4

Kinks in quantum field theory

A particle in a classical harmonic oscillator potential, $m\omega^2 x^2/2$, has minimum energy when it sits at rest at the bottom of the potential. Then the particle's energy vanishes. The Heisenberg uncertainty principle however modifies this picture for the quantum harmonic oscillator. The particle cannot sit at rest (with definite momentum) at the bottom of the potential (a definite location). Indeed, the quantum zero point motion lifts the ground state energy to $\omega/2$. Further, the excited states of the simple harmonic oscillator are discrete and occur at energies $(n + 1/2)\omega$, $n = 0, 1, 2, \ldots$

Just as the classical harmonic oscillator is modified by quantum effects, any classical solution to a field theory is also modified by quantum effects. Quantum effects give corrections to the classical kink energy owing to zero point quantum field fluctuations. These quantum corrections are small provided the coupling constant in the model is weak. To "quantize the kink" means to evaluate all the energy levels of the kink (first quantization) and to develop a framework for doing quantum field theory in a kink background. This involves identifying all excitations in the presence of the kink and their interactions. The field theory of the excitations in the non-trivial background of the kink is akin to second quantization. Finally, one would also like to describe the creation and annihilation of kinks themselves by suitable kink creation and annihilation operators. This would be the elusive third quantization.

Initially we calculate the leading order quantum corrections to the energy of the Z_2 and sine-Gordon kinks. As these two examples illustrate, the precise value of the quantum correction depends on the exact model and kink under consideration. Yet there is one common feature – quantum corrections tend to reduce the energy of the kink. This result is quite general and we prove it using a variational argument in Section 4.5.

The quantum corrections to the kink mass are obtained by using a perturbative analysis where the coupling constant is the expansion parameter, as first done

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in [38, 42]. For fixed values of the masses of particles in the field theory, the energy of the classical solution is proportional to the coupling constant raised to a negative power (for example, see Eq. (1.20)) and so the perturbative analysis holds only if the kink is much more massive than the particles in the model. As the coupling constant is increased, quantum effects become stronger and eventually the perturbative scheme breaks down. Remarkably, the sine-Gordon model is still amenable to analysis in this regime and, at strong coupling, the sine-Gordon kinks become lighter than the particles. Indeed, there exists a weakly coupled description of the model in which perturbative methods can be used: this is the massive Thirring model in which the particles (low energy excitations of a fermionic field) correspond to the sine-Gordon kinks at strong coupling (see Section 4.7).

The phenomenon observed in the sine-Gordon and massive Thirring models, in which solitons of one model (Model 1) are identified with the particles of a second model (Model 2) and vice versa in certain regimes of the coupling constants, is known as "duality." Model 1 is said to be dual to Model 2 if the particle-plus-soliton spectrum of Model 1 maps onto the soliton-plus-particle spectrum of Model 2 and vice versa. Both models describe the same physics, except that the light and heavy degrees of freedom are interchanged.

The Z_2 model does not share the remarkable symmetries of the sine-Gordon model and less is known about the Z_2 kink at strong coupling. However, the mass of the Z_2 kink can be evaluated at strong coupling using lattice field theory. We describe these results in Section 4.8 and conclude, once again, that the kink becomes less massive as the coupling is increased and eventually becomes massless.

In this book, we only describe quantization of the mass of the kink using canonical techniques. A more extensive discussion of various other techniques and issues can be found in [35, 126] and in the series of papers in [38, 42].

4.1 Quantization of kinks: broad outline

In this section, we evaluate the contribution of the zero point fluctuations to the energy of the kink. Then we briefly discuss excited states.

The quantization procedure can be outlined as follows:

- Consider a field theory in two dimensions with compact spatial dimension of size L, assumed large compared to any other length scale in the problem. Periodic boundary conditions are imposed on the fields. Eventually take $L \to \infty$.
- Consider small quantum fluctuations, ψ, about the classical kink background, ϕ_k,

$$\phi(t, x) = \phi_k(x) + \psi(t, x) \tag{4.1}$$

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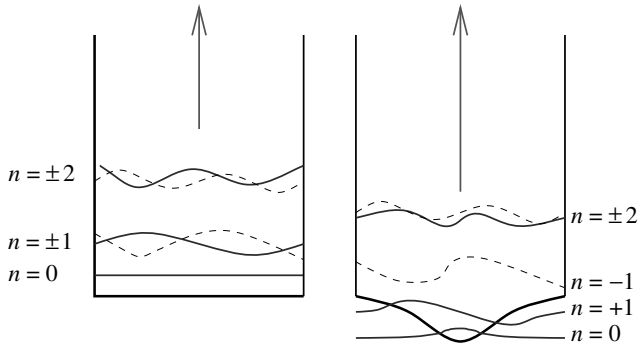

Figure 4.1 A trivial potential on a periodic space with period L is shown on the left. The field modes are labeled by an integer $n = 0, \pm 1, \pm 2, \ldots$ When there is a kink, the potential felt by the modes becomes non-trivial as depicted by the curved bottom of the figure on the right. What used to be the $n = 0$ mode in the trivial potential (on the left) becomes the lowest bound state, the zero mode, in the non-trivial potential. Similarly a linear combination of the $n = \pm 1$ modes in the trivial box may become a second bound state ($n = +1$ in this illustration) and the other states remain unbound but shift in form and energy.

Linearize the equation for ψ and then quantize, that is, regard the field ψ as a quantum operator

$$\psi(t, x) = \sum [a_{\rm k} f_{\rm k}(t, x) + a_{\rm k}^\dagger f_{\rm k}^*(t, x)] \tag{4.2}$$

where $a_{\rm k}^\dagger$ and $a_{\rm k}$ are creation and annihilation operators. The $f_{\rm k}$ are mode functions i.e. orthonormal solutions of the linearized equations of motion for ψ in the kink background. The equation satisfied by $f_{\rm k}$ is

$$\partial_t^2 f_{\rm k} - \partial_x^2 f_{\rm k} + V''(\phi_{\rm k}(x)) f_{\rm k} = 0 \tag{4.3}$$

- Find all the eigenmodes, $f_{\rm k}$, of the fluctuations and their eigenfrequencies $\omega_{\rm k}$. As shown in Fig. 4.1, in the presence of the kink the modes are displaced. Some of the low-lying modes without the kink become bound states in the presence of the kink, and the others become scattering states as $L \to \infty$.
- Each eigenmode corresponds to a quantum harmonic oscillator with zero point fluctuations. Sum up the zero point energies of all the modes to get the quantum correction to the classical kink energy, $E_{\rm cl}$,

$$\tilde{E} = E_{\rm cl} + \sum_i \frac{1}{2} \omega_i \tag{4.4}$$

In the $L \to \infty$ limit, the sum over the modes becomes a sum over bound states and an integral over scattering states. Also note that Eq. (4.4) is only valid to leading order in the quantum corrections since we have ignored interactions of the fluctuation field, ψ.

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In following this procedure, quantum field theoretic subtleties arise.

- The zero point energy of the trivial vacuum (without the kink) must be subtracted from the zero point energy of the kink since we want to define the energy of the trivial vacuum to be zero. Therefore

$$E = E_{\rm cl} + \sum_i \frac{1}{2}\omega_i - \left[E_{\rm cl,0} + \sum_i \frac{1}{2}\omega_i^{(0)}\right]$$

  where $E_{\rm cl,0}$ is the classical energy of the trivial vacuum and is chosen to vanish ($E_{\rm cl,0} = 0$), and $\omega_i^{(0)}$ are the eigenfrequencies of the modes in the trivial vacuum.
- The energy must be expressed in terms of renormalized parameters.

In the trivial vacuum, the energy eigenvalue for the mode with n nodes is

$$\omega_n^{(0)} = \sqrt{k_n^2 + m_\psi^2} \tag{4.5}$$

where $k_n = 2\pi n/L$ and $n \in \mathbf{Z}$, the set of all integers. Now suppose that the kink potential V'' is turned on slowly, i.e. that the potential term in Eq. (4.3) is multiplied by a parameter that vanishes for the free field theory and is continuously increased to one to get to the kink case. As the parameter increases, modes in the trivial box evolve into modes in the kink background. Some of the low-lying modes in the trivial box become the bound states of the kink. Let us label these modes by the index b (for "bound") and the remaining modes by c (for "continuum"). (In the example of Fig. 4.1, $b = 0$, 1, and c is any integer except for 0, 1.) Then

$$E = E_{\rm cl} + \frac{1}{2}\sum_b \left(\omega_b - \omega_b^{(0)}\right) + \frac{1}{2}\sum_c \left[\sqrt{p_c^2 + m_\psi^2} - \sqrt{k_c^2 + m_\psi^2}\right] \tag{4.6}$$

where $\omega_c \equiv \sqrt{p_c^2 + m_\psi^2}$ and m_ψ denotes the mass of the ψ particles. In the limit $L \to \infty$, the sum over continuum states becomes an integral.

The terms in Eq. (4.6) can be understood quite simply. The first term on the right is the classical kink energy, the second contains the excess quantum corrections owing to the zero point motion of the modes bound to the kink, and the third term is the excess energy in the zero point motion of the modes that are not bound to the kink. The wave numbers of the scattering modes in the background of the kink are denoted by p_ν while that of the modes in the trivial vacuum by k_n.

In the trivial vacuum and when $L \to \infty$, the scattering states are plane waves, which are both energy and momentum eigenstates with $k_n = 2\pi n/L$. In the presence of the kink, the scattering states are energy eigenstates but not momentum eigenstates and, in general, an incoming wave gets both reflected and transmitted. Without specifying the field theory, further progress is possible when the scattering potential, $V''(\phi_{\rm k}(x))$, is *reflectionless*. This may seem very restrictive, but it holds for both the Z_2 and sine-Gordon models and we assume it to be

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true for the remainder of this analysis. Then, asymptotically, the scattering states behave as

$$e^{i(px-\alpha(p)/2)} \text{ as } x = -L/2 \to -\infty \tag{4.7}$$

$$e^{i(px+\alpha(p)/2)} \text{ as } x = +L/2 \to +\infty \tag{4.8}$$

where $\alpha(p)$ is a phase shift. Note that on multiplying by $\exp(i\alpha(0)/2)$, the $p = 0$ state can be chosen to be purely real at $x = -\infty$. Since the scattering potential, $V''(\phi_k(x))$, is also real, this implies that the imaginary part of the wavefunction can be taken to be zero everywhere. Therefore $\alpha(0) = 0$.

The phase of the scattering states has a winding number given by the total phase change across the box. Since we have imposed periodic boundary conditions, the total phase winding, $(pL + \alpha(p))/2\pi$, must be an integer. This quantizes p so that

$$p_\nu L + \alpha(p_\nu) = 2\nu\pi \tag{4.9}$$

where $\nu \in \mathbf{Z}$ and we have denoted the νth wave-vector by p_ν.

Now that the scattering states in the soliton potential have been labeled by the integer ν, and those when the potential vanishes by the integer n, correspondence must be drawn between ν and n. To illustrate the problem, consider the $p_\nu = 0$ mode. As discussed above, $\alpha(0) = 0$ and hence, from Eq. (4.9), $\nu = 0$ labels this mode. Further, this mode has the lowest energy of the continuum states. In the specific example of Fig. 4.1, this mode corresponds to the $n = -1$ mode in the trivial box since the $n = 0, 1$ modes have become bound and have dropped out of the set of scattering states. Therefore $n = -1$ corresponds to $\nu = 0$, in this example.

With $k_n = 2\pi n/L$, we can write

$$p_\nu L + \Delta(p_\nu) = 2n\pi = k_n L \tag{4.10}$$

where

$$\Delta(p_\nu) \equiv \alpha(p_\nu) + 2\pi(n - \nu) \tag{4.11}$$

The shift in going from n to ν is the change in the total winding of the phase as the potential evolves from the trivial box to the soliton potential (see Fig. 4.1). As long as there is no change in the relative ordering of the energy levels, the heirarchy of the energy levels is maintained, and the mapping between n and ν is a constant shift. Since some of the low-lying states in the trivial potential have dropped out from the set of continuum states and have been converted into bound states, the set of integers n is partitioned into two subsets – one for the integers that lie above the would-be bound states and another for the integers that lie below the would-be bound states. The map from n in each subset to ν is a constant shift but the shift is different in the two subsets.

Next we think of $n - \nu$ as a function of k. In the $L \to \infty$ limit, $(n - \nu)$ is constant everywhere except at $k = 0$, meaning that the derivative of $n - \nu$ with respect to k is a Dirac delta function at $k = 0$,

$$\frac{\mathrm{d}\Delta}{\mathrm{d}k} = \text{coeff.}\delta(k) + \frac{\mathrm{d}\alpha(k)}{\mathrm{d}k} \tag{4.12}$$

To determine the coefficient of the delta function, let us denote by N_b the number of states in the trivial potential that have dropped out as bound states in the kink potential. In a large interval $n_+ - n_-$ (n_+ is positive and n_- is negative), the corresponding interval in ν is smaller by N_b, and hence the coefficient of the delta function is given by $-2\pi N_b$

$$\frac{\mathrm{d}\Delta}{\mathrm{d}k} = -N_b 2\pi\,\delta(k) + \frac{\mathrm{d}\alpha(k)}{\mathrm{d}k} \tag{4.13}$$

For large momenta (and energy) the modes are unaffected by the deformation of the potential at the bottom of the well. Hence $p_\nu \to k_n$ in this region and $\Delta(p) \to 0$ as $|p| \to \infty$.

The phase shift $\Delta(p_\nu)$ depends on the potential in the equation of motion for $\psi(t, x)$, as in Eq. (4.3). As we explain below, the scattering potential created by the soliton background is non-perturbative. Therefore the phase shifts need not be small owing to factors of the coupling constant. However, note that $\Delta(p_\nu)/L$ is small as $L \to \infty$ and we need only keep terms up to linear order in $1/L$. Therefore

$$\begin{aligned}\sqrt{p_\nu^2 + m_\psi^2} &= \sqrt{\left(k_n - \frac{\Delta(p_\nu)}{L}\right)^2 + m_\psi^2} \\ &= \sqrt{k_n^2 + m_\psi^2} - \frac{k_n \Delta(k_n)}{L\sqrt{k_n^2 + m_\psi^2}} + O\left(\frac{1}{L^2}\right)\end{aligned} \tag{4.14}$$

Note that in the last line $\Delta(p_\nu)$ has been replaced by $\Delta(k_n)$ since $p_\nu = k_n + O(1/L)$.

We now want to express the energy of the kink in terms of renormalized parameters. If we denote the renormalized mass of ψ by $m_{\psi,\mathrm{R}}$ and the bare mass by $m_{\psi,\mathrm{b}}$, then

$$m_{\psi,\mathrm{R}}^2 = m_{\psi,\mathrm{b}}^2 - \delta m_\psi^2 \tag{4.15}$$

where δm_ψ denotes the quantum contribution of vacuum fluctuations to the mass of ψ, and δm_ψ^2 is due to the self-coupling of the field and hence is proportional to the coupling constant.

The expression for the energy in Eq. (4.6) is valid to leading order in quantum corrections. The classical energy is inversely proportional to the coupling constant (e.g. Eq. (1.20)) and so the leading corrections are independent of the coupling

constant. Note that m_ψ in the last two terms in Eq. (4.6) can be freely replaced by $m_{\psi,\mathrm{R}}$ since we are only evaluating the lowest order (coupling constant independent) quantum correction to the energy and δm_ψ^2 is proportional to the coupling constant. Retaining only the terms that are of leading order in the coupling constant, expanding E_{cl} in δm_ψ^2, and using Eq. (4.14), we get

$$E = E_{\mathrm{cl}}[m_{\psi,\mathrm{R}};\lambda_{\mathrm{R}}] + \Delta E_{\mathrm{cl}} + \frac{1}{2}\sum_b \left(\omega_b - \omega_b^{(0)}\right) - \frac{1}{2L}\sum_n \frac{k_n \Delta(k_n)}{\sqrt{k_n^2 + m_{\psi,\mathrm{R}}^2}} \tag{4.16}$$

where ΔE_{cl} denotes the leading order change in E_{cl} when replacing bare parameters by renormalized parameters.

In the limit $L \to \infty$, the sum over n becomes an integral

$$\sum_n \to \frac{L}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k \tag{4.17}$$

Hence,

$$E = E_{\mathrm{cl}}[m_{\psi,\mathrm{R}};\lambda_{\mathrm{R}}] + \Delta E_{\mathrm{cl}} + \frac{1}{2}\sum_b \omega_b - \frac{N_b}{2} m_{\psi,\mathrm{R}} - \frac{1}{4\pi}\int \mathrm{d}k \frac{k\Delta(k)}{\sqrt{k^2 + m_{\psi,\mathrm{R}}^2}} \tag{4.18}$$

where we have made use of the fact that $\omega_b^{(0)} = \sqrt{k_b^2 + m_{\psi,\mathrm{R}}^2} \to m_{\psi,\mathrm{R}}$ as $k_b \propto 1/L \to 0$.

On integration by parts

$$\int \mathrm{d}k \frac{k\Delta(k)}{\sqrt{k^2 + m_{\psi,\mathrm{R}}^2}} = \left[\Delta(k)\sqrt{k^2 + m_{\psi,\mathrm{R}}^2}\right]_{-\infty}^{+\infty} - \int \mathrm{d}k \sqrt{k^2 + m_{\psi,\mathrm{R}}^2}\frac{\mathrm{d}\Delta}{\mathrm{d}k} \tag{4.19}$$

Since $\Delta(k)$ vanishes as $k \to \pm\infty$, the boundary term gives a finite contribution. The last term contains the derivative of $\Delta(k)$ and is given in Eq. (4.13). Therefore the final result is

$$\begin{aligned} E = E_{\mathrm{cl}}[m_{\psi,\mathrm{R}};\lambda_{\mathrm{R}}] + \Delta E_{\mathrm{cl}} + \frac{1}{2}\sum_b \omega_b - \frac{1}{4\pi}\left[|k|\Delta(k)\right]_{-\infty}^{+\infty} \\ + \frac{1}{4\pi}\int \mathrm{d}k \sqrt{k^2 + m_{\psi,\mathrm{R}}^2}\,\frac{\mathrm{d}\alpha}{\mathrm{d}k} \end{aligned} \tag{4.20}$$

Our general calculation can be pushed a little further since, in one spatial dimension, all divergences can be removed by normal ordering a "renormalized potential,"

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$V_{\rm R}$, which can be written in terms of the bare potential, $V(\phi)$ [35]

$$V_{\rm R} = \exp\left\{\frac{1}{8\pi}\left(\ln\frac{4\Lambda^2}{m^2}\right)\frac{{\rm d}^2}{{\rm d}\phi^2}\right\}V(\phi) + \epsilon_0 \tag{4.21}$$

where Λ is a momentum cut-off, and m is the bare mass. The constant ϵ_0 renormalizes the vacuum energy, and is chosen so that the expectation value of the Hamiltonian in the ground state vanishes. For example, in $\lambda\phi^4$ theory (Eq. (1.2)),

$$V_{\rm R} = [\gamma(3\gamma\lambda - m^2) + \epsilon_0] + \frac{1}{2}(6\gamma\lambda - m^2)\phi^2 + \frac{\lambda}{4}\phi^4 \tag{4.22}$$

where

$$\gamma \equiv \frac{1}{8\pi}\ln\left(\frac{4\Lambda^2}{m^2}\right) \stackrel{\Lambda\to\infty}{=} \frac{1}{8\pi}\int_{-\Lambda}^{+\Lambda}\frac{{\rm d}k}{\sqrt{k^2+m^2}} \tag{4.23}$$

Then, the quantum correction to the mass is $\delta m^2 = 6\lambda\gamma$, while the quantum correction to the mass of the excitations in the Z_2 model is:

$$\delta m_\psi^2 = 2\delta m^2 = 12\lambda\gamma \tag{4.24}$$

In the sine-Gordon model (Eq. (1.51))

$$V_{\rm R} = \frac{\alpha}{\beta^2}[1 - {\rm e}^{-\gamma\beta^2}\cos(\beta\phi)] + \epsilon_0 \tag{4.25}$$

and the quantum corrections to the parameters can be read off.

Returning to the general expression in Eq. (4.21), the bare parameters occurring in $V(\phi)$ can be chosen to absorb the cut-off dependent factors. Then the potential $V_{\rm R}$ is given entirely in terms of finite physical parameters. If the classical solution is found for the physical value of the coupling constant, denoted by $\lambda_{\rm R}$, then $\Delta E_{\rm cl}$ only depends on the correction to the mass term, δm_ψ^2,

$$\Delta E_{\rm cl} = \frac{E'_{\rm cl}[m_{\psi,{\rm R}};\lambda_{\rm R}]}{2m_{\psi,{\rm R}}}\delta m_\psi^2 \tag{4.26}$$

where $E'_{\rm cl}$ denotes derivative of $E_{\rm cl}$ with respect to the mass, $m_{\psi,{\rm R}}$. At this stage, we are still left with the last two terms in Eq. (4.20) involving the phase shifts. However, there is no general prescription for finding the phase shifts, and each problem has to be dealt with individually.

Equation (4.20) is our final general expression for the ground state energy of the quantized kink provided that the classical kink solution gives rise to a reflectionless potential. To make further progress one needs to find $E_{\rm cl}[m_{\psi,{\rm R}};\lambda_{\rm R}]$, $\Delta E_{\rm cl}$, ω_b, and the derivative of $\alpha(k)$. These quantities are model specific and we shall find them in the $\lambda\phi^4$ and sine-Gordon models in the next two sections.

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Before going on to some examples, it is helpful to track the coupling constant dependence of the various terms in Eq. (4.16). We write the potential as

$$V(\phi) = -\frac{m^2}{2}\phi^2 + \epsilon S(\phi) \tag{4.27}$$

where m is the mass parameter, ϵ is the small coupling constant, and S is some unspecified function of ϕ, perhaps containing other parameters. The classical energy term in Eq. (4.16) is inversely proportional to the coupling constant. So the leading order correction is independent of the coupling constant. In the second term, δm_ψ^2 is proportional to the coupling constant but $E'_{\rm cl}$ is inversely proportional to the coupling constant. Hence the product is independent of the coupling constant. Next we come to the coupling constant dependence of the energy eigenvalues and the phase shifts. The spectrum of excitations is found by solving for eigenmodes in the kink background. The kink background provides a potential with which the excitations interact. The important point here is that this potential is non-trivial even to zeroth order in the coupling constant. The vacuum expectation value of ϕ, denoted by ϕ_0, is found by minimizing V. Therefore

$$\frac{S'(\phi_0)}{\phi_0} = \frac{m^2}{\epsilon} \tag{4.28}$$

Then

$$V''(\phi_0) = -m^2 + \epsilon S''(\phi_0) \tag{4.29}$$

and approximating $S''(\phi_0)$ as $S'(\phi_0)/\phi_0$,

$$V''(\phi_0) \sim -m^2 + \epsilon \frac{S'(\phi_0)}{\phi_0} \sim m^2 \tag{4.30}$$

Hence the scattering potential in Eq. (4.3) for the mode functions is independent of the coupling constant, and the phase shifts, $\alpha(k)$, are non-trivial even to zeroth order in the coupling constant.

As we see in the specific examples given below, both δm_ψ^2 and the last sum in Eq. (4.16) are divergent. However, the divergences cancel, leading to a finite result for the energy.

4.2 Example: Z_2 kink

We now find the energy of the quantized Z_2 kink by evaluating explicitly the terms in Eq. (4.16).

The classical energy piece is already known from Eq. (1.20)

$$E_{\rm cl}[m_{\psi,\rm R}; \lambda_{\rm R}] = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda_{\rm R}} = \frac{m_{\psi,\rm R}^3}{3\lambda_{\rm R}} \tag{4.31}$$

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Then $\Delta E_{\rm cl}$ is given by Eq. (4.26) and $E'_{\rm cl}$ is found by differentiating Eq. (4.31)

$$E'_{\rm cl}[m_{\psi,\rm R};\lambda_{\rm R}] = \frac{m^2_{\psi,\rm R}}{\lambda_{\rm R}} \tag{4.32}$$

The mass correction δm^2_ψ arises owing to the interaction term $\lambda\phi^4/4$ in this model. The calculation of δm^2_ψ is quite involved since it requires renormalization in a model with spontaneous symmetry breaking, which means that we should find the mass correction from the action in Eq. (1.5). Then there are both cubic and quartic interactions. This calculation can be found in quantum field theory textbooks, for example [119]. The end result is

$$\delta m^2_\psi = \frac{3\lambda_{\rm R}}{2\pi}\int \frac{{\rm d}k}{\sqrt{k^2+m^2_{\psi,\rm R}}} \tag{4.33}$$

The integral in Eq. (4.33) is divergent. However it is only one term in the expression for the quantum kink energy in Eq. (4.18). In particular, the last term with the phase shifts is also divergent, but the quantum kink energy is finite since the two divergences cancel. Note that we can replace m_ψ by $m_{\psi,\rm R}$ in the final integral since we are only evaluating the leading order correction.

Next consider the terms in Eq. (4.16) that involve the spectrum of fluctuations about the classical kink. To find the spectrum, substitute Eq. (4.1) in the field equation, Eq. (1.4), and expand to lowest non-trivial order in ψ. This was already done in Section 3.2 and we now summarize the results

$$\begin{aligned} \omega_0 &= 0, & \chi_0 &= \operatorname{sech}^2 z \\ \omega_1 &= \frac{\sqrt{3}}{2}m_\psi, & \chi_1 &= \sinh z\,\operatorname{sech}^2 z \\ m_\psi &< \omega < \infty, & \chi_k &= {\rm e}^{{\rm i}kx}[3\tanh^2 z - 1 - w^2k^2 - {\rm i}\,3wk\tanh z] \end{aligned}$$

where $z = x/w = m_\psi x/2$, and the dispersion relation is

$$\omega^2_{\rm k} = k^2 + m^2_\psi \tag{4.34}$$

Note that the eigenvalues ω_k are independent of the coupling constant because λ does not occur in Eq. (3.10) if $m_\psi = \sqrt{2\lambda}\eta$ is held fixed. However, this statement is only true to leading order in λ because the mass parameter, the kink width, and indeed the form of the kink solution get modified owing to quantum corrections, and induce λ dependence in the spectrum. Since we are only working to leading order in quantum corrections, the mass parameter m_ψ entering Eq. (3.10) and the definition of the kink width, w, are the same as $m_{\psi,\rm R}$.

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The next step is to impose periodic boundary conditions with period $L \to \infty$ on the scattering state. For this we find the asymptotic behavior of χ_k

$$\chi_k \to \mathrm{e}^{\mathrm{i}kz}(2 - w^2k^2 \mp \mathrm{i}\, 3wk) \propto \exp\{\mathrm{i}(kz \pm \alpha(k)/2)\}, \quad z \to \pm\infty \tag{4.35}$$

from which the phase shifts follow

$$\alpha(k) = 2\tan^{-1}\left[\frac{3wk}{w^2k^2 - 2}\right] \tag{4.36}$$

Hence

$$\frac{\mathrm{d}\alpha}{\mathrm{d}k} = -6w\frac{w^2k^2 + 2}{(w^2k^2 + 1)(w^2k^2 + 4)} \tag{4.37}$$

and

$$\Delta(k) \to \frac{6}{wk}, \quad |k| \to \infty \tag{4.38}$$

Now we combine all the terms in Eq. (4.20)

$$E = \frac{m_{\psi,\mathrm{R}}^3}{3\lambda_\mathrm{R}} + \frac{\sqrt{3}}{4}m_{\psi,\mathrm{R}} - \frac{3}{2\pi}m_{\psi,\mathrm{R}} - \frac{3m_{\psi,\mathrm{R}}^3}{16\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{\sqrt{k^2 + m_{\psi,\mathrm{R}}^2}}\frac{1}{k^2 + m_{\psi,\mathrm{R}}^2/4} \tag{4.39}$$

The last integral is done easily yielding the final result for the kink mass with leading order quantum correction

$$\begin{aligned} E &= \frac{m_{\psi,\mathrm{R}}^3}{3\lambda_\mathrm{R}} - \left(\frac{3}{\pi} - \frac{1}{2\sqrt{3}}\right)\frac{m_{\psi,\mathrm{R}}}{2} \\ &= \frac{m_{\psi,\mathrm{R}}^3}{3\lambda_\mathrm{R}} - 0.33m_{\psi,\mathrm{R}} \end{aligned} \tag{4.40}$$

Note the minus sign in front of the quantum correction to the energy. In Section 4.5 we show that this is a general feature.

4.3 Example: sine-Gordon kink

To quantize the sine-Gordon kink of Section 1.9, we follow the same procedure as for the Z_2 kink. The mode functions now satisfy

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}X^2} + (2\tanh^2 X - 1)\psi = \frac{\omega^2}{m_\psi^2}\psi \tag{4.41}$$

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where $X \equiv m_\psi x$. The kink solution, from Eq. (1.52), is

$$\phi_{\rm k} = \frac{4}{\beta}\tan^{-1}\left(e^{\sqrt{\alpha}x}\right) \equiv \frac{4m_\psi}{\sqrt{\lambda}}\tan^{-1}\left(e^{m_\psi x}\right) \tag{4.42}$$

where $\lambda \equiv \alpha\beta^2$. The classical energy (Eq. (1.55)) is

$$E_{\rm sG,cl} = 8\frac{\sqrt{\alpha}}{\beta^2} \equiv 8\frac{m_\psi^3}{\lambda} \tag{4.43}$$

The spectrum has only one bound state, the translational zero mode given by

$$\omega_1 = 0, \qquad \psi_0 = \frac{{\rm d}\phi_{\rm k}}{{\rm d}x} = \frac{2m_\psi^2}{\sqrt{\lambda}}{\rm sech}(m_\psi x) \tag{4.44}$$

The scattering state with wave-vector k can be written quite generally in terms of hypergeometric functions (see [113], Vol. II, Section 12.3, or Appendix C)

$$\psi_\kappa = N(\cosh X)^{{\rm i}\kappa X} F\left(-{\rm i}\kappa - 1, -{\rm i}\kappa + \frac{1}{2} + \frac{3}{2}|1 - {\rm i}\kappa|\frac{e^{-X}}{e^X + e^{-X}}\right) \tag{4.45}$$

where N is a normalization factor and $\kappa = k/m_\psi$ corresponds to the wave-vector. The phase shifts are found by taking the asymptotic forms of Eq. (4.45)

$$\begin{aligned} \psi_\kappa &\to N e^{{\rm i}\kappa X}, \quad X \to \infty \\ &\to N e^{{\rm i}(\pi+2\theta)} e^{{\rm i}\kappa X}, \quad X \to -\infty \end{aligned} \tag{4.46}$$

where $\tan\theta = \kappa$. Hence the phase shift is

$$\alpha_k = \pi - 2\tan^{-1}\left(\frac{k}{m_\psi}\right) \tag{4.47}$$

Therefore

$$\frac{{\rm d}\alpha_k}{{\rm d}k} = \frac{-2m_\psi}{k^2 + m_\psi^2} \tag{4.48}$$

At large $|k|$, $\Delta(k)$ (as needed in Eq. (4.20)) is given by

$$\Delta(k) = \frac{2m_\psi}{k}, \quad |k| \to \infty \tag{4.49}$$

To find $\Delta E_{\rm cl}$ occurring in Eq. (4.20), we can use the renormalized potential in Eq. (4.25). The parameter β, which occurs in the argument of the cosine function, is taken to be the physical (renormalized) value, while

$$(\sqrt{\alpha})_{\rm b} = (\sqrt{\alpha})_{\rm R}\left(1 + \frac{\beta^2}{2}\gamma\right) \tag{4.50}$$

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to leading order in β^2. The subscripts refer to bare and renormalized quantities and γ is defined in Eq. (4.23). Therefore

$$\Delta E_{\rm cl} = \frac{m_{\psi,\rm R}}{2\pi}\int_{-\Lambda}^{+\Lambda}\frac{{\rm d}k}{\sqrt{k^2+m_\psi^2}} \tag{4.51}$$

Finally, with $\sum\omega_b = 0$, we can put together all the various terms in Eq. (4.20) to get

$$\begin{aligned} E &= \frac{8m_{\psi,\rm R}^3}{\lambda_{\rm R}} + \frac{m_{\psi,\rm R}}{2\pi}\int_{-\Lambda}^{+\Lambda}\frac{{\rm d}k}{\sqrt{k^2+m_\psi^2}} + 0 - \frac{m_{\psi,\rm R}}{\pi} - \frac{m_{\psi,\rm R}}{2\pi}\int_{-\Lambda}^{+\Lambda}\frac{{\rm d}k}{\sqrt{k^2+m_\psi^2}} \\ &= \frac{8m_{\psi,\rm R}^3}{\lambda_{\rm R}} - \frac{m_{\psi,\rm R}}{\pi} \\ &= \frac{8m_{\psi,\rm R}^3}{\lambda_{\rm R}} - 0.32m_{\psi,\rm R} \end{aligned} \tag{4.52}$$

Once again the quantum correction is negative and, coincidentally, quite close to the Z_2 value (see Eq. (4.40)).

4.4 Quantized excitations of the kink

So far we have only calculated the quantum correction to the mass of the kink in its ground state. Now consider the excited states of the kink.

As in the second quantization of a free quantum field theory, particle creation and annhilation operators are introduced for each of the excitation modes of the kink. As we shall see, this is straightforward except for the zero mode. The end result is a procedure for doing quantum field theory with both particles and kinks included in the spectrum of states. Here we only give some introductory remarks. For a more extended discussion see [67, 126].

Let us denote the bound state mode functions by $F_b(t,x)$ and the scattering mode functions by $f_k(t,x)$. The t dependence is of the form $\exp(-{\rm i}\omega_i t)$ where ω_i is the frequency of the bound or scattering mode. Then the second quantized field operator is

$$\phi = \phi_{\rm k}(x) + \sum_b\left[\hat{c}_b F_b(t,x) + \hat{c}_b^\dagger F_b^*(t,x)\right] + \sum_k\left[\hat{a}_k f_k(t,x) + \hat{a}_k^\dagger f_k^*(t,x)\right] \tag{4.53}$$

where $\phi_{\rm k}$ is the classical kink solution, $c_b^\dagger$, c_b are creation/annihilation operators for the bound states, and similarly $a_k^\dagger$, a_k are creation/annihilation operators for the scattering states. Now, for the zero mode, $\omega = 0$ and $F_0(t,x) = F_0^*(t,x)$. Therefore

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the zero mode contribution to the sum is

$$[\hat{c}_0 + \hat{c}_0^\dagger] F_0(x) \tag{4.54}$$

Since c_0 and $c_0^\dagger$ are only present in the combination $c_0 + c_0^\dagger$ let us define $b_0 = \hat{c}_0 + \hat{c}_0^\dagger$ which is then the annihilation operator for the zero mode. However, note that $b_0^\dagger = b_0$ and so $[b_0, b_0^\dagger] = [b_0, b_0] = 0$: the zero mode is classical as the operator b_0 commutes with all other operators. This is to be contrasted with $[a_k, a_p^\dagger] = 2\pi\delta(k-p)$.

Just as the translation mode is a bosonic zero mode, there can also be fermionic zero modes that we discuss in Chapter 5. In that case, the creation and annihilation operators satisfy anticommutation relations leading to $\{b_0, b_0^\dagger\} = 0$. This relation has the remarkable consequence of leading to fractional quantum numbers as we discuss in Chapter 5.

4.5 Sign of the leading order correction

A striking feature of the leading order quantum corrections to the energies of the Z_2 and sine-Gordon kink is that they are negative. In other words, quantum effects reduce the mass of the kink. A variational argument [104] (Coleman, S., 1992, private communication) shows that this observation holds true quite generally in one dimension.[1]

Let the Hamiltonian of the $1+1$ dimensional system be

$$H \equiv \int \mathrm{d}x \mathcal{H} = \int \mathrm{d}x \, [\mathcal{H}_0 + V(\phi)] \tag{4.55}$$

where ϕ is a scalar field,

$$\mathcal{H}_0 \equiv \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 \tag{4.56}$$

and π is the canonical field momenta. Written in this way, the parameters entering the Hamiltonian are bare parameters and subject to renormalization. In one spatial dimension, however, it can be shown that [35]

$$\mathcal{H} = N_m \, [\mathcal{H}_0 + V_\mathrm{R}] \tag{4.57}$$

where N_m denotes normal ordering with respect to free particles of mass m, and the renormalized potential is (Eq. (4.21))

$$V_\mathrm{R} = \exp\left\{ \frac{1}{8\pi} \left(\ln \frac{4\Lambda^2}{m^2} \right) \frac{\mathrm{d}^2}{\mathrm{d}\phi^2} \right\} V(\phi) + \epsilon \tag{4.58}$$

[1] The conclusion may not hold if the model also contains fermionic fields.

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where Λ is an ultraviolet momentum cut-off and ϵ is a constant to be chosen such that $\langle 0|H|0\rangle = 0$ where $|0\rangle$ is the true ground state of the model.

The energy of the kink, including the contribution of quantum fluctuations in the ground state, is

$$E = {}_{\mathrm{k}}\langle 0|H[\phi_{\mathrm{k}} + \psi]|0\rangle_{\mathrm{k}} \tag{4.59}$$

where $|0\rangle_{\mathrm{k}}$ denotes the vacuum for the quantum fluctuations, ψ, around the classical one kink state ϕ_{k}.

Straightforward manipulation now gives the quantum correction to the kink mass

$$\begin{aligned} E - E_{\mathrm{cl,R}} &= {}_{\mathrm{k}}\langle 0|H[\phi_{\mathrm{k}} + \psi] - H[\phi_{\mathrm{k}}]|0\rangle_{\mathrm{k}} \\ &= \int \mathrm{d}x \; {}_{\mathrm{k}}\langle 0|N_m(\mathcal{H}_0[\phi_{\mathrm{k}} + \psi] - \mathcal{H}_0[\phi_{\mathrm{k}}] + V_{\mathrm{R}}[\phi_{\mathrm{k}} + \psi] - V_{\mathrm{R}}[\phi_{\mathrm{k}}])|0\rangle_{\mathrm{k}} \end{aligned}$$

where $E_{\mathrm{cl,R}}$ is the energy of the classical solution obtained with the renormalized potential, V_{R}. Next we use the variational principle, which states that the ground state energy of a system is minimized in its true ground state, and the expectation of the Hamiltonian in any other trial state gives an upper bound to the ground state energy. If we denote the perturbative vacuum state – the state with zero particles of mass m – by $|0, m\rangle$, then

$$\begin{aligned} E &\leq E_{\mathrm{cl,R}} + \int \mathrm{d}x \; \langle 0, m|N_m(\mathcal{H}_0[\phi_{\mathrm{k}} + \psi] - \mathcal{H}_0[\phi_{\mathrm{k}}] + V_{\mathrm{R}}[\phi_{\mathrm{k}} + \psi] - V_{\mathrm{R}}[\phi_{\mathrm{k}}])|0, m\rangle \\ &= E_{\mathrm{cl,R}} \end{aligned}$$

The last line follows since there are no ψ independent terms in the expectation value under the integral,[2] and the annihilation operators of ψ occur to the right owing to normal ordering and annihilate the trial vacuum state.

Note that $E_{\mathrm{cl,R}}$ is the energy of the classical solution found by minimizing $H_{\mathrm{R}}[\phi]$, i.e. the Hamiltonian in Eq. (4.55) but with the potential given in Eq. (4.58). Since the true ground state of the system is not known, the constant ϵ is not known either. The potential V_{R} can be minimized, but there is no guarantee that the minimal value of V_{R} will be zero. Therefore $E_{\mathrm{cl,R}}$ might get an infinite contribution from integrating $\min(V_{\mathrm{R}})$ over all of space. Then the variational bound $E \leq E_{\mathrm{cl,R}}$ is not very useful. However, we do know the value of ϵ to lowest order in the coupling constant and this is precisely so that $\langle 0, m|H|0, m\rangle = 0$. This coincides with choosing ϵ so as to make $\min(V_{\mathrm{R}}) = 0$. Hence the bound

$$E \leq E_{\mathrm{cl,R}} = E_{\mathrm{cl}} \tag{4.60}$$

[2] To see this, note that the expectation value vanishes if $\psi = 0$.

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where $E_{\rm cl}$ denotes the classical energy without any quantum corrections, is meaningful to leading order in perturbation theory and it provides us with the completely general result that the lowest order correction to the soliton energy is negative.

4.6 Boson-fermion connection

A bosonic field, ϕ, in quantum field theory satisfies the equal time commutation relation

$$[\phi(x,t), \dot{\phi}(y,t)] = \delta(x-y) \tag{4.61}$$

Alternatively, a fermionic field, ψ, satisfies the anticommutation relations

$$\{\psi_a(x,t), \psi_b^\dagger(y,t)\} = \delta(x-y)\delta_{ab} \tag{4.62}$$

where $a, b = 1, 2$ label the two components of the spinor in one spatial dimension. It is remarkable that one can construct explicitly a fermionic field ψ satisfying Eq. (4.62) in terms of a bosonic field ϕ that satisfies Eq. (4.61) [108].

The connection between ψ_a and ϕ is

$$\psi_1(x) = C :\mathrm{e}^{P_+(x)}:, \quad \psi_2(x) = -\mathrm{i}C :\mathrm{e}^{P_-(x)}: \tag{4.63}$$

where the c-number C is defined in terms of a mass parameter μ and another cut-off parameter, ϵ,

$$C = \left(\frac{\mu}{2\pi}\right)^{1/2} \mathrm{e}^{\mu/8\epsilon} \tag{4.64}$$

The operators $P_\pm$ contain a free parameter β and are defined by

$$P_\pm(x) = -\mathrm{i}\frac{2\pi}{\beta}\int_{-\infty}^{x} \mathrm{d}\xi\ \dot{\phi}(\xi) \mp \frac{\mathrm{i}\beta}{2}\phi(x) \tag{4.65}$$

The symbol :: in Eq. (4.63) denotes normal ordering with respect to the mass μ. This means that the field ϕ is to be treated as a free field with mass parameter μ and the quantum operator, ϕ, is expanded in terms of creation and annihilation operators that create and destroy particles of this free field theory. A normal ordered operator contains various products of creation and annihilation operators with the annihilation operators always occurring on the right. It is understood that the integral in Eq. (4.65) is cut off at large ξ by a factor $\exp(-\epsilon\xi)$. Note that normal ordering is a symbol and should be treated carefully – normal ordering of strings of operators should be done prior to commuting operators that occur within the string.

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To check if Eq. (4.62) is satisfied for $x \neq y$, we use the identity (see Appendix D)[3]

$$: e^{A+B} := e^{-[A^+,B^-]} : e^A :: e^B := e^{-[B^+,A^-]} : e^B :: e^A : \tag{4.66}$$

where A and B are any two operators that can be written as a linear sum of terms involving only creation or annihilation operators

$$A = A^+ + A^-, \qquad B = B^+ + B^- \tag{4.67}$$

The commutators $[A^+, B^-]$ and $[B^+, A^-]$ are assumed to be c-numbers. Insertion of Eq. (4.66) in Eq. (4.62) gives the commutation relation in Eq. (4.61) for $x \neq y$.

It is harder to check that the commutation relations in Eq. (4.61) hold when $x = y$. Since products of quantum operators at the same point are singular, the commutator must be evaluated at two different points in space, x and y, followed by the coincidence limit $y \to x$. We now outline the scheme employed in [108].

We want to check the anticommutation relation

$$\{\psi_a(x), \psi_b^\dagger(y)\} = Z\delta(x - y) \tag{4.68}$$

where the constant Z, possibly infinite, has been introduced in recognition of the fact that the fields get renormalized. Rather than check Eq. (4.68), we can check the equivalent *commutation* relation

$$[j^\mu(x), \psi(y)] = -\left(g^{0\mu} + \frac{\beta^2}{4\pi}\epsilon^{\mu 0}\gamma^5\right)\psi(x)\delta(x - y) \tag{4.69}$$

where the current j^μ has been regularized using point-splitting and is defined by

$$j^\mu(x) = \lim_{y\to x}\left\{\left[\delta^{\mu 0} + \frac{\beta^2}{4\pi}\delta_1^\mu\right][\mu(x - y)]^\sigma\ \bar{\psi}(x)\gamma^\mu\psi(y) + F^\mu(x - y)\right\} \tag{4.70}$$

where σ is a regularizing parameter and $F^\mu(x - y)$ an unspecified c-valued function. The γ-matrices are defined by the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \gamma^5 = i\gamma^0\gamma^1 \tag{4.71}$$

where $g_{\mu\nu} = \text{diag}(1, -1)$ is the two-dimensional Minkowski metric. An explicit representation of the γ-matrices is given in Eq. (5.15). In Eq. (4.69), $\epsilon^{\mu\nu}$ is the totally antisymmetric tensor.

First the current j^μ is evaluated with ψ_a as given in Eq. (4.63). The evaluation requires

$$[\phi^+(x, t_2), \phi^-(y, t_1)] = \Delta_+[(x - y)^2 - (dt + i\epsilon)^2] \tag{4.72}$$

[3] In the literature it is sometimes incorrectly stated that the identity $e^{A+B} = e^{[B,A]/2}e^A e^B$ (no normal ordering) is being used.

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where $\mathrm{d}t = t_2 - t_1$ and Δ_+ is the propagator. For small $x - y$

$$\Delta_+ = -\frac{1}{4\pi} \ln[\mu^2\{(x-y)^2 - (\mathrm{d}t + \mathrm{i}\epsilon)^2\}] + O((x-y)^2) \tag{4.73}$$

By differentiating Eq. (4.72) we can also obtain the commutators of time derivatives of ϕ^+ and ϕ^-. These appear in the evaluation of j^μ since ψ is defined in terms of $\dot{\phi}$ in Eq. (4.63).

The result for j^μ is singular in the limit $y \to x$ except for a single choice of the regularizing parameter, σ, occurring in the definition of j^μ. This single choice is

$$\sigma = \frac{\beta^2}{8\pi}\left(1 - \frac{4\pi}{\beta^2}\right)^2 \tag{4.74}$$

With this value of σ, the commutator in Eq. (4.69) can be verified. Thus the ψ operator indeed satisfies the anticommutation relations of a fermionic field. Furthermore, the current can be explicitly calculated, leading to

$$j^\mu = -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi \tag{4.75}$$

To summarize, given a quantum scalar field in $1+1$ dimensions, it is possible to construct a fermionic field from it via the relation (4.63). Starting with a fermionic field, a bosonic field may be constructed from it via Eq. (4.75). Note that the transformations from bosons to fermions and vice versa hold at the quantum operator level and not just at the level of expectation values. Further, they hold for any choice of interactions in the bosonic or the fermionic model. However, in the case when the bosonic model is the sine-Gordon model, the fermionic model obtained by transforming to the fermionic variables is another well-known model, namely the massive Thirring model as we now describe.

4.7 Equivalence of sine-Gordon and massive Thirring models

The sine-Gordon model is given by the Lagrangian (Eq. (1.51))

$$L_{\mathrm{sG}} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\alpha}{\beta^2}(1 - \cos(\beta\phi)) \tag{4.76}$$

while the massive Thirring model is

$$L_{\mathrm{mT}} = \mathrm{i}\bar{\psi}\ \partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{g}{2}\bar{\psi}\gamma^\mu\psi\ \bar{\psi}\gamma_\mu\psi \tag{4.77}$$

where ψ is a two-component fermionic field.

In [34] (also see [35]) it is shown that the sine-Gordon model does not have a well-defined ground state for $\beta^2 > 8\pi$. To clarify what this means, consider the

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simple example of a free field theory

$$L_{\text{free}} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\delta}{2}\phi^2 \tag{4.78}$$

This model has a well-defined ground state only in the range $\delta \geq 0$. The model does not have a well-defined ground state for $\delta < 0$. Similarly the sine-Gordon model only has a ground state for a definite range of parameters, though the reasons are much more subtle.[4] The sine-Gordon only has a well-defined ground state if the parameter β^2 is restricted to lie in the interval $(0, 8\pi)$.

In the range, $0 \leq \beta^2 \leq 8\pi$, there is a one-to-one mapping between vacuum expectation values of a string of operators in the sine-Gordon model to those in the massive Thirring model. This means that any vacuum expectation value in the sine-Gordon model has a "corresponding" vacuum expectation value in the massive Thirring model. This strongly suggests that the two models are equivalent, even at the operator level [35].

As we have seen in the last section, there is indeed a two-component fermionic field, ψ, that can be constructed from a bosonic field ϕ (Eq. (4.63)). In [108] it was shown that ψ also obeys the equations of motion of the massive Thirring model if the bosonic field ϕ obeys the equations for the sine-Gordon equation with the coupling constant g written in terms of the coupling constant β as

$$\frac{g}{\pi} = 1 - \frac{4\pi}{\beta^2} \tag{4.79}$$

Note that when the sine-Gordon model is weakly coupled (small β), the massive Thirring model is strongly coupled and vice versa. Hence the sine-Gordon model and the massive Thirring model are completely equivalent as quantum field theories but one is a better description at small β (large g) and the other at large β (small g).

What has the equivalence of the sine-Gordon and massive Thirring models got to do with kinks? Consider the commutation relations between ϕ and ψ. Using Eq. (4.63) and the identity (see Appendix D) $[A, :\mathrm{e}^B:] =: [A, \mathrm{e}^B] :$ with $A = \phi(y)$ and $:\mathrm{e}^B := \psi$ we find

$$[\phi(y), \psi(x)] = \frac{2\pi}{\beta}\psi(x), \quad (y < x) \tag{4.80}$$

$$[\phi(y), \psi(x)] = 0, \quad (y > x) \tag{4.81}$$

Now consider the action of $\psi(x)$ on an eigenstate, $|s\rangle$ of the field operator ϕ. Let us choose this eigenstate to be such that

$$\phi|s\rangle = 0 \tag{4.82}$$

[4] For example, in contrast to the model in Eq. (4.78), the *classical* sine-Gordon model has well-defined global minima for all values of the coupling constant β.

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If we write $|s'\rangle = \psi(x)|s\rangle$, the relation in Eq. (4.80) gives

$$\phi(y)|s'\rangle = \frac{2\pi}{\beta}|s'\rangle, \quad (y < x) \tag{4.83}$$

and Eq. (4.81) gives

$$\phi(y)|s'\rangle = 0, \quad (y > x) \tag{4.84}$$

Therefore the state obtained after action by $\psi(x)$ is one where the value of ϕ is $2\pi/\beta$ for $y < x$ and 0 for $y > x$. In other words, the field $\psi(x)$ creates a step-change of ϕ. The step-function profile is viewed as a "bare kink" which gets dressed by quantum effects that produce a smooth kink profile with some finite width. So the field $\psi(x)$ is the creation operator for a (bare) soliton at location x. In the Thirring model, the field $\psi(x)$ is interpreted as the creation operator for a fermion located at x. Hence the sine-Gordon kink is identified with the fermion in the massive Thirring model.

The topological charge on a sine-Gordon kink is

$$Q_{\rm k} = \int {\rm d}x j^0_B = \int {\rm d}x j^0_F \tag{4.85}$$

where the fermionic current is defined in terms of the bosonic current in Eq. (4.75). Therefore the fermion in the massive Thirring model carries the topological charge of the sine-Gordon kink. In other words, the kink of the strongly coupled sine-Gordon model is better described as a weakly coupled fermion of the massive Thirring model. Here we see the duality between particles and solitons.

Can we also interpret the bosonic particles of the sine-Gordon model in terms of "solitons" of the massive Thirring model? The massive Thirring model only contains fermions, and classical solutions of the Dirac equation do not have the interpretation of solitons. This is because the fermionic fields anticommute and fermions obey the Pauli exclusion principle. Instead a classical solution of the Dirac equation is a state that one (and only one) fermion can occupy. However, there can be bound states of two or more fermions since the force between a fermion and an antifermion is attractive for $g > 0$. A bound state of two fermions can be shown to correspond to a particle of the sine-Gordon field ϕ. If the fermions in the weakly coupled massive Thirring model have mass m, then the bound state energy is approximately $2m$ since it involves two fermions. However, the binding energy decreases (becomes more negative) with increasing interaction strength, g, and eventually the bound state becomes lighter than a single fermion. At this stage, a suitable description of the system is in terms of the bound state being the fundamental degree of freedom as in the sine-Gordon model.

The bound state of two massive Thirring fermions is also a bound state of two sine-Gordon kinks i.e. a breather. Hence it should be possible to interpret

the breather as a particle in the sine-Gordon model. This is seen to be true when the breather is quantized [38–41, 35]. Then, to lowest order, the energy levels of the quantized breather are equal to the mass of one, two, three, etc. particles of the sine-Gordon particle.

4.8 Z_2 kinks on the lattice

Lattice field theory provides another tool to probe the quantum nature of solitons and, in particular, the variation of mass with coupling constant.

The starting point is the action for the Z_2 model defined in Eq. (1.2). The action is to be inserted in the Feynman path integral, which can then be used to find expectation values for any quantum operator. In the Feynman path integral, it is necessary to integrate over field configurations, and this is done numerically on a discretized Euclidean space-time. The reader is referred to the lattice literature for details [37, 112, 141]. Here we shall give the results relevant to the Z_2 kink.

The mass of a Z_2 kink is defined as the expectation value of a suitable operator defined on the lattice in the limit that the lattice spacing, a, goes to zero. One important issue is that there are several different candidate operators on the lattice that all go to the correct limit as $a \to 0$ and, in practice, it is not possible to take the limit all the way to $a = 0$. At best, the numerical analysis gives the expectation of the operator on the lattice for several different values of a and then some scheme must be found for extrapolating the results to $a \to 0$. In [32], the authors evaluate the mass of the Z_2 kink using two different lattice operators. The results are shown in Fig. 4.2. We note that the kink mass decreases monotonically as the coupling constant increases and remains bounded by the classical mass. At a certain coupling, the kink mass goes to zero, and the kink, not the ϕ quanta, is the lightest degree of freedom in the model.

The mass of the sine-Gordon kink has been calculated analytically for a range of parameters in [156] (also see [35]).

4.9 Comments

Several researchers have taken alternate paths to studying quantized kinks. In supersymmetric theories there is greater control over quantum corrections and the mass can, in some cases, be evaluated exactly [51]. Alternate methods to quantize supersymmetric kinks have also been developed in [66]. Variational methods to study the $\lambda\phi^4$ theory have been developed in [49]. The scattering of kinks in classical and quantum theory has been studied in [153]. Kink masses and scattering have also been calculated in [132] using the Hartree approximation.

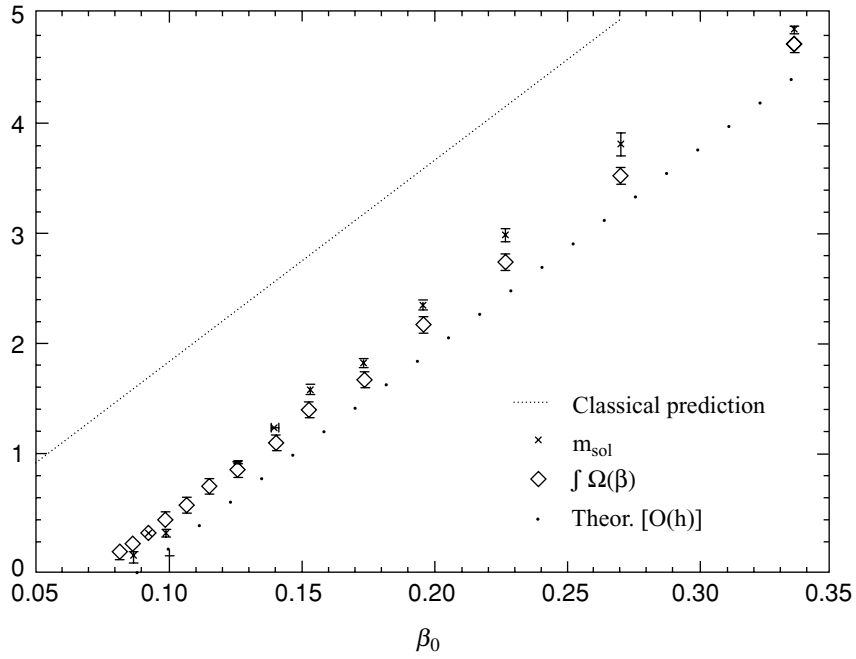

Figure 4.2 The figure shows how the mass of the Z_2 kink depends on the inverse coupling constant, $\beta_0 = 1/\lambda_0$, where $\lambda_0 \equiv 6\lambda a^2$ is the coupling constant in the discrete theory on a 48×48 lattice and a is the lattice spacing. (The factor of 6 is due to our choice of $1/4$ in the λ term in Eq. (1.2) as opposed to $1/4!$ in [32].) The lattice mass parameter, $r_0 \equiv -m^2a^2$, is held fixed at $r_0 = -2.2$. From the plot we see that the classical value of the kink mass is larger than the quantum value. The one-loop corrected mass (see Section 4.2) and the mass found by using two different choices of the lattice mass operator are also shown. The kink mass vanishes at $\beta_0 = 0.0804$. [Figure reprinted from [32].]

The construction of fermion operators from boson operators and vice versa has been discussed and used extensively in condensed matter applications under the name of bosonization. A review, in addition to an historical introduction, may be found in [44]. Finally, the technique of bosonization has also been applied to thermal systems in [69].

4.10 Open questions

1. The quantum corrections to the Z_2 and sine-Gordon kinks were calculated explicitly using the phase shifts. However, the phase shift approach only works if the potential $U(x)$ is reflectionless. What are the conditions necessary for a potential to be reflectionless? Are reflectionless potentials always in factorizable form (see Section 3.3)? (The example of a step-function potential shows that the converse is not true.)

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2. We have shown that the leading order quantum correction to the kink mass is always negative. Can this statement be generalized to all orders? Can one show that the mass of a kink goes to zero in the strong coupling limit? Or perhaps that it is monotonically decreasing as a function of increasing coupling constant?
3. If the Z_2 kink at strong coupling is to be viewed as a particle, then the particle must obey unusual statistics because two kinks cannot be next to each other. Discuss this statistics and its implications for the dual model.
4. From the $SU(5) \times Z_2$ example we learned that a classical kink may be embedded in many different ways in "large" models. On quantization, do the different embeddings correspond to distinct degrees of freedom?
5. Does the addition of fermionic particles change the conclusion that quantum corrections always reduce the energy of a kink?
6. For the sine-Gordon model we have explicitly seen that there is a relation between kinks and particles. It seems reasonable that the connection holds in other models too. In $3 + 1$ dimensions, we could expect the connection to exist between magnetic monopoles and observed particles (e.g. [162, 103]). Construct a model that has families of solitons, similar to the electron, muon, and tau families observed in Nature (see [122]).
7. In Section 3.1 we have discussed the existence of quasi-breather solutions called "oscillons" in the Z_2 model. Can quantum oscillons have an interpretation that is similar to quantized breathers as discussed at the end of Section 4.7?