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GROUPS IN WHICH NORMAL CLOSURES OF ELEMENTS HAVE BOUNDEDLY FINITE RANK

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Abstract. It is proved that if the normal closure of every element of a group G has rank at most r, then the derived subgroup of G has r-bounded rank.

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A well-known result due to B. H. Neumann [**4**] states that a group G in which every element has at most n conjugates, where n is a fixed positive integer, has its derived subgroup finite. Subsequent papers discussed improved bounds for the order of G' in terms of n (see, in particular, [**7**]). On the other hand, it is obvious that if G' has finite-order k, then every element of G has at most k conjugates. In paper [**8**], a related property was discussed, the hypothesis there being that the normal closure of every element of G has (Prüfer or Mal'cev) rank at most r, where r is a fixed positive integer. This means that for every element x of the group G, all finitely generated subgroups of the normal closure $\langle x\rangle^G$ are r-generated. The main result of [**8**] is that a locally soluble-by-finite group G satisfying this hypothesis on normal closures has its derived subgroup of r-bounded rank; that is G' has finite rank at most s for some integer s that depends on r only. It was also pointed out in [**8**] that the same result clearly holds if G belongs to any class $\mathfrak{X}$ of groups with the property that finitely generated $\mathfrak{X}$-groups of finite rank are soluble-by-finite, and as noted in the corollary in that paper we see that $\mathfrak{X}$ here may be chosen to be the class of residually soluble-by-finite groups. (It was shown in [**2**] that a residually finite group of finite rank has a locally soluble subgroup of finite index; the generalization used for the above-mentioned corollary appears in [**1**].) The purpose of this paper is to show that the condition on normal closures is in itself sufficient to establish the finiteness of the rank of G'. It is clear that if G is a group and G' has rank k, then the normal closure of every element of G has rank at most $k+1$. Here we prove the following.

THEOREM. *Let G be a group and r a positive integer, and suppose that $\langle x\rangle^G$ has rank at most r, for every element x of G. Then G' has rank at most s, where s is an integer that depends only on r.*

We note that the hypothesis on G given in the theorem is inherited by subgroups and homomorphic images. As a first step towards proving the theorem we establish a result that exploits the results of [**8**] to an extent that allows us to concentrate on the case in which G' is perfect.

Proposition 1. *Let G be a group that satisfies the hypotheses of the theorem, and let $K = G'$. Then, in order to show that K has r-bounded rank, we may assume that G is finitely generated (so that G has finite rank and K is the normal closure in G of finitely many elements) and that K is perfect and has no proper subgroups of finite index.*

Proof. Let G be as stated. If F is an arbitrary finitely generated subgroup of G', then F is contained in E' for some finitely generated subgroup E of G. Since it is enough to show that E' has suitably bounded rank, we may assume that G is finitely generated and hence of finite rank, and then K is the normal closure in G of some finite set of commutators.

If R is the finite residual of G, then, by [**2**], G/R is soluble-by-finite and hence minimax (see Theorem 10.38 of [**5**]). Now let G/L be an arbitrary soluble-by-finite image of G. Again G/L is minimax, and so its finite residual Y is radicable nilpotent, by (a special case of) Theorem 9.31 of [**5**]. The torsion group T of Y is radicable and Černikov, and Y/T has its derived factor group radicable and minimax and hence periodic, which shows that Y/T is trivial; hence Y is abelian. Thus, if S is the soluble-by-finite residual of G, then R/S is abelian, and it follows that G/S is soluble-by-finite. Observe that S is the soluble-by-finite residual of every subgroup of finite index in G and that SN/N is the soluble-by-finite residual of G/N for every normal subgroup N of G.

Clearly S is perfect. Let V denote the finite residual of S. Then $V \lhd G$ and, by [**2**], S/V has a normal locally soluble subgroup L/V of finite index, and we may choose L/V to be characteristic in S/V; thus $L \lhd G$. Now G/L is finite-by-(soluble-by-finite) and hence soluble-by-finite, and so $L = S$ and S/V is locally soluble. But a non-trivial locally soluble group of finite rank cannot be perfect [**5**, Lemma 10.39], and it follows that $V = S$ and hence that S has no proper subgroups of finite index.

By [**8**], K/S has r-bounded rank r_1, say. Also, by Theorem 3.25 of [**5**], G/S is nilpotent-by-abelian-by-finite, and so there is a normal subgroup U of finite index in G such that U'/S is nilpotent. Since G/U' is finitely presented we see that U'/S is finitely generated as a G-group, and so there is an r_1-generated subgroup X of U' such that $U' = S\langle X\rangle^G$. As $\langle X\rangle^G$ has rank at most $r_1 r$ we may factor and hence assume that $U' = S$. If U' has rank r_2, then $K = G'$ has rank at most $r + r_2$; thus we may replace G by U and hence assume that $K = S$. This concludes the proof of the proposition. □

If K is a perfect normal subgroup of a group G and $N_1, \ldots, N_s$ are distinct maximal G-invariant subgroups of K with intersection N, then K/N is a direct product of s G-invariant subgroups M_j/N isomorphic to the groups K/N_j. This is well known and easily established by induction on s, and we shall require this result in the proof of the theorem. But we shall require too the following more general version of this result (which is also probably well known), where N_j are not necessarily maximal.

Lemma 2. *Let K be a perfect group and s an integer greater than 2, and suppose that $N_1, \ldots, N_s$ are normal subgroups of K such that $K = N_iN_j$ whenever $i \neq j$. Let $N = \bigcap_{i=1}^{s} N_i$, and for each $j = 1, \ldots, s$, let $M_j = \bigcap_{i\neq j}\{N_i : 1 \leq i \leq s\}$. Then $K = \prod_{j=1}^{s} M_j$, and so $K = M_jNj$ for each j and $K/N = Dr_{j=1}^{s} M_j/N$.*

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Proof. Suppose first that $s = 3$. Then we have

$$[N_3, K] = [N_3, N_1][N_3, N_2] \leq (N_3 \cap N_1)(N_3 \cap N_2)$$

and, similarly,

$$[N_2, K] \leq (N_2 \cap N_1)(N_2 \cap N_3),$$

so that $K = K' = [N_2N_3, K] \leq M_1M_2M_3$, and the result is established in this case.

Now suppose that the result holds for some $t \geq 3$ and that $N_1, \ldots, N_{t+1}$ are normal subgroups of K such that $K = N_iN_j$ whenever $i \neq j$. Let $L = \bigcap_{i=2}^{t} N_i$. Then $K = N_1N_{t+1}$, and by two applications of the case $s = t$, we have $K = N_1L = N_{t+1}L$. By the case $s = 3$ it follows that $K = N_1(L \cap N_{t+1}) = N_1M_1$. By symmetry we thus have $K = N_jM_j$ for each $j = 1, \ldots, t+1$, and so

$$\begin{aligned} K &= (N_1M_1) \cap (N_2M_2) \cap \cdots \cap (N_{t+1}M_{t+1}) \\ &= ((M_1M_2)(N_1 \cap N_2)) \cap (N_3M_3) \cap \cdots \cap (N_{t+1}M_{t+1}) \\ &= ((M_1M_2M_3)(N_1 \cap N_2 \cap N_3)) \cap (N_4M_4) \cap \cdots \cap (N_{t+1}M_{t+1}) \\ &= \cdots = (M_1M_2 \ldots M_{t+1})N = M_1M_2 \ldots M_{t+1}, \end{aligned}$$

and the result follows by induction. □

We need two further lemmas. The first of these may be established by a routine application of Zorn's lemma, and we omit the proof.

LEMMA 3. *Let G be a group and K a subgroup that is the normal closure in G of a finite set of elements but not the normal closure of a single element. Then there is a G-invariant subgroup M of K that is maximal with the property that K/M is not the normal closure of a single element.*

LEMMA 4. *Let G be a group and K a perfect subgroup of G that is the normal closure in G of finitely many elements x_i, each of finite order, and suppose that K has finite rank. Then $K = \langle a \rangle^G$ for some $a \in K$.*

Proof. Let m be the rank of K; suppose that $N_1, \ldots, N_t$ are distinct maximal G-invariant subgroups of K; and let N denote the intersection of the $N_j, j = 1, \ldots, t$. By the remarks preceding the statement of Lemma 2, K/N is a direct product of t G-invariant subgroups M_j/N, where $M_j/N \cong K/N_j$ for each j. At most m of these direct factors are non-periodic, since K/N has no free abelian section with rank greater than m. Now let π be the set of primes p such that p divides $|x_i|$ for some i. If, for some j, K/N_j is periodic, then it contains an element of order p for some $p \in \pi$. But as for non-periodic elements, at most m of the factors K/N_j can contain an element of order p, and we deduce that $t \leq m(1 + |\pi|)$.

Thus we may assume that $\{N_1, \ldots, N_t\}$ is the set of *all* maximal G-invariant subgroups of K. Now, with the M_j as defined above, choose $a_j \in M_j \setminus N$ for each j, and set $a = a_1 \ldots a_t$. Since K/N is perfect we deduce that $N\langle a \rangle^G = K$ (see Section 3.3.12 of [**6**]). If $\langle a \rangle^G < K$, then, since K is finitely generated as a G-group, $\langle a \rangle^G$ is contained in some maximal G-invariant subgroup N_j of K, and we have the contradiction $K = N\langle a \rangle^G \leq N_j$. The result follows. □

We are now in a position to complete the proof of the theorem. In view of Proposition 1, it will be enough to establish the following.

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PROPOSITION 2. *Let G be a group and K a perfect subgroup that is the normal closure in G of finitely many elements. Suppose that K has finite rank and that K has no proper subgroups of finite index. Then there are elements x, y, z of K such that $K = \langle x, y, z\rangle^G$.*

Proof. Assuming that the result is false, we shall prove that for every integer $s \geq 2$, the following holds:

(*) There exist G-invariant subgroups $N_1, \ldots, N_s$ of K such that $K = N_i N_j$ whenever $i \neq j$, and for each $j = 1, \ldots, s$, K/N_j is not the normal closure in G of a single element and N_j is maximal with respect to this property.

By Lemma 3 there is a G-invariant subgroup N_1 of K such that K/N_1 is not the normal closure of a single element and N_1 is maximal subject to this condition. If $x \in K \setminus N_1$, then there is an element $y \in K$ such that $K = N_1\langle x, y\rangle^G$, and $K/\langle x, y\rangle^G$ is not the normal closure in G of a single element, by hypothesis. Thus, again by Lemma 3, there is a G-invariant subgroup N_2 of K, containing x and y, such that K/N_2 is not the normal closure of a single element and N_2 is maximal such. Then $K = N_1N_2$, and (*) holds for $s = 2$.

Assume now that, for some $t \geq 2$, we have subgroups $N_1, \ldots, N_t$ such that (*) holds. Each of the indices $|K : N_j|$ is infinite, and so by a well-known result from [**3**], some element x of K is not contained in $\bigcup_{j=1}^{t} N_j$. For each $j = 1, \ldots, t$ we have $K = N_j\langle x, a_j\rangle^G$, for some a_j; moreover, by Lemma 2 we may choose a_j to lie in $M_j := \bigcap_{i\neq j}\{N_i : 1 \leq i \leq s\}$. Write $a = a_1 \ldots a_t$.

Since $K/\langle a, x\rangle^G$ is not the normal closure of a single element, we may apply Lemma 3 once more to obtain a G-invariant subgroup N_{t+1} of K such that N_{t+1} contains a and x, K/N_{t+1} is not the normal closure of a single element and N_{t+1} is maximal with this condition. We show now that $K = N_jN_{t+1}$ for all $j = 1, \ldots, t$. Indeed, for each such j and for every $i \in \{1, \ldots, t\} \setminus \{j\}$, we have $a_i \in N_j \leq N_jN_{t+1}$, and since $a \in N_jN_{t+1}$ it follows that $a_j \in N_jN_{t+1}$ and hence that $K = N_j\langle x, a_j\rangle^G \leq N_jN_{t+1}$, as required. This in turn establishes that (*) holds for all $s \geq 2$.

Now fix an integer $s \geq 2$, and set $N = \bigcap_{i=1}^{s} N_i$. From Lemma 2 we see that K/N is the direct product of s G-invariant subgroups, each isomorphic to some K/N_j. But, by Lemma 4 (with K/N_j in place of K), none of the factors K/N_j is periodic, and it follows that K/N has a free abelian subgroup of rank s. Since s was arbitrary we have a contradiction to the fact that K has finite rank, and the proposition is proved, as therefore is the theorem. □

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