle}{z-w}\psi^*_{\alpha}(w);$$

see [KRW03]. For any smooth $\widehat{\mathfrak{g}}$-module M, we can twist the action of $C(V^k(\mathfrak{g}))$ on $C(M)$ by the correspondence

$$\sigma_{\check{\mu}} : Y(A,z) \mapsto Y_{C(M)}(\Delta(J^{\{\check{\mu}\}},z)A,z). \tag{38}$$

for any $A\in C(V^{\kappa}(\mathfrak{g}))$. Since we have

$$\sigma_{\check{\mu}}(x_{\alpha}(z)) = z^{-\langle\alpha,\check{\mu}\rangle}x_{\alpha}(z), \quad \sigma_{\check{\mu}}(\Phi_{\alpha}(z)) = z^{-\langle\alpha,\check{\mu}\rangle}\Phi_{\alpha}(z),$$
$$\sigma_{\check{\mu}}(\psi_{\alpha}(z)) = z^{-\langle\alpha,\check{\mu}\rangle}\psi_{\alpha}(z), \quad \sigma_{\check{\mu}}(\psi^*_{\alpha}(z)) = z^{\langle\alpha,\check{\mu}\rangle}\psi^*_{\alpha}(z),$$

it follows that the resulting complex $(C(M), \sigma_{\check{\mu}}(Q_{(0)}))$ is naturally identified with $(C_{\check{\mu}}(M), (Q_{\check{\mu}})_{(0)})$. Therefore, $(C_{\check{\mu}}(M), (Q_{\check{\mu}})_{(0)})$ has the structure of a differential graded module over the differential vertex algebra $(C(V^k(\mathfrak{g})), Q_{(0)})$, and hence, each cohomology $H^i_{DS,f,\check{\mu}}(M), i\in\mathbb{Z}$, is a module over $\mathscr{W}^k(\mathfrak{g},f) = H^0_{DS,f}(M)$.

The functor $H^0_{DS,f,\check{\mu}}(?)$ was introduced in [AF19] in the case that f is a principal nilpotent element.

Let V be vertex algebra equipped with a vertex algebra homomorphism $V^k(\mathfrak{g}) \to V$, and let M be a V-module. Then the same construction as above gives $H^i_{DS,f,\check{\mu}}(M)$ a structure of a module over the vertex algebra $H^0_{DS,f}(V)$.

Theorem 7.1. *Let V be a quotient of the universal affine vertex algebra $V^k(\mathfrak{g})$, $\ell\in\mathbb{Z}_{\geqslant 0}$, and let $\check{\mu}\in\check{P}$. For any V-module M, $\mathbb{L}_{\ell}(\mathfrak{g})$-module N and $i\in\mathbb{Z}$, there is an isomorphism*

$$H^i_{DS,f,\check{\mu}}(M\otimes N) \cong H^i_{DS,f,\check{\mu}}(M)\otimes\sigma^*_{\check{\mu}}N$$

as modules over $H^0_{DS,f}(V\otimes\mathbb{L}_\ell(\mathfrak{g}))\cong H^0_{DS,f}(V)\otimes\mathbb{L}_\ell(\mathfrak{g})$*, where* $\sigma^*_{\check{\mu}}N$ *is the twist of* N *on which* $A\in\mathbb{L}_\ell(\mathfrak{g})$ *acts as* $\sigma_{\check{\mu}}(A(z))$*.*

Proof. Since $\mathbb{L}_\ell(\mathfrak{g})$ is rational, both $A_{(0)}$ and $\tilde{A}_{(0)}$ acts semisimply on $C_\mu(M\otimes N)$ with respect to the twisted action of $C(V^k(\mathfrak{g})\otimes\mathbb{L}_\ell(\mathfrak{g}))$ described above, where A is defined in equation (25). Therefore, the assertion follows immediately from Lemma 2.3. □

More generally, let

$$w=(y,\check{\mu})$$

be an element of the extented affine Weyl group $\tilde{W}=W\ltimes P^\vee$, where W is the Weyl group of $\mathfrak{g}$, and let $\tilde{y}$ be a Tits lifting of y to and automorphism of $\mathfrak{g}$ so that $\tilde{y}(x_\alpha)=c_\alpha x_{y(\alpha)}$ for some $c_\alpha\in\mathbb{C}^*$. Then

$$\sigma_w : x_\alpha t^n\mapsto\tilde{y}(x_\alpha)t^{n-\langle\alpha,\check{\mu}\rangle}$$

defines a Tits lifting of w to an automorphism of $\widehat{\mathfrak{g}}$. Set

$$F_{\chi,w}=U(\tilde{w}(\mathfrak{g}_{>0}[t,t^{-1}]))\otimes_{U(\tilde{w}(\mathfrak{g}_{>0}[t]+\mathfrak{g}_{\geqslant 1}[t,t^{-1}]))}\mathbb{C}_{\hat{\chi}_{\check{\mu}}},$$

where $\mathbb{C}_{\hat{\chi}_{\check{\mu}}}$ is the one-dimensional representation of $\tilde{w}(\mathfrak{g}_{>0}[t]+\mathfrak{g}_{\geqslant 1}[t,t^{-1}])$ on which $\tilde{w}(\mathfrak{g}_{\geqslant 1}[t,t^{-1}])$ acts by the character $\hat{\chi}_w : x_\alpha t^n\mapsto(x_\alpha t^n|\tilde{w}(ft^{-1}))$ and $\tilde{w}(\mathfrak{g}_{>0}[t])$ acts trivially. Then

$$H^i_{DS,f,w}(M)=H^{\infty/2+i}(y(\mathfrak{g}_{>0})[t,t^{-1}],M\otimes F_{\chi,w})\tag{39}$$

is equipped with a $\mathscr{W}^k(\mathfrak{g},f)$-module structure.

The proof of the following assertion is the same as that of Theorem 7.1.

Theorem 7.2. *Let* V *be a quotient of the universal affine vertex algebra* $V^k(\mathfrak{g})$*,* $\ell\in\mathbb{Z}_{\geqslant 0}$*, and let* $w\in\tilde{W}$*. For any* V*-module* M*,* $\mathbb{L}_\ell(\mathfrak{g})$*-module* N*, and* $i\in\mathbb{Z}$*, there is an isomorphism*

$$H^i_{DS,f,w}(M\otimes N)\cong H^i_{DS,f,w}(M)\otimes\sigma^*_w N$$

as modules over $H^0_{DS,f}(V\otimes\mathbb{L}_\ell(\mathfrak{g}))\cong H^0_{DS,f}(V)\otimes\mathbb{L}_\ell(\mathfrak{g})$*.*

In the above, $\sigma^*_w N$ is the twist of N on which $A\in\mathbb{L}_\ell(\mathfrak{g})$ acts as $\sigma_w(A(z))$, which is again an integrable representation of $\widehat{\mathfrak{g}}$ of level ℓ.

Suppose that the grading (11) is even, that is, $\mathfrak{g}_j=0$ unless $j\in\mathbb{Z}$. Then $x_0\in P^\vee$. For a smooth $\widehat{\mathfrak{g}}$-module M of level k, "–"-Drinfeld–Sokolov reduction $H^\bullet_{DS,f,-}(M)$ ([FKW92, Ara11]) is nothing but the twisted reduction for $w=(w_0,-x_0)$, where w_0 is the longest element of W:

$$H^\bullet_{DS,f,-}(M)=H^\bullet_{DS,f,(w_0,-x_0)}(M)\tag{40}$$

Corollary 7.3. *Let* V *be a quotient of the universal affine vertex algebra* $V^k(\mathfrak{g})$*,* $\ell\in\mathbb{Z}_{\geqslant 0}$*. Suppose that the grading (11) is even. For any* V*-module* M*,* $\mathbb{L}_\ell(\mathfrak{g})$*-module* N *and* $i\in\mathbb{Z}$*, there is an isomorphism*

$$H^i_{DS,f,-}(M\otimes N)\cong H^i_{DS,f,-}(M)\otimes\sigma^*_{(w_0,-x_0)}N$$

as modules over $H^0_{DS,f}(V\otimes\mathbb{L}_\ell(\mathfrak{g}))\cong H^0_{DS,f}(V)\otimes\mathbb{L}_\ell(\mathfrak{g})$*.*

8. Higher-rank Urod algebras

In the case that f is a principal nilpotent element f_{prin}, we denote by $\mathscr{W}^k(\mathfrak{g})$ the principal W-algebra $\mathscr{W}^k(\mathfrak{g},f_{prin})$ and by $\mathscr{W}_k(\mathfrak{g})$ the unique simple quotient of $\mathscr{W}^k(\mathfrak{g})$. We have the *Feigin–Frenkel duality*

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([FF91, AFO18]; see also [ACL19]) which states that

$$\mathscr{W}^k(\mathfrak{g}) \cong \mathscr{W}^{\check{k}}({}^L\mathfrak{g}), \tag{41}$$

where ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra and $\check{k}$ is defined by the formula $r^\vee(k+h^\vee)(\check{k}+{}^Lh^\vee)=1$. Here, $r^\vee$ is the lacety of $\mathfrak{g}$, and ${}^Lh^\vee$ is the dual Coxeter number of ${}^L\mathfrak{g}$.

Suppose that $\mathfrak{g}$ is simply laced and $k+h^\vee-1\notin\mathbb{Q}_{\leqslant 0}$. By [ACL19], we have

$$\mathscr{W}^\ell(\mathfrak{g}) \cong (V^{k-1}(\mathfrak{g})\otimes\mathbb{L}_1(\mathfrak{g}))^{\mathfrak{g}[t]};$$

that is, $\mathscr{W}^\ell(\mathfrak{g})$ is isomorphic to the commutant of $V^k(\mathfrak{g})$ in $V^{k-1}(\mathfrak{g})\otimes\mathbb{L}_1(\mathfrak{g})$, where $V^k(\mathfrak{g})$ is considered as a vertex subalgebra of $V^{k-1}(\mathfrak{g})\otimes\mathbb{L}_1(\mathfrak{g})$ by the diagonal embedding and ℓ is defined by equation (6). In other words, we have a conformal vertex algebra embedding

$$V^k(\mathfrak{g})\otimes\mathscr{W}^\ell(\mathfrak{g}) \hookrightarrow V^{k-1}(\mathfrak{g})\otimes\mathbb{L}_1(\mathfrak{g}). \tag{42}$$

By Main Theorem 1, (42) induces the conformal vertex algebra homomorphism

$$\mathscr{W}^k(\mathfrak{g},f)\otimes\mathscr{W}^\ell(\mathfrak{g}) \to H^0_{DS,f}(V^{k-1}(\mathfrak{g})\otimes\mathbb{L}_1(\mathfrak{g})) \overset{\varphi^{-1}}{\tilde{\to}} \mathscr{W}^{k-1}(\mathfrak{g},f)\otimes\mathcal{U}(\mathfrak{g},f), \tag{43}$$

which is again embedding by Theorem 3.1. Here, $\mathcal{U}(\mathfrak{g},f)$ is the vertex algebra $\mathbb{L}_1(\mathfrak{g})$ equipped with the Urod conformal vector ω_{Urod}, which we call the *Urod algebra* associated with $(\mathfrak{g},f)$. We set

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g},f_{prin}).$$

8.1. Generic decompositions

In this subsection, we assume that k is irrational.

For $(\lambda,\check{\mu})\in P_+\times\check{P}_+$, define

$$T^k_{\lambda,\check{\mu}} = H^0_{DS,f_{prin},\check{\mu}}(\mathbb{V}^k_\lambda) \in \mathscr{W}^k(\mathfrak{g})\text{-Mod}.$$

It was shown in [AF19] that the $\mathscr{W}^k(\mathfrak{g})$-modules $T^k_{\lambda,\check{\mu}}$ are simple, and the isomorphism

$$T^k_{\lambda,\check{\mu}} \cong T^{\check{k}}_{\check{\mu},\lambda} \tag{44}$$

holds under the Feigin–Frenkel duality (41).

Let ℓ be nonnegative integer. Then $\{\mathbb{L}^\ell_\lambda \mid \lambda\in P^\ell_+\}$ gives the complete set of isomorphism classes of simple $\mathbb{L}_\ell(\mathfrak{g})$-modules, where

$$P^\ell_+ = \{\lambda\in P_+ \mid \langle\lambda,\theta^\vee\rangle\leqslant\ell\}\subset P_+$$

is the set of integrable dominant weight of $\mathfrak{g}$ of level ℓ. In particular, $\{\mathbb{L}^1_\lambda \mid \lambda\in P^1_+\}$ gives the complete set of isomorphism classes of simple $\mathcal{U}(\mathfrak{g},f)$-modules.

Let

$$\mathbb{V}^k_{\mu,f} = H^0_{DS,f}(\mathbb{V}^k_\mu) \in \mathscr{W}^k(\mathfrak{g},f)\text{-Mod}. \tag{45}$$

Theorem 8.1. *Let $\mathfrak{g}$ be simply laced, and let f be any nilpotent element of $\mathfrak{g}$. For $\lambda,\mu\in P_+$ and $\nu\in P^1_+$, we have*

$$\mathbb{V}^{k-1}_{\mu,f}\otimes\mathbb{L}^1_\nu \cong \bigoplus_{\substack{\lambda\in P_+\\ \lambda-\mu-\nu\in Q}} \mathbb{V}^k_{\lambda,f}\otimes T^\ell_{\lambda,\mu}$$

as $\mathscr{W}^k(\mathfrak{g},f)\otimes\mathscr{W}^\ell(\mathfrak{g})$-modules (see equation (43)).

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In the case $f = f_{prin}$, we have the following more general statement.

Theorem 8.2. *Let $\mathfrak{g}$ is simply laced. For $\lambda, \mu, \mu' \in P_+$ and $\nu \in P_+^1$, we have*

$$T_{\mu,\mu'}^{k-1} \otimes \mathbb{L}_\nu^1 = \bigoplus_{\substack{\lambda \in P_+ \\ \lambda-\mu-\mu'-\nu \in Q}} T_{\lambda,\mu'}^k \otimes T_{\lambda,\mu}^\ell \tag{46}$$

as $\mathscr{W}^k(\mathfrak{g}) \otimes \mathscr{W}^\ell(\mathfrak{g})$-modules.

Note that Theorem 8.2 is compatible with equation (44) since $(k \check{-} 1) = \ell - 1$.

Proof of Theorem 8.1 and Theorem 8.2. Let $\lambda, \mu \in P_+$ and $\nu \in P_+^1$. By [ACL19, Main Theorem 3], we have

$$\mathbb{V}_\mu^{k-1} \otimes \mathbb{L}_\nu^1 = \bigoplus_{\substack{\lambda \in P_+ \\ \lambda-\mu-\nu \in Q}} \mathbb{V}_\lambda^k \otimes T_{\lambda,\mu}^\ell \tag{47}$$

as $V^{k+1}(\mathfrak{g}) \otimes \mathscr{W}^\ell(\mathfrak{g})$-modules. Applying Main Theorem 1 to $\mathbb{V}_\mu^{k-1} \otimes \mathbb{L}_\nu^1$, we obtain Theorem 8.1. Next, under the identification $P/Q \cong P_+^1$, we have $\sigma_{\mu'}^* \mathbb{L}_{\nu+\mu'+Q}^1 \cong \mathbb{L}_{\nu+Q}^1$. Hence, we obtain Theorem 8.2 by applying Theorem 7.1. □

Let π^k be the Heisenberg vertex subalgebra of $V^k(\mathfrak{g})$ generated by $h(z)$, $h \in \mathfrak{h}$, and let π_λ^k be the irreducible highest weight representation of π^k with highest weight λ. There is a vertex algebra embedding

$$\mathscr{W}^k(\mathfrak{g}) \hookrightarrow \pi^{k+h^\vee}$$

called the *Miura map* ([FF90b]; see also [Ara17]), and hence, each $\pi_\lambda^{k+h^\vee}$ is a $\mathscr{W}^k(\mathfrak{g})$-module.

Theorem 8.3. *Let $\mathfrak{g}$ is simply laced, $\mu \in \mathfrak{h}^*$ be generic, $\nu \in P_+^1$. We have the isomorphism*

$$\pi_\mu^{k-1+h^\vee} \otimes \mathbb{L}_\nu^1 \cong \bigoplus_{\substack{\lambda \in \mathfrak{h}^* \\ \lambda-\mu-\nu \in Q}} \pi_\lambda^{k+h^\vee} \otimes \pi_{\lambda-(\ell+h^\vee)\mu}^{\ell+h^\vee}$$

as $\mathscr{W}^k(\mathfrak{g}) \otimes \mathscr{W}^\ell(\mathfrak{g})$-modules.

For a generic $\lambda \in \mathfrak{h}^*$, the $\mathscr{W}^k(\mathfrak{g})$-module $\pi_\lambda^{k+h^\vee}$ is irreducible and isomorphic to a Verma module $\mathbb{M}^k(\chi_\lambda)$ with highest weight ([FF90a, Fre92, Ara07]). Here, $\chi_\lambda : \mathrm{Zhu}(\mathscr{W}^k(\mathfrak{g})) \to \mathbb{C}$ is described in [ACL19, (27)], where $\mathrm{Zhu}(\mathscr{W}^k(\mathfrak{g}))$ is the Zhu algebra of $\mathscr{W}^k(\mathfrak{g})$. Hence, the following assertion immediately follows from Theorem 8.3.

Corollary 8.4. *Let $\mathfrak{g}$ is simply laced, $\mu \in \mathfrak{h}^*$ be generic, $\nu \in P_+^1$. We have the isomorphism*

$$\mathbb{M}^{k-1}(\chi_\mu) \otimes \mathbb{L}_\nu^1 \cong \bigoplus_{\substack{\lambda \in \mathfrak{h}^* \\ \lambda-\mu-\nu \in Q}} \mathbb{M}^k(\chi_\lambda) \otimes \mathbb{M}^\ell(\chi_{\lambda-(\ell+h^\vee)\mu})$$

as $\mathscr{W}^k(\mathfrak{g}) \otimes \mathscr{W}^\ell(\mathfrak{g})$-modules.

Let $\mathbb{W}_\lambda^k$ be the Wakimoto module [FF90b] of $\widehat{\mathfrak{g}}$ at level k with highest weight $\lambda \in \mathfrak{h}^*$. For $\lambda = 0$, $\mathbb{W}^k := \mathbb{W}_0^k$ is a vertex algebra, and we have an embedding $V^k(\mathfrak{g}) \hookrightarrow \mathbb{W}^k$ of vertex algebras ([FF90b, Fre05]). We have $H_{DS,f_{prin}}^i(\mathbb{W}_\lambda^k) \cong \delta_{i,0} \pi_\lambda^{k+h^\vee}$, and the Miura map is by definition [FF90b] the map $\mathscr{W}^k(\mathfrak{g}) \to H_{DS,f_{prin}}^0(\mathbb{W}^k) \cong \pi^{k+h^\vee}$ induced by the embedding $V^k(\mathfrak{g}) \hookrightarrow \mathbb{W}^k$.

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Proof of Theorem 8.3. Since ν is generic, $\mathbb{W}^{k-1}_{\mu}\otimes\mathbb{L}^{1}_{\nu}$ is a direct sum of irreducible Verma modules as a diagonal $\widehat{\mathfrak{g}}$-module or, equivalently, a direct sum of irreducible Wakimoto modules $\mathbb{W}^{k}_{\lambda}$. So we can write $\mathbb{W}^{k-1}_{\mu}\otimes\mathbb{L}^{1}_{\nu} = \bigoplus_{\lambda\in\mathfrak{h}^*}\mathbb{W}^{k}_{\lambda}\otimes m^{\lambda}_{\mu,\nu}$, where $m^{\lambda}_{\mu,\nu}$ be the multiplicity of $\mathbb{W}^{k}_{\lambda}$ in $\mathbb{W}^{k-1}_{\mu}\otimes\mathbb{L}^{1}_{\nu}$. Note that $m^{\lambda}_{\mu,\nu}$ is a $\mathscr{W}^{\ell}(\mathfrak{g})$-module by equation (42). We have

$$m^{\lambda}_{\mu,\nu} \cong \mathrm{Hom}_{\pi^{k+1+h^\vee}}(\pi^{k+1+h^\vee}_{\lambda}, H^{\infty/2+0}(\mathfrak{n}[t,t^{-1}], \mathbb{W}^{k-1}_{\mu}\otimes\mathbb{L}^{1}_{\nu}) \\ \cong \begin{cases}\pi^{\ell+h^\vee}_{\mu-(\ell+h^\vee)\lambda} & \text{if } \lambda-\mu-\nu\in Q,\\ 0 & \text{otherwise}\end{cases}$$

as $\mathscr{W}^{\ell}(\mathfrak{g})$-modules by [ACL19, Proposition 8.3]. The assertion follows by applying Main Theorem 1 to

$$\mathbb{W}^{k-1}_{\mu}\otimes\mathbb{L}^{1}_{\nu} = \bigoplus_{\substack{\lambda\in\mathfrak{h}^*\\ \lambda-\mu-\nu\in Q}} \mathbb{W}^{k}_{\lambda}\otimes\pi_{\mu-(\ell+h^\vee)}. \tag{48}$$

□

8.2. Decomposition at admissible levels

Let k be an admissible number for $\widehat{\mathfrak{g}}$, that is, $\mathbb{L}_k(\mathfrak{g})$ is admissible ([KW89]) as a $\widehat{\mathfrak{g}}$-module. In the case that $\mathfrak{g}$ is simply laced, this condition is equivalent to

$$k+h^\vee = \frac{p}{q}, \quad p,q\in\mathbb{Z}_{\geqslant 1},\ (p,q)=1,\ p\geqslant h^\vee. \tag{49}$$

For an admissible number k, a simple module over $\mathbb{L}_k(\mathfrak{g})$ need not be ordinary, that is, in KL, unless k is a nonnegative integer. The classification of simple highest weight representations of $\mathbb{L}_k(\mathfrak{g})$ was given in [Ara16a]. For our purpose, we need only the ordinary representations of $\mathbb{L}_k(\mathfrak{g})$. By [Ara16a], the complete set of isomorphism classes of ordinary simple $\mathbb{L}_k(\mathfrak{g})$-modules is given by

$$\{\mathbb{L}^{k}_{\lambda} \mid \lambda\in Adm^{k}_{\mathbb{Z}}\},$$

where $Adm^{k}_{\mathbb{Z}} = \{\lambda\in P_+ \mid \mathbb{L}^{k}_{\lambda} \text{ is admissible}\}$. We have

$$Adm^{k}_{\mathbb{Z}} = P^{p-h^\vee}_{+} \tag{50}$$

if $\mathfrak{g}$ is simply laced and k is of the form (49).

Let k be an admissible number and $\lambda\in Adm^{k}_{\mathbb{Z}}$. By [Ara15a], the associated variety $X_{\mathbb{L}^{k}_{\lambda}}$ [Ara12] of $\mathbb{L}^{k}_{\lambda}$ is the closure of some nilpotent orbit $\mathbb{O}_k$ which depends only on the denominator q of k. More explicitly, we have

$$X_{\mathbb{L}^{k}_{\lambda}} = \{x\in\mathfrak{g} \mid (\mathrm{ad}\,x)^{2q}=0\} \tag{51}$$

in the case that $\mathfrak{g}$ is simply laced. By [Ara15a], we have

$$H^{0}_{DS,f}(\mathbb{L}^{k}_{\lambda})\neq 0 \quad\Longleftrightarrow\quad f\in X_{\mathbb{L}^{k}_{\lambda}} = \overline{\mathbb{O}}_k. \tag{52}$$

An admissible number k is called *nondegenerate* if $X_{\mathbb{L}^{k}_{\lambda}}$ equals to the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ for some $\lambda\in Adm^{k}_{\mathbb{Z}}$ or, equivalently, for all $Adm^{k}_{\mathbb{Z}}$. In the case that $\mathfrak{g}$ is simply laced, this happens if and only if the denominator q of k is equal or greater than $h^\vee$. If this is the case, the simple principal W-algebra $\mathscr{W}_k(\mathfrak{g}) = \mathscr{W}_k(\mathfrak{g}, f_{prin})$ is rational and lisse ([Ara15a, Ara15b]), and the complete set of the isomorphism

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classes of $\mathscr{W}_k(\mathfrak{g})$-module is given by

$$\{\mathbf{L}^k_{[\lambda,\check{\mu}]} \mid [\lambda,\check{\mu}] \in (Adm^k_{\mathbb{Z}} \times Adm^{\check{k}}_{\mathbb{Z}})/\tilde{W}_+\}, \tag{53}$$

where

$$\mathbf{L}^k_{[\lambda,\check{\mu}]} := H^0_{DS,f_{prin}}(\mathbb{L}^k_{\lambda-(k+h^\vee)\check{\mu}}), \tag{54}$$

and $\tilde{W}_+$ is the subgroup of the extended affine Weyl group of $\mathfrak{g}$ consisting of elements of length zero that acts on the set $Adm^k_{\mathbb{Z}} \times Adm^{\check{k}}_{\mathbb{Z}}$ diagonally. We have

$$\mathbf{L}^k_{[\lambda,\check{\mu}]} \cong \mathbf{L}^{\check{k}}_{[\check{\mu},\lambda]} \tag{55}$$

under the Feigin–Frenkel duality.

The following assertion is new for nonzero $\check{\mu}$.

Theorem 8.5. *Let k be a nondegenerate admissible number, and let $\lambda \in Adm^k_{\mathbb{Z}}$, $\check{\mu} \in Adm^{\check{k}}_{\mathbb{Z}}$. We have*

$$H^i_{DS,f_{prin},\check{\mu}}(\mathbb{L}^k_\lambda) \cong \begin{cases} \mathbf{L}^k_{[\lambda,\check{\mu}]} & \text{for } i = 0,\\ 0 & \text{for } i \neq 0 \end{cases}$$

as $\mathscr{W}^k(\mathfrak{g})$-modules.

Proof. The case $\check{\mu} = 0$ has been proved in [Ara04, Ara07]. In particular,

$$H^0_{DS,f_{prin}}(\mathbb{L}_k(\mathfrak{g})) \cong \mathscr{W}_k(\mathfrak{g}). \tag{56}$$

By [AF19, Theorem 2.1], we have $H^i_{DS,f_{prin},\check{\mu}}(\mathbb{V}^k_\lambda) = 0$ for all $i \neq 0$, $\lambda \in P_+$. It follows that $H^{i\neq 0}_{DS,f_{prin},\check{\mu}}(M) = 0$, $i \neq 0$, for any object M in KL that admits a Weyl flag. This implies that

$$H^i_{DS,f_{prin},\check{\mu}}(M) = 0 \quad \text{for } i > 0,\ M \in \mathrm{KL}; \tag{57}$$

see the proof of [Ara04, Theorem 8.3].

By [Ara14], the admissible representation $\mathbb{L}^k_\lambda$ admits a two-sided resolution

$$\dots C^{-1} \to C^0 \to C^1 \to \dots \tag{58}$$

of the form $C^i = \bigoplus_{\substack{w\in\widehat{W}(\lambda+k\Lambda_0)\\ \ell^{\infty/2}(w)=i}} \mathbb{W}^k_{w\circ\lambda}$, where $\widehat{W}(\lambda+k\Lambda_0)$ is the integral Weyl group of $\lambda+k\Lambda_0$ and $\ell^{\infty/2}(w)$ is the semi-infinite length of w. By [AF19, Lemma 3.2], we have

$$H^0_{DS,f_{prin},\check{\mu}}(\mathbb{W}^k_\lambda) \cong \begin{cases} \pi^k_{\lambda-(k+h^\vee)\check{\mu}} & \text{for } i = 0,\\ 0 & \text{for } i \neq 0 \end{cases}$$

as $\mathscr{W}^k(\mathfrak{g})$-modules. It follows that $H^i_{DS,f_{prin},\check{\mu}}(\mathbb{L}^k_\lambda)$ is the i-th cohomology of the complex obtained by applying the functor $H^0_{DS,f_{prin},\check{\mu}}(?)$ to the resolution (58). In particular, $H^i_{DS,f_{prin},\check{\mu}}(\mathbb{L}^k_\lambda)$ is a subquotient of the $\mathscr{W}^k(\mathfrak{g})$-module

$$H^0_{DS,f_{prin},\check{\mu}}(C^i) \cong \bigoplus_{\substack{w\in\widehat{W}(\lambda+k\Lambda_0)\\ \ell^{\infty/2}(w)=i}} \pi^k_{w\circ\lambda-(k+h^\vee)\check{\mu}}.$$

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On the other hand, since $\mathbb{L}^k_\lambda$ is a $\mathbb{L}_k(\mathfrak{g})$-module, each $H^i_{DS,f_{prin},\check{\mu}}(\mathbb{L}^k_\lambda)$ is a module over the simple W-algebra $\mathscr{W}_k(\mathfrak{g}) = H^0_{DS,f_{prin}}(\mathbb{L}_k(\mathfrak{g}))$. Since $\mathscr{W}_k(\mathfrak{g})$ is rational, $H^i_{DS,f_{prin},\check{\mu}}(\mathbb{L}^k_\lambda)$ is a direct sum of simple modules of the form (53). However, such a module appears in the local composition factor of $\pi^k_{w\circ\lambda-(k+h^\vee)\check{\mu}}$ if and only if $w \in W$ ([Ara07]), where $W \subset \widehat{W}(k\Lambda_0)$ is the Weyl group of $\mathfrak{g}$. As

$$\ell^{\infty/2}(w) \geqslant 0 \text{ for } w \in W \text{ and the equality holds if and only if } w = 1,$$

$H^i_{DS,f_{prin},\check{\mu}}(\mathbb{L}^k_\lambda)$ must vanish for $i < 0$. Together with equation (57), we get $H^i_{DS,f_{prin},\check{\mu}}(\mathbb{L}^k_\lambda) = 0$ for $i \neq 0$. Finally, since $\mathbf{L}^k_{[\lambda,\check{\mu}]}$ is the unique simple $\mathscr{W}_k(\mathfrak{g})$-module that appears in the local composition factor of $\pi^k_{\lambda-(k+h^\vee)\check{\mu}}$ and it appears with multiplicity one ([Ara07]), $H^i_{DS,f_{prin},\check{\mu}}(\mathbb{L}^k_\lambda)$ is either zero or isomorphic to $\mathbf{L}^k_{[\lambda,\check{\mu}]}$. The assertion follows since the Euler character of $H^\bullet_{DS,f_{prin}}(\mathbb{L}_k(\mathfrak{g}))$ is equal to the character of $\mathbf{L}^k_{[\lambda,\check{\mu}]}$. □

For an admissible number k and $\lambda \in Adm^k_{\mathbb{Z}}$, define

$$\mathbb{L}^k_{\lambda,f} = H^0_{DS,f}(\mathbb{L}^k_\lambda) \in \mathscr{W}^k(\mathfrak{g},f)\text{-Mod}. \tag{59}$$

Note that $\mathbb{L}^k_{\lambda,f_{prin}} = \mathbf{L}^k_{[\lambda,0]} \cong \mathbf{L}^{\check{k}}_{[0,\lambda]}$.

Let $\mathfrak{g}$ be simply laced. Observe that if $k-1$ is an admissible number, then so is k, and ℓ is a nondegenerate admissible number. Moreover, we have

$$Adm^{k-1}_{\mathbb{Z}} = Adm^{\check{\ell}}_{\mathbb{Z}} = P^{p-h^\vee}_+, \quad Adm^k_{\mathbb{Z}} = Adm^\ell_{\mathbb{Z}} = P^{p+q-h^\vee}_+$$

if $k - h^\vee + 1 = p/q$ with $p \geqslant h^\vee$, $q \geqslant 1$, $(p,q) = 1$.

Theorem 8.6. *Let $\mathfrak{g}$ be simply laced, and let $k-1$ be admissible. Suppose that $f \in X_{\mathbb{L}_k(\mathfrak{g})} = X_{\mathbb{L}_{k-1}(\mathfrak{g})}$. For $\mu \in Adm^{k-1}_{\mathbb{Z}}$, $\nu \in P^1_+$, we have*

$$\mathbb{L}^{k-1}_{\mu,f} \otimes \mathbb{L}^1_\nu \cong \bigoplus_{\substack{\lambda\in Adm^k_{\mathbb{Z}} \\ \lambda-\mu-\nu\in Q}} \mathbb{L}^k_{\lambda,f} \otimes \mathbf{L}^\ell_{[\lambda,\mu]}$$

as $\mathscr{W}^k(\mathfrak{g},f)\otimes\mathscr{W}^\ell(\mathfrak{g})$-modules.

In the case $f = f_{prin}$, we have the following more general statement.

Theorem 8.7. *Let $\mathfrak{g}$ be simply laced, and let $k-1$ be nondegenerate admissible. For $\mu \in Adm^{k-1}_{\mathbb{Z}}$, $\mu' \in Adm^{\check{k}-1}_{\mathbb{Z}} = Adm^{\check{k}}_{\mathbb{Z}}$, $\nu \in P^1_+$, we have*

$$\mathbf{L}^{k-1}_{[\mu,\mu']} \otimes \mathbb{L}^1_\nu \cong \bigoplus_{\substack{\lambda\in Adm^k_{\mathbb{Z}} \\ \lambda-\mu-\mu'-\nu\in Q}} \mathbf{L}^k_{[\lambda,\mu']} \otimes \mathbf{L}^\ell_{[\lambda,\mu]}$$

as $\mathscr{W}_k(\mathfrak{g})\otimes\mathscr{W}_\ell(\mathfrak{g})$-modules.

Corollary 8.8. *Let $\mathfrak{g}$ be simply laced. Let $k + h^\vee = (2h^\vee+1)/h^\vee$ so that $\ell + h^\vee = (2h^\vee+1)/(h^\vee+1)$.*

1. *There is an embedding of vertex algebras*

$$\mathscr{W}_k(\mathfrak{g})\otimes\mathscr{W}_\ell(\mathfrak{g}) \hookrightarrow \mathbb{L}_1(\mathfrak{g}),$$

and $\mathscr{W}_k(\mathfrak{g})$ and $\mathscr{W}_\ell(\mathfrak{g})$ form a dual pair in $\mathbb{L}_1(\mathfrak{g})$.

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2. *For* $\nu \in P^1_+$, *we have*

$$\mathbb{L}^1_\nu \cong \bigoplus_{\lambda \in Adm^k_{\mathbb{Z}} \cap (\nu + Q)} \mathbf{L}^k_{[\lambda,0]} \otimes \mathbf{L}^\ell_{[\lambda,0]}$$

as $\mathscr{W}_k(\mathfrak{g}) \otimes \mathscr{W}_\ell(\mathfrak{g})$*-modules.*

Proof. The assertion follows from Theorem 8.7 noting that $\mathscr{W}_{k-1}(\mathfrak{g}) = \mathbf{L}^{k-1}_{[0,0]} = \mathbb{C}$ if $k + h^\vee - 1 = (h^\vee + 1)/h^\vee$ or $h^\vee/(h^\vee + 1)$. □

Proof of Theorem 8.6 and Theorem 8.7. By [ACL19, Main Theorem 3], we have

$$\mathbb{L}^{k-1}_\mu \otimes \mathbb{L}^1_\nu \cong \bigoplus_{\substack{\lambda \in Adm^k_{\mathbb{Z}} \\ \lambda - \mu - \nu \in Q}} \mathbb{L}^k_\lambda \otimes \mathbf{L}^\ell_{[\lambda,\mu]}$$

as $\mathbb{L}_k(\mathfrak{g}) \otimes \mathscr{W}_\ell(\mathfrak{g})$-modules. Hence, the assertion follows from Main Theorem 1 and Theorem 7.1. □

9. Application to the extension problem of vertex algebras

In this section, we apply Theorem 8.2 to prove the existence of the extensions of vertex algebras that are expected by four-dimensional supersymmetric gauge theories ([CG17, CGL18]).

For $\lambda \in P_+$, let $\lambda^* = -w_0(\lambda)$ so that $E^*_\lambda \cong E_{\lambda^*}$.

Theorem 9.1. *Let* $\mathfrak{g}$ *be simply laced, and let* k, k' *be irrational complex numbers satisfying*

$$\frac{1}{k + h^\vee} + \frac{1}{k' + h^\vee} = n$$

for $n \in \mathbb{Z}_{\geqslant 1}$. *Then*

$$A^n[\mathfrak{g}] := \bigoplus_{\lambda \in P_+ \cap Q} T^k_{\lambda,0} \otimes T^{k'}_{\lambda^*,0}$$

can be equipped with a structure of simple vertex operator algebra of central charge

$$2\,\mathrm{rk}\,\mathfrak{g} + 4h \dim \mathfrak{g} - nh \dim \mathfrak{g} \left(1 + \frac{\psi^2}{n\psi - 1}\right),$$

where $\psi = k + h^\vee$ *and* h *is the Coxeter number (which equals to* $h^\vee$*). The vertex operator algebra* $A^n[\mathfrak{g}]$ *is of CFT type if* $n \geqslant 2$.

Proof. We shall prove the assertion on induction on n using the fact that

$$T^{k'}_{\mu,\mu'} \otimes \mathcal{U}(\mathfrak{g}) \cong \bigoplus_{\substack{\lambda \in P_+ \\ \lambda - \mu - \mu' \in Q}} T^{k'+1}_{\lambda,\mu'} \otimes T^\ell_{\lambda,\mu} \cong \bigoplus_{\substack{\lambda \in P_+ \\ \lambda - \mu - \mu' \in Q}} T^{k'+1}_{\lambda,\mu'} \otimes T^{\check{\ell}}_{\mu,\lambda}, \tag{60}$$

with $\check{\ell}$ satisfying the relation

$$\frac{1}{\check{\ell} + h^\vee} = \frac{1}{k' + h^\vee} + 1$$

which follows from Theorem 8.2 and equation (44).

Let us show the assertion for $n = 1$. Let $\mathbf{I}^k_G$ be the *chiral universal centralizer* on G at level k ([Ara18]), which was introduced earlier in [FS06] for the $\mathfrak{g} = \mathfrak{sl}_2$ case as the *modified regular* *representation of the Virasoro algebra*. By definition, $\mathbf{I}_G^k$ is obtained by taking the principal Drinfeld–Sokolov reduction with respect to two commuting actions of $\widehat{\mathfrak{g}}$ on the algebra of the chiral differential operators [MSV99, BD04] $\mathcal{D}_{G,k}^{ch}$ on G at level k. The $\mathbf{I}_G^k$ is a conformal vertex algebra of central charge $2\,\mathrm{rk}\,\mathfrak{g} + 48(\rho|\rho^\vee) = 2\,\mathrm{rk}\,\mathfrak{g} + 4h^\vee \dim\mathfrak{g}$, equipped with a conformal vertex algebra homomorphism $\mathscr{W}^k(\mathfrak{g})\otimes\mathscr{W}^{k^*}(\mathfrak{g}) \to \mathbf{I}_G^k$, where k^* the dual level of k defined by the equation

$$\frac{1}{k+h^\vee} + \frac{1}{k^*+h^\vee} = 0.$$

As explained in [Ara18], $\mathbf{I}_G^k$ is a strict chiral quantization ([Ara18]) of the *universal centralizer* $\mathcal{S}\times_{\mathfrak{g}^*}(G\times\mathcal{S})$, where $\mathcal{S}$ is the Kostant–Slodowy slice. Since $\mathcal{S}\times_{\mathfrak{g}^*}(G\times\mathcal{S})$ is a smooth symplectic variety, $\mathbf{I}_G^k$ is simple ([AM]). For an irrational k, we have the decomposition

$$\mathbf{I}_G^k \cong \bigoplus_{\lambda\in P_+} T_{\lambda,0}^k \otimes T_{\lambda^*,0}^{k^*}$$

as $\mathscr{W}^k(\mathfrak{g})\otimes\mathscr{W}^{k^*}(\mathfrak{g})$-modules, which follows from the decomposition [AG02, Zhu08] of $\mathcal{D}_{G,k}^{ch}$ as $\widehat{\mathfrak{g}}\times\widehat{\mathfrak{g}}$-modules. Hence, by equation (60),

$$\mathbf{I}_G^k\otimes\mathcal{U}(\mathfrak{g}) \cong \bigoplus_{\lambda\in P_+} T_{\lambda,0}^k\otimes T_{\lambda^*,0}^{k^*}\otimes\mathcal{U}(\mathfrak{g}) \cong \bigoplus_{\substack{\lambda\in P_+,\ \mu\in P_+\\ \mu-\lambda^*\in Q}} T_{\lambda,0}^k\otimes T_{\mu,0}^{k^*+1}\otimes T_{\lambda^*,\mu}^{\check{\ell}},$$

and $\check{\ell}$ satisfies the relation

$$\frac{1}{k+h^\vee} + \frac{1}{\check{\ell}+h^\vee} = 1.$$

It follows that

$$A^1[\mathfrak{g}] := \mathrm{Com}(\mathscr{W}^{k^*+1}(\mathfrak{g}), \mathbf{I}_G^k\otimes\mathcal{U}(\mathfrak{g})) \cong \bigoplus_{\lambda\in P_+\cap Q} T_{\lambda,0}^k\otimes T_{\lambda^*,0}^{\check{\ell}}.$$

Moreover, since $\mathbf{I}_G^k\otimes\mathcal{U}(\mathfrak{g})$ is simple $A^1[\mathfrak{g}]$ is simple as well by [CGN, Proposition 5.4].

Assuming that the statement is true for $n\in\mathbb{Z}_{\geqslant 1}$, we find similarly that

$$A^{n+1}[\mathfrak{g}] := \mathrm{Com}(\mathscr{W}^{k'+1}(\mathfrak{g}), A^n[\mathfrak{g}]\otimes\mathcal{U}(\mathfrak{g}))$$

has the required decomposition.

For the central charge computation, it is useful to introduce

$$\psi := k+h^\vee \qquad \text{and} \qquad \phi_n := \frac{\psi}{n\psi-1}$$

so that

$$\frac{1}{\psi} + \frac{1}{\phi_n} = n.$$

By equation (34) and the fact that the central charge of $\mathscr{W}^{k+1}(\mathfrak{g})$ is

$$(1-h(h+1)(k+h)^2/(k+h+1))\,\mathrm{rk}\,\mathfrak{g} = \mathrm{rk}\,\mathfrak{g} - \dim\mathfrak{g}\,h\frac{\psi^2}{\psi+1}$$

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(here we used that $(h+1)\operatorname{rk}\mathfrak{g} = \dim\mathfrak{g}$), we have

$$c_{A^{n+1}[\mathfrak{g}]} = c_{A^n[\mathfrak{g}]} - \frac{h^2+h-1}{h+1}\dim\mathfrak{g} - \operatorname{rk}\mathfrak{g} + h\frac{\psi^2}{(n\psi-1)((n+1)\psi-1)})\dim\mathfrak{g}$$
$$= -\dim\mathfrak{g}h + h\phi_n\phi_{n+1}\dim\mathfrak{g},$$

where c_V is the central charge of V, and we have put $A^0[\mathfrak{g}] = \mathbf{I}^k_G$. Note that

$$\phi_n - \phi_{n+1} = \phi_n\phi_{n+1} = \psi(n\phi_n - (n+1)\phi_{n+1}),$$

and so by induction for n, we have

$$c_{A^n[\mathfrak{g}]} = 2\operatorname{rk}\mathfrak{g} + 4h\dim\mathfrak{g} - nh\dim\mathfrak{g}(1+\psi\phi_n).$$

The conformal dimension of $T^k_{\lambda,0} \otimes T^{k'}_{\lambda^*,0}$ is

$$\frac{n}{2}(|\lambda+\rho|^2 - |\rho|^2) - 2(\lambda|\rho) = \frac{n}{2}|\lambda|^2 + (n-2)(\lambda|\rho),$$

which is an integer for $\lambda \in Q$. If $n \geqslant 2$, this is clearly nonnegative and is equal to zero if and only if $\lambda = 0$, whence the last assertion. □

Remark 9.2. More generally, it is expected [CG17, CGL18] that if k and k' are irrational numbers related by

$$\frac{1}{k+h^\vee} + \frac{1}{k'+h^\vee} = n \in \mathbb{Z},$$

then

$$\bigoplus_{\lambda\in Q\cap P^+} \mathbb{V}^k_{\lambda,f} \otimes \mathbb{V}^{k'}_{\lambda^*,f'}$$

can be given the structure of a simple vertex operator algebra for any nilpotent elements f, f'.

10. Fusion categories of modules over quasi-lisse W-algebras

For a vertex operator algebra V, let $\mathcal{C}^{ord}_V$ be the full subcategory of the category of finitely generated V-modules consisting of modules M on which L_0 acts locally finitely, the L_0-eigenvalues of M are bounded from below and all the generalized L_0-eigenspaces are finite dimensional. A simple object in $\mathcal{C}^{ord}_V$ is called an *ordinary representation* of V.

Recall that a finitely strongly generated vertex algebra V is called *quasi-lisse* [AK18] if the associate variety X_V has finitely many symplectic leaves. By [AK18, Theorem 4.1], if V is quasi-lisse, then $\mathcal{C}^{ord}_V$ has only finitely many simple objects.

Conjecture 1. *Let V a finitely strongly generated, self-dual, quasi-lisse vertex operator algebra of CFT type. Then the category $\mathcal{C}^{ord}_V$ has the structure of a vertex tensor category in the sense of [HL94].*

In the case that V is *lisse*, that is, X_V is zero dimensional, Conjecture 1 has been proved in [Hua09]. Conjecture 1 is true if one can show that every object in $\mathcal{C}^{ord}_V$ is C_1-cofinite and if grading-restricted generalized Verma modules for V are of finite length [CY20]. If V is a vertex algebra that has an affine vertex subalgebra at admissible level, then another possibility of proving Conjecture 1 is the following. First, show that the affine vertex subalgebra is simple; second, prove that a suitable category of modules of the coset by the affine subalgebra in V has vertex tensor category structure. Then use the theory of vertex algebra extensions [CKM17] to deduce vertex tensor category structure on $\mathcal{C}^{ord}_V$. This

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is a promising direction as, for example, many simple quotients of cosets of $\mathcal{W}$-algebras of type A are rational and lisse by [CL20, Cor. 6.13], and similar results for $\mathcal{W}$-algebras of type B, C and D are work in progress.

In the case that Conjecture 1 is true, we denote by $M \boxtimes_V N$ the tensor product of V-modules M and N in $\mathcal{C}_V^{ord}$, or simply by $M \boxtimes N$ if no confusion should occur.

Now let k be an admissible number for $\widehat{\mathfrak{g}}$. As explained in Subsection 8.2,

$$X_{\mathbb{L}_k(\mathfrak{g})} = \overline{\mathbb{O}_k}$$

for some nilpotent orbit $\mathbb{O}_k$, and hence, $\mathbb{L}_k(\mathfrak{g})$ is quasi-lisse. By the conjecture of Adamović and Milas [AM95] that was proved in [Ara16a], the category $\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})}$ is semisimple and $\{\mathbb{L}^k_\lambda | \lambda \in Adm^k_{\mathbb{Z}}\}$ gives a complete set of isomorphism classes in $\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})}$. Note that, in the case k is nonnegative integer, this is a well-known fact [FZ92], and if this is the case, $\mathbb{L}_k(\mathfrak{g})$ is rational and lisse, and hence, Conjecture 1 holds by [HL95]. In the case k is admissible but not an integer, $\mathbb{L}_k(\mathfrak{g})$ is neither rational nor lisse anymore. Nevertheless, Conjecture 1 has been proved for $V = \mathbb{L}_k(\mathfrak{g})$ in [CHY18] provided that $\mathfrak{g}$ is simply laced. Moreover, it was shown in [Cre19] that $\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})}$ is a fusion category, i.e., any object is rigid, and we have an isomorphism

$$K[\mathcal{C}^{ord}_{\mathbb{L}_{p-h^\vee}(\mathfrak{g})}] \cong K[\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})}], \quad [\mathbb{L}^{p-h^\vee}_\lambda] \mapsto [\mathbb{L}^k_\lambda] \tag{61}$$

of Grothendieck rings of our fusion categories, where p is the numerator of $k + h^\vee$.

Let f be a nilpotent element of $\mathfrak{g}$. By [Ara15a],

$$X_{H^0_{DS,f}(\mathbb{L}_k(\mathfrak{g}))} = X_{\mathbb{L}_k(\mathfrak{g})} \cap \mathcal{S}_f,$$

where $\mathcal{S}_f$ is the Slodowy slice at f in $\mathfrak{g}$. Therefore, in the case that k is admissible, $X_{H^0_{DS,f}(\mathbb{L}_k(\mathfrak{g}))} = \overline{\mathbb{O}_k} \cap \mathcal{S}_f$ is a *nilpotent Slodowy slice*, that is, the intersection of the Slodowy slice with a nilpotent orbit closure, provided that $H^0_{DS,f}(\mathbb{L}_k(\mathfrak{g})) \neq 0$ or, equivalently, $\overline{\mathbb{O}_k} \cap \mathcal{S}_f \neq \emptyset$, that is, $f \in \overline{\mathbb{O}_k}$. In particular, $H^0_{DS,f}(\mathbb{L}_k(\mathfrak{g}))$ is quasi-lisse if $f \in \overline{\mathbb{O}_k}$ and so is the simple W-algebra $\mathscr{W}_k(\mathfrak{g}, f)$. If $f \in \mathbb{O}_k$, then $\overline{\mathbb{O}_k} \cap \mathcal{S}_f$ is a point by the transversality of the Slodowy slices, and therefore, $\mathscr{W}_k(\mathfrak{g}, f)$ is lisse.

The good grading (11) is called *even* of $\mathfrak{g}_j = 0$ unless $j \in \mathbb{Z}$. If this is the case, $\mathscr{W}^k(\mathfrak{g}, f)$ is $\mathbb{Z}_{\geqslant 0}$-graded and thus of CFT type.

The following assertion was stated in the case that $f \in \mathbb{O}_k$ in [AvE], but the same proof applies.

Theorem 10.1. *Let k be admissible and $f \in \overline{\mathbb{O}_k}$. Suppose that f admits a good even grading. Then, for $\lambda \in Adm^k_{\mathbb{Z}}$, $\mathbb{L}^k_{\lambda,f} = H^0_{DS,f}(\mathbb{L}^k_\lambda)$ is simple. In particular, $H^0_{DS,f}(\mathbb{L}_k(\mathfrak{g})) = \mathscr{W}_k(\mathfrak{g}, f)$.*

Lemma 10.2. *Let f be an even nilpotent element. Then $\mathscr{W}_k(\mathfrak{g}, f)$ is self-dual and of CFT type with respect to the Dynkin grading.*

Proof. Since f is an even nilpotent element, the Dynkin grading is even, and so $\mathscr{W}_k(\mathfrak{g}, f)$ is of CFT type. The self-duality follows from the formula in [AvE, Proposition 6.1]. □

Remark 10.3. If the grading is not Dynkin, $\mathscr{W}_k(f, g)$ need not be self-dual; see [AvE, Proposition 6.3].

Theorem 10.4. *Let $\mathfrak{g}$ be simply laced, k admissible, and let f be an even nilpotent element in $\overline{\mathbb{O}_k}$. Suppose Conjecture 1 is true for $\mathscr{W}_{k+1}(\mathfrak{g}, f)$ and also that $\mathscr{W}_{k+1}(\mathfrak{g}, f)$ is self-dual. Then the functor*

$$\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})} \to \mathcal{C}^{ord}_{\mathscr{W}_k(\mathfrak{g},f)}, \quad M \mapsto H^0_{DS,f}(M),$$

is a unital braided tensor functor. In particular, the modules $\mathbb{L}^k_{\lambda,f}$, $\lambda \in Adm^k_{\mathbb{Z}}$, are rigid.

Remark 10.5. Note that by Theorem 8.6, we have that $\mathscr{W}_k(\mathfrak{g}, f) \otimes \mathbb{L}_1(\mathfrak{g})$ is an extension of $\mathscr{W}_{k+1}(\mathfrak{g}, f) \otimes \mathscr{W}_\ell(\mathfrak{g})$, where $\mathscr{W}_\ell(\mathfrak{g})$ is rational and lisse if k is admissible. Especially every ordinary module of $\mathscr{W}_k(\mathfrak{g}, f) \otimes \mathbb{L}_1(\mathfrak{g})$ is an object in the category of ordinary modules of $\mathscr{W}_{k+1}(\mathfrak{g}, f) \otimes \mathscr{W}_\ell(\mathfrak{g})$, and so if the latter has vertex tensor category structure, then so does the former by [CKM17]. In other words, if Conjecture 1 is true for $\mathscr{W}_{k+1}(\mathfrak{g}, f)$ and if k is admissible, then Conjecture 1 is also true for $\mathscr{W}_k(\mathfrak{g}, f)$.

Moreover, [CKM19, Thm. 5.12] applies to our setting if $\mathscr{W}_k(\mathfrak{g}, f)$ is $\mathbb{Z}$-graded. Thus, if the category of ordinary modules of $\mathscr{W}_{k+1}(\mathfrak{g}, f) \otimes \mathscr{W}_\ell(\mathfrak{g})$ is a fusion category, then so is the one of $\mathscr{W}_k(\mathfrak{g}, f) \otimes \mathbb{L}_1(\mathfrak{g})$. If $\mathscr{W}_k(\mathfrak{g}, f)$ is $\mathbb{Z}$-graded by conformal weight, it thus also follows that $\mathscr{W}_k(\mathfrak{g}, f)$ is rational provided that $\mathscr{W}_{k+1}(\mathfrak{g}, f)$ is rational and lisse.

Note that if $f \in \mathbb{O}_k$, then Conjecture 1 holds since $\mathscr{W}_k(\mathfrak{g}, f)$ is lisse.

In the case that $f = f_{prin}$, $f \in \overline{\mathbb{O}_k}$ if and only if $\overline{\mathbb{O}_k} = \mathcal{N}$, the nilpotent cone of $\mathfrak{g}$. An admissible number k such that $\overline{\mathbb{O}_k} = \mathcal{N}$ is called *nondegenerate*. If this is the case, $\mathscr{W}^k(\mathfrak{g})$ is rational, and the complete fusion rule $\mathscr{W}_k(\mathfrak{g}, f)$ has been determined previously in [FKW92, AvE19, Cre19].

In the case that $\mathbb{O}_k$ is a subregular nilpotent orbit and $f \in \mathbb{O}_k$, then $\mathscr{W}_k(\mathfrak{g}, f)$ is rational [AvE], and the fusion rules of $\mathscr{W}_k(\mathfrak{g}, f)$ has been determined in [AvE].

The following assertion, which follows immediately from Theorem 10.4, is new except for type A ([AvE]) and the above cases since the conjectural rationality [KW08, Ara15a] of $\mathscr{W}_k(\mathfrak{g}, f)$ with $f \in \mathbb{O}_k$ is open otherwise.

Corollary 10.6. *Let $\mathfrak{g}$ be simply laced and k admissible, and suppose that $\mathbb{O}_k$ is an even nilpotent orbit, $f \in \mathbb{O}_k$. Then $\mathbb{L}^k_{\lambda,f}$ is rigid for all $\lambda \in Adm^k_{\mathbb{Z}}$.*

Remark 10.7. Let $\mathcal{C}^{KL}_{\mathscr{W}_k(\mathfrak{g},f)}$ denote the fusion category consisting of objects $H^0_{DS,f}(M)$, $M \in \mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})}$. By Theorem 10.4,

$$\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})} \to \mathcal{C}^{KL}_{\mathscr{W}_k(\mathfrak{g},f)}, \quad M \mapsto H^0_{DS,f}(M)$$

is a quotient functor between fusion categories. It gives an equivalence if and only if the modules $\mathbb{L}^k_{\lambda,f}$, $\lambda \in Adm^k_{\mathbb{Z}}$ are distinct.

The rest of this section is devoted to the proof of Theorem 10.4.

Suppose Conjecture 1 holds for V, and let $\mathcal{C}$ be a monoidal full subcategory of $\mathcal{C}^{ord}_V$. Recall that $M, N \in \mathcal{C}$ are said to *centralize each other* [Mü03] if the monodromy of M and N is trivial, i.e., is equal to the identity on $M \boxtimes_V N$, where the monodromy is the double braiding

$$M \boxtimes_V N \xrightarrow{b_{M,N}} N \boxtimes_V M \xrightarrow{b_{N,M}} M \boxtimes_V N.$$

Suppose that V is a vertex operator subalgebra of another quasi-lisse vertex operator algebra W and that W as a V-module is an object of $\mathcal{C}^{ord}_V$. We assume that Conjecture 1 holds for W as well. Let $\mathcal{D}$ be a monoidal full subcategory of $\mathcal{C}$ such that W and any object in $\mathcal{D}$ centralize each other. Then

$$\mathcal{F}(M) := W \boxtimes_V M$$

can be equipped with a structure of a module for the vertex operator algebra W, giving rise to the *induction functor* [CKM17]

$$\mathcal{D} \to \mathcal{C}^{ord}_W, \quad M \mapsto \mathcal{F}(M).$$

Theorem 10.8 ([Cre19]). *Let $\mathfrak{g}$ be simply laced, and let $\lambda \in Adm^k_{\mathbb{Z}}$, $\mu \in Adm^{\check{k}}_{\mathbb{Z}}$. We have $\mathbf{L}^k_{[\lambda,0]} \boxtimes \mathbf{L}^k_{[0,\mu]} \cong \mathbf{L}^k_{[\lambda,\mu]}$. Moreover, $\mathbf{L}^k_{[\lambda,0]}$ and $\mathbf{L}^{\check{k}}_{[0,\mu]}$ centralize each other if $\lambda \in Adm^k_{\mathbb{Z}} \cap Q$.*

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Let f be admissible, and let f be an even nilpotent element in $\overline{\mathbb{O}_k}$. Then $f \in \overline{\mathbb{O}_{k+1}}$ as well since $\overline{\mathbb{O}_k}$ depends only on the denominator of k. Let ℓ be the number defined by

$$\frac{1}{k+h^\vee+1} + \frac{1}{\ell+h^\vee} = 1,$$

that is,

$$\ell + h^\vee = \frac{k+h^\vee+1}{k+h^\vee}.$$

Then ℓ is a nondegenerate admissible number. Note that

$$Adm^{\ell}_{\mathbb{Z}} = Adm^{k+1}_{\mathbb{Z}} = P_+^{p+q-h^\vee}, \quad Adm^{\check{\ell}}_{\mathbb{Z}} = Adm^{k}_{\mathbb{Z}} = P_+^{p-h^\vee}, \tag{62}$$

where we have put $k + h^\vee = p/q$.

Consider the $\mathscr{W}_\ell(\mathfrak{g})$-modules

$$\mathbf{L}^{\ell}_{[0,\mu]} \cong \mathbf{L}^{\check{\ell}}_{[\mu,0]} = H^0_{DS,f_{prin}}(\mathbb{L}^{\check{\ell}}_{\mu}), \quad \mu \in Adm^{\ell}_{\mathbb{Z}}.$$

Since the stabilizer of $0 \in Adm^{\ell}_{\mathbb{Z}}$ of the $\tilde{W}_+$-action is trivial, the simple $\mathscr{W}_\ell(\mathfrak{g})$-modules $\mathbf{L}^{\ell}_{[0,\mu]}$, $\mu \in Adm^{\ell}_{\mathbb{Z}}$, are distinct. Therefore, by Theorem 10.4 (that is proved for $f = f_{prin}$ in [Cre19]), the modules $\mathbf{L}^{\ell}_{[0,\mu]}$, $\mu \in Adm^{\ell}_{\mathbb{Z}}$ form a fusion full subcategory of $\mathcal{C}^{ord}_{\mathscr{W}_\ell(\mathfrak{g})}$ that is equivalent to a category that can be called a *s*imple current twist of $\mathcal{C}^{ord}_{\mathbb{L}_{\check{\ell}}(\mathfrak{g})} \cong \mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})}$ (see [Cre19, Thm.7.1] for the details). This simple current twist of $\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})}$ is the fusion subcategory of $\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})} \boxtimes \mathcal{C}^{ord}_{\mathbb{L}_1(\mathfrak{g})}$ whose simple objects are the $\mathbb{L}^k_{\mu,f} \otimes \mathbb{L}^1_{-\mu+Q}$. Call this category $\mathcal{C}^{ord,tw}_{\mathbb{L}_k(\mathfrak{g})}$.

Now set

$$W := \mathscr{W}_k(\mathfrak{g}, f) \otimes \mathbb{L}_1(\mathfrak{g}) = \mathbb{L}^k_{0,f} \otimes \mathbb{L}^1_{\nu}, \quad V := \mathscr{W}_{k+1}(\mathfrak{g}, f) \otimes \mathscr{W}_\ell(\mathfrak{g}) = \mathbb{L}^{k+1}_{0,f} \otimes \mathbf{L}_{[0,0]}.$$

By Theorem 8.6,

$$W \cong \bigoplus_{\lambda \in Adm^{k+1}_{\mathbb{Z}} \cap Q} \mathbb{L}^{k+1}_{\lambda,f} \otimes \mathbf{L}^{\ell}_{[\lambda,0]}. \tag{63}$$

Hence, V is a vertex subalgebra of W. Moreover, each direct summand of W is an ordinary V-module and the sum is finite, and so W is an object of $\mathcal{C}^{ord}_V$.

Clearly, the V-modules

$$\mathbb{L}^{k+1}_{0,f} \otimes \mathbf{L}^{\ell}_{[0,\mu]} = \mathscr{W}_{k+1}(\mathfrak{g}, f) \otimes \mathbf{L}^{\ell}_{[0,\mu]}, \quad \mu \in Adm^{\check{\ell}}_{\mathbb{Z}}$$

form a monoidal full subcategory of $\mathcal{C}^{ord}_V$ that is equivalent to $\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})}$. We denote by $\mathcal{D}$ this fusion category.

By Theorem 10.8 and equation (63), we find that W centralizes any object of $\mathcal{D}$. Hence, we have the induction functor

$$\mathcal{F} \colon \mathcal{D} \to \mathcal{C}^{ord}_W, \quad M \mapsto W \boxtimes_V M.$$

Theorem 10.9. *The induction functor* $\mathcal{F} : \mathcal{D} \to \mathcal{C}^{ord}_W$ *is a fully faithful tensor functor that sends* $\mathbb{L}^{k+1}_{0,f} \otimes \mathbf{L}^{\ell}_{[0,\mu]}$ *to* $\mathbb{L}^k_{\mu,f} \otimes \mathbb{L}^1_{-\mu+Q}$*, where* $-\mu + Q$ *denotes the class in* $P/Q \cong P^1_+$.

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Proof. Thanks to [Cre19, Theorem 3.5], it is sufficient to show that $\mathcal{F}(\mathbb{L}^{k+1}_{0,f}\otimes\mathbf{L}^{\ell}_{[0,\mu]})$ is a simple W-module that is isomorphic to $\mathbb{L}^{k}_{\mu,f}\otimes\mathbb{L}^{1}_{-\mu+Q}$ for all $\mu\in Adm^{\check{\ell}}_{\mathbb{Z}}$. As V-modules we have

$$\mathcal{F}(\mathbb{L}^{k+1}_{0,f}\otimes\mathbf{L}^{\ell}_{[0,\mu]}) = \bigoplus_{\lambda\in Adm^{k+1}_{\mathbb{Z}}\cap Q} (\mathbb{L}^{k+1}_{\lambda,f}\otimes\mathbf{L}^{\ell}_{[\lambda,0]})\boxtimes_V(\mathbb{L}^{k+1}_{0,f}\otimes\mathbf{L}^{\ell}_{[0,\mu]})$$
$$\cong \bigoplus_{\lambda\in Adm^{k+1}_{\mathbb{Z}}\cap Q}\mathbb{L}^{k+1}_{\lambda,f}\otimes\mathbf{L}^{\ell}_{[\lambda,\mu]}\cong\mathbb{L}^{k}_{\mu,f}\otimes\mathbb{L}^{1}_{-\mu+Q}$$

by Theorem 8.6 and Theorem 10.8. We claim this is indeed an isomorphism of W-modules. To see this, it is sufficient to show that there is a nontrivial W-module homomorphism $\mathcal{F}(\mathbb{L}^{k+1}_{0,f}\otimes\mathbf{L}^{\ell}_{[0,\mu]})\to\mathbb{L}^{k}_{\mu,f}\otimes\mathbb{L}^{1}_{-\mu+Q}$ since $\mathbb{L}^{k}_{\mu,f}\otimes\mathbb{L}^{1}_{-\mu+Q}$ is simple. By the Frobenius reciprocity [KO02, CKM17], we have

$$\begin{aligned}&\mathrm{Hom}_{W\text{-Mod}}(\mathcal{F}(\mathbb{L}^{k+1}_{0,f}\otimes\mathbf{L}^{\ell}_{[0,\mu]}),\mathbb{L}^{k}_{\mu,f}\otimes\mathbb{L}^{1}_{-\mu+Q})\\&\cong\mathrm{Hom}_{V\text{-Mod}}(\mathbb{L}^{k+1}_{0,f}\otimes\mathbf{L}^{\ell}_{[0,\mu]},\mathbb{L}^{k}_{\mu,f}\otimes\mathbb{L}^{1}_{-\mu+Q})\\&\cong\bigoplus_{\lambda\in Adm^{k+1}_{\mathbb{Z}}\cap Q}\mathrm{Hom}_{V\text{-Mod}}(\mathbb{L}^{k+1}_{0,f}\otimes\mathbf{L}^{\ell}_{[0,\mu]},\mathbb{L}^{k+1}_{\lambda,f}\otimes\mathbf{L}^{\ell}_{[\lambda,\mu]}).\end{aligned}$$

It follows that there is a nontrivial homomorphism corresponding to the identity map $\mathbb{L}^{k+1}_{0,f}\otimes\mathbf{L}^{\ell}_{[0,\mu]}\to\mathbb{L}^{k+1}_{0,f}\otimes\mathbf{L}^{\ell}_{[0,\mu]}$. □

Proof of Theorem 10.4. By Theorem 10.9, the correspondence

$$\mathbb{L}^{k}_{\mu}\otimes\mathbb{L}^{1}_{-\mu+Q}\mapsto\mathbf{L}^{\ell}_{[0,\mu]}\mapsto\mathbb{L}^{k+1}_{0,f}\otimes\mathbf{L}^{\ell}_{[0,\mu]}\mapsto\mathbb{L}^{k}_{\mu,f}\otimes\mathbb{L}^{1}_{-\mu+Q},\quad \mu\in Adm^{k}_{\mathbb{Z}}=Adm^{\check{\ell}}_{\mathbb{Z}}$$

gives a tensor functor

$$\mathcal{C}^{ord,tw}_{\mathbb{L}_k(\mathfrak{g})}\to\mathcal{C}^{ord}_{\mathscr{W}_k(\mathfrak{g},f)\otimes\mathbb{L}_1(\mathfrak{g})}=\mathcal{C}_{\mathscr{W}_k(\mathfrak{g},f)}\boxtimes\mathcal{C}^{ord}_{\mathbb{L}_1(\mathfrak{g})},$$

where the last $\boxtimes$ denotes the Deligne product. This functor is fully faithful if and only if all the $\mathbb{L}^{k}_{\mu,f}$ are nonisomorphic. Since $\mathcal{C}^{ord}_{\mathbb{L}_1(\mathfrak{g})}$ is semisimple and its simple objects are all invertible, i.e., simple currents, this corresponds extends to a surjective tensor functor

$$\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})}\boxtimes\mathcal{C}^{ord}_{\mathbb{L}_1(\mathfrak{g})}\to\mathcal{C}_{\mathscr{W}_k(\mathfrak{g},f)}\boxtimes\mathcal{C}^{ord}_{\mathbb{L}_1(\mathfrak{g})},$$

which restricts to a surjective tensor functor

$$\mathcal{C}^{ord}_{\mathbb{L}_k(\mathfrak{g})}\to\mathcal{C}_{\mathscr{W}_k(\mathfrak{g},f)},\qquad\mathbb{L}^{k}_{\mu}\to\mathbb{L}^{k}_{\mu,f}=H^{0}_{DS,f}(\mathbb{L}^{k}_{\mu})$$

that again is fully faithful if and only if all the $\mathbb{L}^{k}_{\mu,f}$ are nonisomorphic. □

Let us conclude with a remark on the general case.

Remark 10.10. It is a well-known result of Kazhdan–Lusztig that ordinary modules of affine vertex algebras at generic level have vertex tensor category [KL1, KL2, KL3, KL4], and it is reasonable to expect that a similar result might hold for $\mathscr{W}$-algebras as well. However, this is only proven for the Virasoro algebra [CJORY20] and the $N=1$ super Virasoro algebra [CMY20].

At generic levels one has to deal with infinite order extensions of vertex algebras, and so one needs to consider completions of vertex tensor categories. The theory of vertex algebra extension also works for such completions [CMY20]. This means that if one can prove the existence of vertex tensor category structure on categories of ordinary modules of $\mathscr{W}$-algebras at generic levels, then one can also derive results for generic levels that are similar to the statements of this section.

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