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Mr W. L. THOMSON, President, in the Chair.

## Contact between a Curve and its Envelope.

By EDWARD B. ROSS, M.A.

This paper deals with a few of the simpler specialisations of the intersections of a plane curve and the envelope of the family to which it belongs. It follows the method adopted by Professor Chrystal in dealing with the $p$-discriminant of a differential equation of the first order. This method is specially applicable to definite problems; in these it is safer to work out the result than to rely on theory.

A summary of the results is given at the end.

The family is taken to be $\phi(x, y, t) = 0$; $t$ is the parameter. The $t$-discriminant locus $\Delta_t = 0$, with which rather than with the somewhat vague envelope we must deal, is given by

$$\phi = 0, \quad \phi_t = 0.$$

$\Delta_t = 0$ should be found by a formal process, not by a short cut.

Our work will be simplified if we take the curve under consideration to be $t = 0$; the point where it meets the envelope to be $x = 0$, $y = 0$; and the tangent to it there the $x$-axis. We assume that $\phi$ is expansible in powers of $t$, $x$, $y$ in the neighbourhood of the point $x = 0$, $y = 0$, $t = 0$.

Let $$\phi = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + \dots,$$

where $$A_0 = a_0 + b_0 x + c_0 y + d_0 x^2 + e_0 xy + f_0 y^2 + \dots,$$

$$A_1 = a_1 + b_1 x + c_1 y + \dots,$$

$$A_2 = a_2 + b_2 x + \dots,$$

$$A_3 = a_3 + \dots,$$

etc., etc.

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Since $\phi(0)=0$, *i.e.*, $A_0=0$ touches the $x$-axis at the origin, $a_0=0$, $b_0=0$; as $\Delta_t=0$ also passes through the origin, $a_1=0$.

A sufficient approximation to the envelope in the ordinary case is

$$4a_2(c_0y+d_0x^2)=(b_1x)^2.$$

This shows ordinary 2-pointic contact between the envelope and the curve. This approximation is not sufficient if $b_1$, $a_2$, or $c_0$ be zero.

We shall find it useful to write $[\lambda]$ for a quantity of order $\lambda$ in the variable $x$ of the *envelope*, in terms of which we may suppose $y$ and $t$ expanded. If $A_0$ is of order 4 in this variable (and $c_0 \neq 0$), the expansions of $y$ in terms of $x$ derived from $A_0=0$ and from the envelope would first differ in the coefficient of $x^4$, and so the curves would have 4-pointic contact; if $c_0=0$, more information is required before the species of contact can be assigned.

Take $\alpha_0$, $\alpha_1$, ... as the orders of $A_0$, $A_1$, ..., $\eta$ of $y$, $\tau$ of $t$.

$$\phi=0 \text{ or } 0=A_0+A_1t+A_2t^2+\ldots,$$

may be written $\quad 0=[\alpha_0]+[\alpha_1+\tau]+[2\tau]+\ldots,$

and $\phi_t=0$, $\quad 0=\qquad [\alpha_1]+[\tau]+\ldots\quad,$

provided $a_2 \neq 0$.

(1) Let $b_1=0$. We get $\alpha_1=\tau$, $\alpha_0=2\tau$. In the simplest case $\tau=2$ so that $\alpha_0=4$.

So we do not get in this way 3-pointic contact. By the theory of implicit functions if $a_2$, $c_0 \neq 0$, $y$ is expansible in integral powers of $x$; so $\eta$ and $\alpha_1$ are integral, and $\alpha_0$ even. So from this case we can get only even-pointic contact.

(2) Let $a_2=0$, $b_1 \neq 0$.

Our equations take the form

$$0=[\alpha_0]+[\alpha_1+\tau]+[2\tau+1]+[3\tau]+\ldots,$$
$$0=\qquad [\alpha_1]\quad+\quad[\tau+1]+[2\tau]+\ldots,$$

$\tau=\frac{1}{2}$, $\alpha_1=1$, $\alpha_0=\frac{3}{2}$.

There is a cusp on the envelope, the approximation being

$$4(b_1x)^3+27a_3(c_0y)^2=0;$$

so that it breaks down if $b_1$, $c_0$, or $a_3=0$.

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(3) Let $a_2 = 0$, $b_1 = 0$.

We find $a_0 = 3$; the algebraic approximation is

$$-4A_1^3A_3 + A_1^2A_2^2 + 18A_0A_1A_2A_3 - 4A_0A_2^3 - 27A_0^2A_3^2 = 0.$$

From this we find that there are two branches of the envelope each having 3-pointic contact with the curve. In some cases the branches coincide, *e.g.*, a curve enveloping its circles of curvature.

The condition for 4-pointic contact between the branches appears to be

$$b_2^2c_0 - 3a_3(d_1c_0 - c_1d_0) = 0.$$

In the important particular case of a family of straight lines, one of the branches is accurately $y = 0$, and the other is approximately $(4b_2x)^3 + 27a_3^2c_0y = 0$, which appears to reduce to $y = 0$ accurately for $b_2 = 0$.

(4) Let $c_0 = 0$. This corresponds to a double point on the original curve; the envelope has the double-point indicated by

$$4a_2A_0 = A_1^2.$$

If $d_0 = 0$ and $b_1 = 0$, one tangent of the envelope coincides with one of the curve.

(5) If the double point be a cusp,

$$(c_0 = 0 \text{ and say } d_0 = 0,\ e_0 = 0),$$

the cuspidal tangent bisects the angle between the tangents to the envelope

$$4a_2(f_0y^2 + g_0x^3) = (b_1x)^2.$$

When in addition $b_1 = 0$,

$$4a_2(f_0y^2 + g_0x^3) = (c_1y)^2$$

indicates that the envelope has a cusp with the same cuspidal tangent.

(6) If in (4) $c_1$ also is zero,

$$4a_2A_0 = A_1^2;$$

*i.e*, the two branches of the discriminant have 3-pointic contact with the branches of the curve.

In (5) if $c_1 = 0$ the envelope coincides more closely with the curve.

(7) If $c_0 = 0,\ a_2 = 0,$
we get for the envelope, supposing $a_3 \neq 0$,

$$4A_1^3 + 27a_3A_0^2 = 0,$$

*i.e.*, $A_1$ is of order $\frac{4}{3}$.

(8) If $c_0 = 0,\ d_0 = 0,\ b_1 = 0,\ a_2 = 0,$
the envelope has three branches touching the $x$-axis.

## Results.

We may now write down our results in a form independent of our choice of origin and axes.

(1) If in addition to $\phi = 0,\ \phi_t = 0,$

$$\begin{vmatrix} \phi_x, & \phi_y \\ \phi_{xt}, & \phi_{yt} \end{vmatrix} = 0, \quad - \quad - \quad \text{(i)}$$

we have 4-pointic contact.

(2) If $\phi_{tt} = 0$ (ii), the envelope has a cusp.

(3) If conditions (i) and (ii) hold, the envelope has two branches which have each 3-pointic contact with the curve.

(4) To a double point $\phi_x = 0,\ \phi_y = 0$ - - - (iii)
corresponds a double-point.

If, in addition to (iii),

$$\begin{vmatrix} \phi_{xx}, & 2\phi_{xy}, & \phi_{yy} \\ \phi_{tx}, & \phi_{ty}, & \\ & \phi_{tx}, & \phi_{ty} \end{vmatrix} = 0, \quad - \quad - \quad \text{(iv)}$$

one tangent is common.

(5) The tangent at a cusp,

$$\text{(iii) with} \quad \begin{vmatrix} \phi_{xx}, & \phi_{xy} \\ \phi_{xy}, & \phi_{yy} \end{vmatrix} = 0 \quad - \quad \text{(v)},$$

bisects the angle between the tangents at the double point on the envelope.

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If in addition (iv) holds,

$$i.e., \text{ (iii) and } \left\| \begin{matrix} \phi_{xx}, & \phi_{xy} \\ \phi_{xy}, & \phi_{yy} \\ \phi_{tx}, & \phi_{ty} \end{matrix} \right\| = 0 \quad - \quad - \quad \text{(vi)},$$

the envelope has a cusp with the same cuspidal tangent.

(6) If in addition to (iii)

$$\phi_{tx} = 0, \ \phi_{ty} = 0, \quad - \quad - \quad - \quad \text{(vii)}$$

(*i.e.*, (i) and (iii)) the branches of the discriminant have 3-pointic contact with those of the curve.

(7) If (ii) and (iii) hold, the envelope has a singularity of the form $\eta^3 = \lambda\xi^4$, where $\eta = 0$ is the tangent to $\phi_t = 0$.

(8) But if this tangent should coincide with one of the two tangents to the curve at the double-point, *i.e.*, (iv), the form is $\eta = \lambda\xi^2$ thrice.

## A Proof of the Theorem that the Arithmetic Mean of $n$ positive quantities is not less than their Harmonic Mean.

By W. A. Lindsay, M.A., B.Sc.

## Two Theorems on the factors of $2^p - 1$.

By George D. Valentine, M.A.

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