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GENERATORS FOR THE BOUNDED AUTOMORPHISMS OF INFINITE-RANK FREE NILPOTENT GROUPS

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To B.H. Neumann on his eightieth birthday

It is shown that the natural generalisations of the elementary Nielsen transformations of a free group to the infinite-rank case, furnish generators for the subgroup of "bounded" automorphisms of any relatively free nilpotent group of infinite rank. This settles the nilpotent analogue of a question of D. Solitar concerning the "bounded" automorphisms of absolutely free groups of infinite rank.

1. INTRODUCTION

In the search for a simple set of generators of the automorphism group of a free group F on an infinite set $\{x_i \mid i \in I\}$ of free generators, the following natural generalisations of the elementary Nielsen transformations (see for example [**7**]) suggest themselves:

(i) automorphisms permuting the x_i;
(ii) automorphisms inverting any subset of the x_i's and leaving the remainder fixed;
(iii) automorphisms of the following form: given any partition $I = I_1 \coprod I_2$, set $x_{i_1} \mapsto x_{i_1}x_{i_2}$ for each $i_1 \in I_1$ and any choice of $i_2 \in I_2$; and $x_{i_2} \mapsto x_{i_2}$ for all $i_2 \in I_2$. (Thus each x_{i_1} is post-multiplied by some x_{i_2}, where i_2 may vary with i_1.)

It will be convenient to consider also automorphisms of the following type, though generated by those of types (ii) and (iii):

(iv) as in (iii) except that now pre- as well as post-multiplication is permitted, and by any $x_{i_2}^{-1}$, $i_2 \in I_2$, as well as x_{i_2}; thus $x_{i_1} \mapsto x_{i_1}x_{i_2}^{\pm 1}$ or $x_{i_2}^{\pm 1}x_{i_1}$.

We call automorphisms of these four types *generalised elementary Nielsen transformations* (briefly, *GENTs*).

However, the automorphisms of types (i), (ii), (iii) (and so also (iv)) do not generate the full automorphism group of F, since clearly any automorphism φ which is a finite

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product of these will be *(freely) bounded* in the sense that the lengths of the words $x_i\varphi$ and $x_i\varphi^{-1}$ are bounded: there is an N such that $|x_i\varphi|$, $|x_i\varphi^{-1}| \leqslant N$ for all $i \in I$. Since the bounded automorphisms form a subgroup of $\operatorname{Aut} F$, it is natural to ask the following

Question. (Solitar) Is the group of all automorphisms of F that are bounded relative to the given basis $\{x_i \mid i \in I\}$, generated by the GENTs?

R. Cohen [4] has shown that if φ is bounded by $N \leqslant 3$, then φ is a product of finitely many GENTs. The question remaining unsettled for larger N, it becomes natural to consider its analogue for relatively free groups F/V all of whose automorphisms are induced by automorphisms of F. This is known to be the case when F/V is nilpotent (see [1]), and it is in this context that we give an affirmative answer.

To formulate our result a few preliminaries are needed. With F, as above, absolutely free with infinite free basis $\{x_i \mid i \in I\}$, let V be a characteristic subgroup of F containing $\gamma_{c+1}(F)$, the $(c+1)$st term of the lower central series of F. (By [2] V must then in fact be fully invariant in F.) It is well known (see for example [5]) that every element of F/V may be written in the form

$$\left(x_{i_1}^{m_1} x_{i_2}^{m_2} \ldots x_{i_k}^{m_k} c_{j_1}^{n_1} c_{j_2}^{n_2} \ldots c_{j_l}^{n_l}\right) V, \tag{1}$$

where $k, l \geqslant 0$, the m_s are non-zero integers, the n_t are positive integers, $i_1, i_2, \ldots, i_k$ are distinct elements of I, and the c_{j_t} are distinct left-normed commutators in the x_i, whose weights satisfy $\operatorname{wt} c_{j_1} \leqslant \operatorname{wt} c_{j_2} \leqslant \ldots \leqslant \operatorname{wt} c_{j_l} \leqslant c$. We shall say that an automorphism θ of F/V is *(nilpotently) bounded* if there is an integer $N = N(\theta)$ such that for every $i \in I$ there are expressions for $(x_i V)\theta$ and $(x_i V)\theta^{-1}$ of the form (1) satisfying

$$|m_1| + \ldots + |m_k| + n_1 + \ldots + n_l \leqslant N.$$

Our main result is as follows:

THEOREM 1.1. *Every bounded automorphism θ of F/V is induced by an automorphism of F which is a finite product of GENTs of type (iv).*

We shall in the sequel use the word *genetic* to refer to automorphisms of F expressible as finite products of GENTs of type (iv).

Remark 1. It is not difficult to see that every bounded automorphism of F induces a bounded automorphism of F/V. On the other hand by our theorem every bounded automorphism of F/V lifts to a genetic automorphism of F. Hence we draw the following conclusion, which may be of interest in connection with Solitar's question:

COROLLARY 1.2. *Let φ be any bounded automorphism of the infinite-rank free group F. Corresponding to each characteristic subgroup V of F such that F/V is nilpotent, there are automorphisms ψ_V and η_V of F such that ψ_V is genetic, η_V induces the identity automorphism of F/V, and $\varphi = \psi_V\eta_V$.*

Remark 2. The case where F/V is free abelian of countable rank has been established by Macedońska-Nosalska [6]. Our proof in the abelian situation (more general than in [6] since we allow finite exponent and make no countability requirement of I) is obtained by modifying mildly an argument of Swan appearing in [3].

Remark 3. Automorphisms of F resembling those of type (iv) except that we allow $x_{i_1} \mapsto x_{i_1}x_{i_2}^n$ or $x_{i_2}^n x_{i_1}$, that is pre- or post-multiplication of the x_{i_1}, $i_1 \in I_1$, by arbitrary powers of the x_{i_2}, are clearly unbounded precisely if the exponents n are unbounded. A more complex, though standard, example of an unbounded automorphism of F (or F/V) is the following one:

$$x_1 \mapsto x_1,\ x_2 \mapsto x_1x_2,\ x_3 \mapsto x_2x_3,\ \ldots.$$

We do not consider here the question as to how the automorphisms of types (i), (ii), (iii) (and (iv)) might be supplemented to obtain a full generating set for $\mathrm{Aut}(F/V)$ (although in the case F/V abelian, Theorem 2.2 below provides one answer).

2. THE ABELIAN CASE

Let F be as above, free on the infinite set $\{x_i \mid i \in I\}$, and write $A = F/[F,F]$, the free abelianisation of F, and A_m for F/V where here V is generated by $[F,F]$ together with all mth powers of elements of F ($m > 1$); thus A_m is the free abelian group of exponent m and rank $|I|$. Our aim is to prove the abelian case of Theorem 1.1, namely:

THEOREM 2.1. (Compare Macedońska-Nosalska [6].) *Every bounded automorphism of A or A_m is induced by an automorphism of F which is a finite product of GENTs of type (iv), that is, genetic.*

(Note that in this, the abelian situation, the distinction between pre- and post-multiplication becomes immaterial.)

This theorem is a direct consequence of a result essentially due to Swan. To formulate this result we need the following concept. Let $\{y_i \mid i \in I\}$ be a free basis for A (or A_m), and for $S \subseteq I$ denote by $\langle S\rangle$ the subgroup generated by the y_i, $i \in S$. We shall say that an automorphism θ of A (or A_m) is 2×2 *block-unitriangular* relative to the given basis if there is a partition $I = I_1 \coprod I_2$ such that $|I_1| = |I| = |I_2|$, and with respect to the direct decomposition

$$A(\text{or } A_m) = \langle I_1\rangle \oplus \langle I_2\rangle,$$

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θ has the block matrix form

$$\theta = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}, \tag{3}$$

where the 1's denote the identity maps on $\langle I_1 \rangle$ and $\langle I_2 \rangle$. (We are assuming the convention that the vectors on which θ acts are written as row-vectors to the left of the matrix (3).)

THEOREM 2.2. (Swan; see [**3**, Section 2]) *Every (bounded) automorphism* φ *of* A *(or* A_m*) is a product of* $\leqslant 22$ *(bounded)* 2×2 *block-unitriangular automorphisms all relative to a fixed basis* $\{y_i \mid i \in I\}$.

Theorem 2.1 follows readily from this since an automorphism of the form (3) where the column-sums of the absolute values of the entries of the (infinite) matrix g are bounded, by N say, is the product of $\leqslant N$ GENTs of type (iv). (Note incidentally that in the more general situation where the columns of g have bounded numbers of non-zero entries, θ is a finite product of automorphisms of the type mentioned at the beginning of Remark 3 above.)

PROOF OF THEOREM 2.2: This largely follows Swan's proof in [**3**]. We give the proof for A only, in the "bounded" situation. The proofs for A_m, and of the "unbounded" versions, differ insignificantly from this one.

(a) Consider first the case that our bounded automorphism φ fixes all y_{i_1} with i_1 in some subset $I_1 \subseteq I$ of the same cardinality as I. Clearly, we may suppose that $I_2 = I - I_1$ also has cardinality $|I|$, by reducing I_1 appropriately, if necessary. Then

$$A = \langle I_1 \rangle \oplus \langle I_2 \rangle,$$

and relative to this decomposition we may write φ in the block-matrix form

$$\varphi = \begin{pmatrix} 1 & 0 \\ h & k \end{pmatrix},$$

with the understanding, as always, that elements of A are to be written as row-vectors to the left of such matrices. Since φ is bounded relative to the given basis, the maps h and k, regarded as infinite matrices relative to the bases $\{y_{i_1} \mid i_1 \in I_1\}$ and $\{y_{i_2} \mid i_2 \in I_2\}$, are *column-bounded* in the sense, already noted, that the column-sums of the absolute values of their entries are bounded. Similarly, since

$$\varphi^{-1} = \begin{pmatrix} 1 & 0 \\ -k^{-1}h & k^{-1} \end{pmatrix},$$

the same holds for k^{-1} (and $k^{-1}h$). Since

$$\varphi = \begin{pmatrix} 1 & 0 \\ h & k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k^{-1}h & 1 \end{pmatrix}, \tag{4}$$

and the last matrix in this equation has the form (3) and is bounded, it suffices to show that bounded automorphisms of the form

$$\psi = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$$

are finite products of bounded automorphisms of the form (3).

For such automorphisms the result now follows by using "tricks of Whitehead and Eilenberg" ([**3**]): Partition I_1 into countably many subsets $E_1, E_2, \dots$, each of cardinality $|I|$. Relative to the direct decomposition

$$A = \langle I_2 \rangle \oplus \langle E_1 \rangle \oplus \langle E_2 \rangle \oplus \dots,$$

we may write the automorphism ψ as

$$k \oplus 1 \oplus 1 \oplus \dots = (k \oplus k^{-1} \oplus k \oplus \dots) \circ (1 \oplus k \oplus k^{-1} \oplus k \oplus \dots),$$

where the actions of the various k's on the corresponding $\langle E_j \rangle$'s are similar to the action of k on $\langle I_2 \rangle$. The desired conclusion then follows from

$$k \oplus k^{-1} = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} = \begin{pmatrix} 1 & k-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & k^{-1}-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}.$$

(It follows that ψ is expressible as a product of $\leqslant 8$ (bounded) automorphisms of the form (3), so that by (4) φ is a product of $\leqslant 9$ such automorphisms.)

(b) Now let φ be an arbitrary bounded automorphism of A relative to the basis $\{y_i \mid i \in I\}$. Well-order I so that it is order-isomorphic to the least ordinal of cardinality $|I|$. Analogously to [**3**, Section 2] we write I as the union of an ascending chain

$$\phi = I_0 \subseteq J_0 \subseteq I_1 \subseteq J_1 \subseteq \dots$$

of initial segments, defined by transfinite induction as follows: Assuming I_{r-1} defined, choose J_{r-1} to be the smallest initial segment of I such that

$$\langle I_{r-1} \rangle \cup (\langle I_{r-1} \rangle \varphi) \subseteq \langle J_{r-1} \rangle,$$

and consider

$$\varphi\colon \langle I_{r-1} \rangle \oplus \langle I - I_{r-1} \rangle \to \langle J_{r-1} \rangle \oplus \langle I - J_{r-1} \rangle.$$

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Let i_r be the least element of $I - J_{r-1}$, and write

$$\varphi^{-1}(y_{i_r}) = b_r + a_r, \quad \text{where } b_r \in \langle I_{r-1} \rangle,\ a_r \in \langle I - I_{r-1} \rangle. \tag{5}$$

Then

$$\varphi(a_r) = c_r + y_{i_r}, \quad \text{where } c_r = \varphi(-b_r) \in \langle J_{r-1} \rangle, \tag{6}$$

and since $c_r + y_{i_r}$ is a member of a free basis for A which includes the y_j with $j \in J_{r-1}$, it follows that a_r is a member of a free basis for A including the y_i with $i \in I_{r-1}$. Writing $a_r = \sum \alpha_t y_{i_t}$, define I_r to be the shortest initial segment of the well-ordered set I, containing i_r (and therefore J_{r-1}), all those i_t for which $\alpha_t \neq 0$, and the least element k_r exceeding both i_r and these i_t. Then a_r is a member of a free basis for $\langle I_r - (I_{r-1} \cup \{k_r\}) \rangle$.

This defines I_r when r is not a limit ordinal. If r is a limit ordinal, set $I_r = \bigcup_{j<r} I_j$.

Writing $H_r = \langle I_r - I_{r-1} \rangle$ for non-limit ordinals $r \geqslant 1$, we have $A = \bigoplus_r H_r$ where the sum is over all non-limit ordinals $0 < r < |I|$, and where for each such r, the element a_r defined as above is a member of a free basis for $\langle I_r - (I_{r-1} \cup \{k_r\}) \rangle = H'_r$ say. Hence for each such r we have $H_r = H'_r \oplus \langle y_{k_r} \rangle$ with a_r a free generator of H'_r, so that there is an automorphism μ_r of H_r of the form $\mu'_r \oplus 1$ relative to this decomposition, satisfying $y_{i_r}\mu_r = y_{i_r}\mu'_r = a_r$ (and $y_{k_r}\mu_r = y_{k_r}$). Since $\sum |\alpha_t|$ is bounded (over all r) the μ_r may moreover be chosen so that the automorphism $\mu = \bigoplus_r \mu_r$ is bounded. Thus with respect to the direct decomposition $A = \langle I - K \rangle \oplus \langle K \rangle$, where K is the set of all k_r, the automorphism μ of A has the form $\mu' \oplus 1$, and so by the conclusion of part (a) is a product of $\leqslant 4$ bounded automorphisms of the form (3).

Consider next the map ν given by

$$y_i \nu = \begin{cases} y_i & \text{for } i \neq \text{any } i_r, \\ c_r + y_{i_r} & \text{for } i = i_r. \end{cases}$$

Since $c_r \in \langle J_{r-1} \rangle$, while $i_r > j$ for all $j \in J_{r-1}$, this defines an automorphism ν of A, bounded in view of (5) and (6). Since ν fixes the y_{k_r}, it fixes $|I|$ elements of the basis $\{y_i \mid i \in I\}$, and so by part (a) of the proof ν is a product of $\leqslant 9$ bounded 2×2 block-unitriangular automorphisms relative to that basis. Since $y_{i_r}\mu\varphi\nu^{-1} = y_{i_r}$, the same is true of $\mu\varphi\nu^{-1}$. Hence φ is a product of $\leqslant 9 + 9 + 4 = 22$ such automorphisms, as claimed. ∎

3. The nilpotent case

Having disposed of the case $c = 1$ of Theorem 1.1, we proceed to the proof of the full result, using induction on c, and basing our argument essentially on Section 3 of [1].

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Thus suppose $c > 1$ and that the conclusion of Theorem 1.1 holds for smaller classes. Let θ' be the automorphism induced by θ on $F/\gamma_c(F)V$. By the inductive hypothesis θ' can be lifted to a genetic automorphism μ_1 of F. Denoting by θ_1 the automorphism of F/V induced by μ_1, we consider the automorphism $\theta_2 = \theta\theta_1^{-1}$ of F/V. Since μ_1 is certainly (freely) bounded, θ_1 is bounded, and therefore so is θ_2. (It is easy to show, by induction on the class, that the product of two bounded automorphisms is again bounded.) Since θ_1 lifts to a genetic automorphism of F, it suffices to show that θ_2 can be so lifted. Since θ_2 induces the identity map on $F/\gamma_c(F)V$, the image under θ_2 of each free generator x_iV of F/V may be put in the form

$$(x_iV)\theta_2 = (x_ic_{i,1}c_{i,2}\dots c_{i,N})V, \tag{7}$$

where each $c_{i,\lambda}$, $\lambda = 1, 2, \dots, N$, is either trivial or a left-normed commutator of weight c in the x_j, $j \in I$, and N is the bound on θ_2. Since $\theta_2 = \theta_2^{(1)}\theta_2^{(2)}\dots\theta_2^{(N)}$, where $\theta_2^{(\lambda)}$ is defined by

$$(x_iV)\theta_2^{(\lambda)} = (x_ic_{i,\lambda})V, \; i \in I,$$

it suffices to consider automorphisms φ of F/V of the form

$$(x_iV)\varphi = (x_i[x_{i_1}, x_{i_2}, \dots, x_{i_c}])V, \; i \in I. \tag{8}$$

Before proceeding further with the general inductive step, we need to consider separately, as in [1, Section 3], the case $c = 2$. By the above we may in this case suppose

$$(x_iV)\varphi = (x_i[x_{i_1}, x_{i_2}])V, \; i \in I, \tag{9}$$

where now we are assuming $V \geqslant \gamma_3(F)$.

Consider any fixed i in (9) for which $i_1 \neq i_2$. If $i_1 = i$, then $x_i[x_{i_1}, x_{i_2}] = x_i[x_i, x_{i_2}] = x_{i_2}^{-1}x_ix_{i_2}$, while if $i_2 = i$, then

$$x_i[x_{i_1}, x_{i_2}]V = x_i[x_{i_1}, x_i]V = x_i[x_i, x_{i_1}^{-1}]V = \left(x_{i_1}x_ix_{i_1}^{-1}\right)V.$$

Hence if either $i_1 = i$ or $i_2 = i$, then the automorphism η_1 of F/V induced by the automorphism η of F conjugating x_i by $x_{i_2}^{-1}$ or by x_{i_1}, as the case may be, and leaving fixed all x_j, $j \neq i$, satisfies:

$$(x_iV)\varphi\eta_1 = x_iV; \quad (x_jV)\varphi\eta_1 = (x_jV)\varphi, \; j \neq i. \tag{10}$$

If $i_1 \neq i \neq i_2$, let η be the automorphism of F sending x_i to $x_i[x_{i_1}, x_{i_2}]^{-1}$ and fixing all x_j, $j \neq i$. The automorphism η_1 of F/V induced by this η will again satisfy (10).

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Since in every case η is a product of $\leqslant 8$ elementary Nielsen transformations of F, we see that individual $(x_iV)\varphi$ of the form (9) may be transformed to x_iV by applying a free automorphism which is a product of $\leqslant 8$ elementary Nielsen transformations.

The question remains as to how such transformations can be effected simultaneously for all $(x_iV)\varphi$ by means of a single genetic automorphism of F. To answer this question we reformulate it in terms of graph theory. Consider the directed graph Γ whose vertices are just the elements $(x_iV)\varphi$, $i \in I$, and whose directed edges are defined as follows: For $v_i = (x_iV)\varphi$ as in (9) where $i_1 \neq i_2$, introduce an edge leading from v_i to v_{i_1} if $i \neq i_1$, and also from v_i to v_{i_2} if $i \neq i_2$. Now in transforming $(x_iV)\varphi$ to x_iV in the manner described above, one executes a succession of (at most 4) pre- and/or post-multiplications of $v_i = (x_iV)\varphi$ by $v_{i_1}^{\pm 1}$ and/or $v_{i_2}^{\pm 1}$ (both if $i_1 \neq i \neq i_2$). (This is the effect of following φ by η_1 in (10).) We encode this operation in Γ by colouring the $\leqslant 2$ edges beginning at v_i with a single colour. Now in using v_{i_1} and/or v_{i_2} to reduce v_i we are prohibited from simultaneously carrying out an analogous reduction of v_{i_1} and/or v_{i_2}, but this is the only restriction on carrying out a set of such reducing operations simultaneously. In terms of Γ this translates into the following criterion: A partial edge-colouring of Γ by a single colour (say red) corresponds to the application of an automorphism of F/V induced by a genetic automorphism of F which on certain v_i acts like η_1 (see (10)), precisely if the $\leqslant 2$ edges out of these v_i are coloured red, and no others, and no directed path of length 2 is coloured red. It follows that all $(x_iV)\varphi \neq x_iV$ can be reduced by means of a finite product of such genetic automorphisms if and only if Γ can be edge-coloured with finitely many colours in such a way that the $\leqslant 2$ edges beginning at each vertex receive the same colour, and no directed path of length 2 is monochromatic.

That Γ can be so coloured is the case $c = 2$ of the following simple combinatorial lemma, whose proof we relegate to the end of the paper.

LEMMA 3.1. *Let Γ be a (possibly infinite) directed graph without loops each of whose vertices is the initial vertex of $\leqslant c$ edges. Then the edges of Γ can be coloured with $\leqslant 3c$ colours in such a way that no directed path of length 2 is monochromatic, and furthermore so that all edges with a common initial vertex are coloured with the same colour.*

We now sketch the inductive step in the proof of Theorem 1.1, from $c - 1 (\geqslant 2)$ to c. We shall need the following lemma (valid for all $c \geqslant 2$).

LEMMA 3.2. *Let F, free on $\{x_i \mid i \in I\}$, I infinite, and $V \geqslant \gamma_{c+1}(F)$ be as before, and let $p, p_1, \ldots, p_c$ be (not necessarily distinct) elements of I. The automorphism π of F/V defined by:*

$$(x_pV)\pi = (x_p[x_{p_1}, x_{p_2}, \ldots, x_{p_c}])V; \quad (x_iV)\pi = x_iV,\ i \neq p,$$

can be lifted to an automorphism of F which is a product of $< 8^c$ GENTs of type (iv) each of which fixes those of $x_{p_1}, \ldots, x_{p_c}$ different from x_p.

Remark. Note for later use that, assuming this lemma true, then given any infinite subset $S \subset I$ containing $p, p_1, \ldots, p_c$, π may in fact be lifted to an automorphism of F of the form $f * 1$ relative to the free decomposition $F = \langle S \rangle * \langle I - S \rangle$, where f is an automorphism of the free group $\langle S \rangle$ which is a finite product of GENTs of type (iv) as in the lemma.

PROOF OF LEMMA 3.1: We adapt the basic idea of [1, Section 3]. The case $c = 2$ having been dealt with above, we suppose $c > 2$ and that the statement of the lemma is true with $c - 1$ in place of c. We may assume also that $p_1 \neq p_2$ since otherwise the lemma is trivial. We define a sequence of elements $q_1, q_2, \ldots$ of I, and a sequence of automorphisms π_1, ξ_1, π_2, $\xi_2, \ldots$ of F, as follows: Partition $I - \{p, p_1, \ldots, p_c\}$ into countably many infinite subsets $Q_1, Q_2, \ldots$, write $P_1 = Q_1 \cup \{p, p_1, \ldots, p_c\}$, and choose $q_1 \in Q_1$. By the inductive hypothesis the automorphism of $F/\gamma_c(F)V$ defined by:

$$x_p(\gamma_c(F)V) \mapsto x_p[x_{q_1}, x_{p_3}, \ldots, x_{p_c}]\gamma_c(F)V; \quad x_i(\gamma_c(F)V) \text{ fixed for } i \neq p, \tag{11}$$

can be lifted to an automorphism π_1 of F of the form $\pi_1' * 1$ relative to the free decomposition $F = \langle P_1 \rangle * \langle I - P_1 \rangle$ (see the above remark) such that π_1' is a product of $< 8^{c-1}$ GENTs of type (iv) each fixing those of $x_{p_1}, \ldots, x_{p_c}$ different from x_p. Define ξ_1 by:

$$x_{q_1}\xi_1 = x_{q_1}[x_{p_1}, x_{p_2}]; \quad x_i\xi = x_i,\ i \neq q_1. \tag{12}$$

A direct calculation shows that, working now modulo V, the automorphism $\pi_1\xi_1\pi_1^{-1}\xi_1^{-1} = [\pi_1^{-1}, \xi_1^{-1}]$ induces the automorphism of F/V given by:

$$\begin{aligned} &x_pV \mapsto x_p[x_{p_1}, x_{p_2}, \ldots, x_{p_c}]V; \\ &x_{q_1}V \mapsto \begin{cases} x_{q_1}V, & \text{if } p_1 \neq p \neq p_2, \\ x_{q_1}[x_{p_3}, x_{q_1}, x_{p_4}, \ldots, x_{p_c}, x_{p_2}]V, & \text{if } p_1 = p, \\ x_{q_1}[x_{q_1}, x_{p_3}, x_{p_4}, \ldots, x_{p_c}, x_{p_1}]V, & \text{if } p_2 = p; \end{cases} \\ &x_iV \mapsto x_iV, \qquad \text{if } i \neq p,\ q_1. \end{aligned} \tag{13}$$

If $p_1 \neq p \neq p_2$, then $[\pi_1^{-1}, \xi_1^{-1}]$ induces π and we have the desired conclusion. Otherwise define $P_2 = (Q_2 \cup \{q_1, p_1, \ldots, p_c\}) - \{p\}$, choose $q_2 \in Q_2$, and let π_2 be an automorphism of F (guaranteed by the inductive hypothesis) of the form $\pi_2' * 1$ relative to the free decomposition $\langle P_2 \rangle * \langle I - P_2 \rangle$ such that π_2' is a product of $< 8^{c-1}$ GENTs of type (iv) each fixing those of $x_{p_1}, \ldots, x_{p_c}$ different from x_p, which induces the automorphism of $F/\gamma_c(F)V$ given by (compare (11)):

$$(14)\qquad \begin{aligned} &x_{q_1}(\gamma_c(F)V) \mapsto \begin{cases} x_{q_1}[x_{q_2}, x_{p_4}, \dots, x_{p_c}, x_{p_2}]^{-1}\gamma_c(F)V, & \text{if } p_1 = p, \\ x_{q_1}[x_{q_2}, x_{p_4}, \dots, x_{p_c}, x_{p_1}]^{-1}\gamma_c(F)V, & \text{if } p_2 = p; \end{cases} \\ &x_i(\gamma_c(F)V) \mapsto x_i(\gamma_c(F)V) \qquad \text{for } i \neq q_1. \end{aligned}$$

Define ξ_2 by (compare (12)):

$$(15)\qquad \begin{aligned} &x_{q_2}\xi_2 = \begin{cases} x_{q_2}[x_{p_3}, x_{q_1}], & \text{if } p_1 = p, \\ x_{q_2}[x_{q_1}, x_{p_3}], & \text{if } p_2 = p; \end{cases} \\ &x_i\xi_2 = x_i \qquad \text{for } i \neq q_2. \end{aligned}$$

Then the free automorphism $[\pi_2^{-1}, \xi_2^{-1}]$ induces the automorphism of F/V defined by (compare (13)):

$$(16)\qquad \begin{aligned} &x_{q_1}V \mapsto \begin{cases} x_{q_1}[x_{p_3}, x_{q_1}, x_{p_4}, \dots, x_{p_c}, x_{p_2}]^{-1}V, & \text{if } p_1 = p, \\ x_{q_1}[x_{q_1}, x_{p_3}, x_{p_4}, \dots, x_{p_c}, x_{p_1}]^{-1}V, & \text{if } p_2 = p; \end{cases} \\ &x_{q_2}V \mapsto \begin{cases} x_{q_2}[x_{q_2}, x_{p_4}, \dots, x_{p_c}, x_{p_2}, x_{p_3}]^{-1}V & \text{if } p_1 = p, \\ x_{q_2}[x_{q_2}, x_{p_4}, \dots, x_{p_c}, x_{p_1}, x_{p_3}]V, & \text{if } p_2 = p; \end{cases} \\ &x_iV \mapsto x_iV \qquad \text{for } i \neq q_1, q_2. \end{aligned}$$

From this and (13) we see that the product $[\pi_1^{-1}, \xi_1^{-1}][\pi_2^{-1}, \xi_2^{-1}]$ induces the automorphism of F/V defined by (compare (13)):

$$\begin{aligned} &x_pV \mapsto x_p[x_{p_1}, x_{p_2}, \dots, x_{p_c}]V; \\ &x_{q_2}V \mapsto \begin{cases} x_{q_2}[x_{q_2}, x_{p_4}, \dots, x_{p_c}, x_{p_2}, x_{p_3}]^{-1}V, & \text{if } p_1 = p, \\ x_{q_2}[x_{q_2}, x_{p_4}, \dots, x_{p_c}, x_{p_1}, x_{p_3}]V, & \text{if } p_2 = p; \end{cases} \\ &x_iV \mapsto x_iV \qquad \text{for } i \neq p, q_2. \end{aligned}$$

One continues (inductively) in this way, choosing $q_n \in Q_n$, writing $P_n = (Q_n \cup \{q_{n-1}, p_1, \dots, p_c\}) - \{p\}$, and using the inductive hypothesis to obtain π_n, in the form $\pi_n' * 1$ relative to the free decomposition $F = \langle P_n \rangle * \langle I - P_n \rangle$, as a product of $< 8^{c-1}$ GENTs of type (iv) each involving only the x_i with $i \in P_n$ and fixing those of $x_{p_1}, \dots, x_{p_c}$ different from x_p, and inducing an automorphism of $F/\gamma_c(F)V$ moving only $x_{q_{n-1}}(\gamma_c(F)V)$ (analogously to (14)). The automorphism ξ_n is defined, analogously to (15), to move only x_{q_n}, by post-multiplying it by a commutator $[x_{i_n}, x_{j_n}]$ where one of i_n, j_n is q_{n-1} and the other is one of $p_1, \dots, p_c$. Since $q_n \notin \{p_1, \dots, p_c\}$ and the subsets of $\{x_i \mid i \in I\}$ involved in the GENTs composing the automorphisms

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π_n, π_{n+k}, $k \geqslant 2$, intersect in a subset of $\{p_1, \dots, p_c\} - \{p\}$ (contained as they are in P_n, P_{n+k}), the automorphisms $\pi_1, \xi_1, \pi_2, \xi_2, \dots$ ultimately obtained have the following properties:

(α) the automorphism π is induced by the infinite product $\prod_{n=1}^{\infty} [\pi_n^{-1}, \xi_n^{-1}]$ (which makes sense since each x_i is fixed by almost every π_n, ξ_n);

(β) the commutators $[\pi_n^{-1}, \xi_n^{-1}]$ induce pairwise commuting automorphisms of F/V;

(γ) for every $n = 1, 2, \dots$, and every $k \geqslant 2$, the automorphisms π_n, ξ_n both commute with π_{n+k}, ξ_{n+k};

(δ) each of the infinite products

$$\pi_1^{\pm 1} \pi_3^{\pm 1} \pi_5^{\pm 1} \dots, \quad \pi_2^{\pm 1} \pi_4^{\pm 1} \pi_6^{\pm 1} \dots$$

is expressible as a product of $< 8^{c-1}$ GENTs of type (iv), and each of

$$\xi_1^{\pm 1} \xi_3^{\pm 1} \xi_5^{\pm 1} \dots, \quad \xi_2^{\pm 1} \xi_4^{\pm 1} \xi_6^{\pm 1} \dots$$

is expressible as a product of $\leqslant 4$ GENTs of type (iv).

Writing $\widehat{\pi}_n$, $\widehat{\xi}_n$ for the induced automorphisms of F/V, we obtain from (α), (β) and (γ) that

$$\pi = [\widehat{\pi}_1^{-1} \widehat{\pi}_3^{-1} \widehat{\pi}_5^{-1} \dots, \widehat{\xi}_1^{-1} \widehat{\xi}_3^{-1} \widehat{\xi}_5^{-1} \dots] \circ [\widehat{\pi}_2^{-1} \widehat{\pi}_4^{-1} \widehat{\pi}_6^{-1} \dots, \widehat{\xi}_2^{-1} \widehat{\xi}_4^{-1} \widehat{\xi}_6^{-1} \dots],$$

and it then follows from (δ) that π is induced by a product of $< 4(8^{c-1} + 4) < 8^c$ GENTs of type (iv), completing the proof of the lemma. $\square$

We are now in a position to complete the proof of Theorem 1.1 in the case $c > 2$. We have already observed that by induction it suffices to consider automorphisms φ of F/V of the form (8):

$$(x_i V)\varphi = x_i [x_{i_1}, x_{i_2}, \dots, x_{i_c}] V, \; i \in I.$$

Let $I = I_1 \coprod I_2$ where $|I_1| = |I| = |I_2|$, and define φ_1 to agree with φ on $\langle I_1 \rangle$ and fix the elements of $\langle I_2 \rangle$, and vice versa for φ_2. Then $\varphi = \varphi_1 \varphi_2$, so that it suffices to show that φ_1 is genetic.

Let J denote the subset of indices $j \in I$ for which $(x_j V)\varphi_1 \neq x_j V$. Partition I_2 into $|J|$ infinite subsets A_j, $j \in J$, and write $B_j = A_j \cup \{j, j_1, j_2, \dots, j_c\}$. For each $j \in J$ consider the automorphism φ_j of F/V agreeing with φ_1 on $x_j V$ and fixing all $x_i V$ with $i \neq j$:

$$(x_j V)\varphi_j = x_j [x_{j_1}, x_{j_2}, \dots, x_{j_c}] V; \quad (x_i V)\varphi_j = x_i V \text{ for } i \neq j.$$

Then by Lemma 3.2 (and the remark following it) each φ_j is induced by an automorphism f_j of F of the form $f_j' * 1$ relative to the free decomposition $F = \langle B_j \rangle * \langle I - B_j \rangle$, where f_j' is expressible as a product of $< 8^c$ GENTs of type (iv) each fixing those of $x_{j_1}, \ldots, x_{j_c}$ different from x_j.

We now introduce a directed graph Γ_1 analogous to that exploited in the case $c = 2$. The vertices v_i of Γ_1 are the $(x_i V)\varphi$, $i \in I$, and each v_j, $j \in J$, is joined by an edge directed to every v_i with $i \in B_j - \{j\}$. Much as before we encode the automorphism φ_j in Γ_1 as a colouring of all the edges emanating from v_j with the same colour (red, say); a partial edge-colouring of Γ_1 with the colour red then corresponds to a genetic automorphism executing various of the φ_j simultaneously, if all edges out of each vertex are red (or else none is coloured) and no directed path of length 2 is red. Hence φ_1 is genetic if Γ_1 can be coloured with finitely many colours so that all edges out of each vertex receive the same colour and no directed path of length 2 is monochromatic. Now each vertex v_s with $s \in I_2$, has no edges leading out of it (since $x_s V$ is fixed by φ_1) and precisely one edge terminating at it (namely that from v_j to v_s if $s \in A_j$); call this edge e_s. If an edge-colouring of Γ_1 is such that all edges out of each vertex receive the same colour and there is a monochromatic (red, say) directed path of length 2 of which e_s is one of the edges, then v_j must be the midpoint of that path. Denote by e the first edge of that directed path. Since φ_1 does not fix $x_j V$ there is another edge e' out of v_j (terminating in one of $v_{j_1}, \ldots, v_{j_c}$), and by the condition on our edge-colouring this edge will also be red. Hence the pair e, e' makes up a directed red path of length 2 no vertex of which is a v_s with $s \in I_2$. The upshot is that in deciding whether or not an edge-colouring of Γ_1 of the desired sort is possible we may delete all vertices v_s with $s \in I_2$, together with the edges terminating at them. The resulting graph Γ then has $\leqslant c$ edges emanating from each vertex, and Lemma 3.1 now applies. ∎

4. Proof of Lemma 3.1

We first show, by induction on c, that Γ can be coloured with $\leqslant 3c$ colours in such a way that no directed path of length 2 is monochromatic. In the case $c = 1$ it is easy to see that each connected component of Γ is either a tree directed downwards towards its root, or else consists of a single directed closed path without repeated edges, possibly with "downwards - directed" trees attached to it by their roots. It is not difficult to see that such a graph can be coloured in the desired way using $\leqslant 3$ colours.

Suppose now that $c > 1$. For each vertex v with > 0 edges leading out of it, choose exactly one of these edges e_v, and let $\triangle$ denote the (directed) subgraph consisting of all the edges e_v together with their initial and terminal vertices. Then by the case $c = 1$, $\triangle$ can be coloured appropriately with $\leqslant 3$ colours, and by the inductive hypothesis the complementary subgraph Ω consisting of the edges outside Δ together with their initial and terminal vertices, can be coloured with $\leqslant 3(c-1)$ colours appropriately. Hence Γ itself can be coloured in the desired way using $\leqslant 3(c-1)+3=3c$ colours.

It remains to show that there is such an edge-colouring of Γ with the additional property that edges with a common initial vertex have the same colour. Perform the following operation on Γ: for each vertex v, identify with a single directed edge all edges emanating from v (and identify their terminal vertices with a single vertex), doing this simultaneously for all vertices. The resulting directed graph $\widehat{\Gamma}$ again has the property that each vertex is initial for $\leqslant c$ edges, so that, by the first part of the proof, $\widehat{\Gamma}$ can be edge-coloured with $\leqslant 3c$ colours with no directed path of length 2 monochromatic. If we now colour the edges of the original graph Γ so that the natural identification map $E(\Gamma) \to E\left(\widehat{\Gamma}\right)$ between the edge-sets of Γ and $\widehat{\Gamma}$ is colour-preserving, then this edge-colouring has both desired properties. $\square$

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