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SUBMANIFOLDS SATISFYING SOME CURVATURE CONDITIONS IMPOSED ON THE WEYL TENSOR

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In this paper we define Weyl semiparallel ($\mathcal{C} \cdot h = 0$) and Weyl 2-semiparallel ($\mathcal{C} \cdot \overline{\nabla} h = 0$) submanifolds. We consider n-dimensional normally flat submanifolds satisfying these curvature conditions in $(n + d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. We also consider normally flat submanifolds in $(n + d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$ satisfying the condition $\mathcal{C} \cdot h = LQ(g, h)$ and $\mathcal{C} \cdot \overline{\nabla} h = LQ(g, \overline{\nabla} h)$.

1. INTRODUCTION

Let (M, g) be an n-dimensional submanifold in $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. Let ξ be a local normal section on M. The formulas of Gauss and Weingarten are given by $\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and $\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$ for vector fields X, Y, Z which are tangent to M. Here $\widetilde{\nabla}$ is the Euclidean connection on $\mathbb{E}^{n+d}$, ∇ is the Levi-Civita connection on M, and $\nabla^\perp$ is the normal connection of M in $\mathbb{E}^{n+d}$. The second fundamental form h and A_ξ are related by $\langle h(X, Y), \xi \rangle = g(A_\xi X, Y)$. For the second fundamental form h the covariant derivative of h is defined by $(\overline{\nabla}_X h)(Y, Z) = \nabla^\perp_X \big(h(Y, Z)\big) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$, for any vector fields X, Y, Z tangent to M. Then $\overline{\nabla} h$ is a normal bundle valued tensor of type (0,3) and is called the *third fundamental form* of M. The equation of Codazzi implies that $\overline{\nabla} h$ is symmetric hence

$$(\overline{\nabla}_X h)(Y, Z) = (\overline{\nabla}_Y h)(X, Z) = (\overline{\nabla}_Z h)(X, Y). \tag{1.1}$$

$\overline{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M, that is, $\overline{\nabla}$ is the connection in $TM \oplus T^\perp M$ built with ∇ and $\nabla^\perp$. If $\overline{\nabla} h = 0$ then M is said to have *parallel second fundamental form* [**2**]. In the third chapter sometimes we will use $(\overline{\nabla}_X h)(Y, Z) = (\overline{\nabla} h)(X, Y, Z)$.

Let $X \wedge Y$ denote the endomorphism defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{1.2}$$

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where g is the metric tensor on M. Then the curvature operator $\mathcal{R}$ of M is given by the equation of Gauss:

$$\mathcal{R}(X,Y)Z = \sum_{i=1}^{d}(A_iX \wedge A_iY)Z, \tag{1.3}$$

and the *curvature tensor* R of M is defined by $R(X,Y,Z,W) = g\big(\mathcal{R}(X,Y)Z,W\big)$, where $A_i := A_{\xi_i}$ and $\{\xi_1,\xi_2,\dots,\xi_d\}$ is a local orthonormal basis for $T^{\perp}M$ and X,Y,Z,W are vector fields tangent to M. The equation of Ricci becomes

$$R^{\perp}(X,Y,\xi,\eta) = g\big([A_\xi,A_\eta]X,Y\big), \tag{1.4}$$

for the vector fields ξ and η normal to M.

A submanifold M is said to have *flat normal connection* (or trivial normal connection) if $R^{\perp} = 0$. If M has flat normal connection then shortly we call it normally flat. The relation (1.4) shows that the triviality of the normal connection of M into Euclidean space $\mathbb{E}^{n+d}$ (and more generally for submanifolds in a locally conformally flat space) is equivalent to the fact that all second fundamental tensors mutually commute and to the simultaneous diagonasability of all second fundamental tensors ([**2**]).

The *Ricci tensor* S, the *Ricci operator* $\mathcal{S}$ and the *scalar curvature* κ of M are defined by

$$S(X,Y) = \sum_{k=1}^{n} g\big(R(e_k,X)Y,e_k\big), \tag{1.5}$$

$S(X,Y) = g(\mathcal{S}X,Y)$ and $\kappa = tr(\mathcal{S})$, respectively ([**2**]).

The *Weyl conformal curvature operator* $\mathcal{C}$ is defined by

$$\mathcal{C}(X,Y) = \mathcal{R}(X,Y) - \frac{1}{n-2}\Big(X \wedge \mathcal{S}Y + \mathcal{S}X \wedge Y - \frac{\kappa}{n-1}X \wedge Y\Big) \tag{1.6}$$

and the *Weyl conformal curvature tensor* C is defined by $C(X,Y,Z,W) = g\big(\mathcal{C}(X,Y)Z,W\big)$. If $C = 0$, $n \geqslant 4$, then M is called *conformally flat*.

For a $(0,k)$-tensor field T ($k \geqslant 1$) and the metric tensor g on M we can define the tensor $Q(g,T)$ by

$$\begin{aligned} Q(g,T)(X_1,X_2,\dots,X_k;X,Y) = &-T\big((X\wedge Y)X_1,X_2,\dots,X_k\big) \\ &-\dots - T\big(X_1,X_2,\dots,(X\wedge Y)X_k\big), \end{aligned} \tag{1.7}$$

for all $X_1,X_2,\dots,X_k$, $X,Y \in TM$ (see [**6**]).

2. Weyl Semiparallel Submanifolds

Let M be an n-dimensional submanifold in $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. Denote the curvature tensor of $\overline{\nabla}$ by $\bar{R}$. In [**4**] and [**5**], Deprez defined and investigated semiparallel surfaces (that is, surfaces satisfying the condition $\bar{R}\cdot h = 0$) and semiparallel hypersurfaces in Euclidean space respectively.

Similar to Deprez's definition we can give the following:

DEFINITION 2.1. Let M be a normally flat submanifold in an $(n+d)$-dimensional Riemannian manifold $\widetilde{M}$. We define

$$(2.1) \qquad \big(\mathcal{C}(X,Y)\cdot h\big)(U,V) = -h\big(\mathcal{C}(X,Y)U,V\big) - h\big(U,\mathcal{C}(X,Y)V\big)$$

for $X, Y, U, V \in TM$. If for every point $p \in M$ and for every vector fields $X, Y \in TM$, the tensor $\mathcal{C}(X,Y)\cdot h = 0$ then M is called *Weyl-semiparallel.*

THEOREM 2.1. ([5]) *Let M be a hypersurface in $\mathbb{E}^{n+1}$, $(n \geqslant 4)$. Then the following assertions are equivalent;*

(i) $\mathcal{C}\cdot h = 0$ *(that is; M is Weyl semiparallel),*

(ii) *M is conformally flat.*

It is well known that all hypersurfaces are always normally flat. Our problem is the following:

Is it possible to find the natural generalisation of Theorem 2.1 to all submanifolds with flat normal connections in $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$?

Firstly we have;

THEOREM 2.2. *Let M be an n-dimensional, $(n \geqslant 4)$, normally flat submanifold in $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. Then locally the following assertions are equivalent;*

(i) $\mathcal{C}\cdot h = 0$ *(that is; M is Weyl semiparallel),*

(ii) *M is conformally flat.*

PROOF: Assume that M is an n-dimensional, $(n \geqslant 4)$, normally flat submanifold in $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. Let $\{\xi_1, \dots, \xi_d\}$ be an orthonormal basis of the normal space $T_p^{\perp}M$ at a point $p \in M$. On the other hand, by a result of Cartan, we know that the flatness of the normal connection of M is equivalent to the simultaneous diagonazability of all shape operator matrices A_{ξ_i} for all ξ_i $(1 \leqslant i \leqslant d)$ of the normal space $T_p^{\perp}M$. So we can choose an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space T_pM at $p \in M$ of eigenvectors of A_{ξ_i} such that $h(e_i, e_j) = 0$ for all $i \neq j$.

Using (2.1) we can write

$$(2.2) \qquad \big(\mathcal{C}(e_i,e_j)\cdot h\big)(e_i,e_j) = -h\big(\mathcal{C}(e_i,e_j)e_i,e_j\big) - h\big(e_i,\mathcal{C}(e_i,e_j)e_j\big),$$

for $e_i, e_j \in T_pM$, $1 \leqslant i, j \leqslant n$.

We denote by K_{ij} the sectional curvature of a plane Π spanned by the vectors e_i and

e_j. An easy calculation shows us $S(e_i, e_j) = 0$ for all $i \neq j$ and

$$\text{(2.3)} \qquad \mathcal{C}(e_i, e_j)e_i = \Big[-K_{ij} + \frac{1}{n-2}\Big(S(e_i, e_i) + S(e_j, e_j) - \frac{\kappa}{n-1}\Big)\Big]e_j,$$

$$\text{(2.4)} \qquad \mathcal{C}(e_i, e_j)e_j = -\Big[-K_{ij} + \frac{1}{n-2}\Big(S(e_i, e_i) + S(e_j, e_j) - \frac{\kappa}{n-1}\Big)\Big]e_i,$$

at $p \in M$. So substituting (2.3) and (2.4) into (2.2) we obtain

$$\text{(2.5)} \qquad \big(\mathcal{C}(e_i, e_j) \cdot h\big)(e_i, e_j) = \big[h(e_i, e_i) - h(e_j, e_j)\big]\Big[-K_{ij} + \frac{1}{n-2}\Big(S_{ii} + S_{jj} - \frac{\kappa}{n-1}\Big)\Big],$$

where $S_{ii} = S(e_i, e_i)$ and $S_{jj} = S(e_j, e_j)$.

Since M is Weyl semiparallel $\big(\mathcal{C}(e_i, e_j) \cdot h\big)(e_i, e_j) = 0$, which gives

$$\text{(2.6)} \qquad \big[h(e_i, e_i) - h(e_j, e_j)\big]\Big[-K_{ij} + \frac{1}{n-2}\Big(S_{ii} + S_{jj} - \frac{\kappa}{n-1}\Big)\Big] = 0$$

at $p \in M$. If $h(e_i, e_i) = h(e_j, e_j)$ then M is totally umbilical at p, so by ([**3**] and[**8**]) $C = 0$ at p. If $-K_{ij} + \Big(S_{ii} + S_{jj} - \big(\kappa/(n-1)\big)\Big)/(n-2) = 0$ then by (2.3) and (2.4) we have $\mathcal{C}(e_i, e_j)e_i = \mathcal{C}(e_i, e_j)e_j = 0$. Moreover it can be easily seen that $\mathcal{R}(e_i, e_j)e_k = 0$ and $\mathcal{C}(e_i, e_j)e_k = 0$ for different i, j, k. Therefore the vanishing of $\mathcal{C}(e_i, e_j)e_i$ and $\mathcal{C}(e_i, e_j)e_j$ give us $C = 0$ at p, which proves the theorem. ∎

Now we give an extension of Theorem 2.2.

Theorem 2.3. *Let M be an n-dimensional, $(n \geqslant 4)$, normally flat submanifold in $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. If the condition $\mathcal{C} \cdot h = LQ(g, h)$ holds on M, where $L : M \to \mathbb{R}$ is a function, then locally the relation $\mathcal{C}(e_i, e_j) = L(e_i \wedge e_j)$ holds on M for $e_i, e_j \in T_pM$.*

Proof: Let M be an n-dimensional, $(n \geqslant 4)$, normally flat submanifold in $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. Choose the same bases as in the proof of the previous theorem, with $\{\xi_1, \dots, \xi_d\}$ and $\{e_1, \dots, e_n\}$ of the normal space $T_p^{\perp}M$ and the tangent space T_pM respectively at a point $p \in M$.

Using (1.7) we have

$$\text{(2.7)} \qquad Q(g, h)(e_i, e_j; e_i, e_j) = -h\big((e_i \wedge e_j)e_i, e_j\big) - h\big(e_i, (e_i \wedge e_j)e_j\big).$$

Since e_i, $e_j \in T_pM$ are orthonormal vectors, the equation (2.7) can be written as

$$\text{(2.8)} \qquad Q(g, h)(e_i, e_j; e_i, e_j) = h(e_j, e_j) - h(e_i, e_i).$$

Suppose that the condition $\mathcal{C} \cdot h = 0$ is satisfied on M. Then the condition $\mathcal{C} \cdot h = LQ(g, h)$ is trivially realised at p. Now assume $\mathcal{C} \cdot h \neq 0$ and the condition $\mathcal{C} \cdot h = LQ(g, h)$ holds at $p \in M$. Then by the use of (2.5) and (2.8)

$$\text{(2.9)} \qquad \big[h(e_i, e_i) - h(e_j, e_j)\big]\Big[L - K_{ij} + \frac{1}{n-2}\Big(S(e_i, e_i) + S(e_j, e_j) - \frac{\kappa}{n-1}\Big)\Big] = 0.$$

Since we suppose $\mathcal{C} \cdot h \neq 0$ at p, we obtain $L - K_{ij} + \Big(S(e_i, e_i) + S(e_j, e_j) - \kappa(n-1)\Big)/(n-2) = 0$ at p. So, by a similar discussion in the proof of the previous theorem, we obtain the relation $\mathcal{C}(e_i, e_j) = L(e_i \wedge e_j)$ holds at p. This completes the proof of the theorem. ∎

3. Weyl 2-Semiparallel Submanifolds

In [**1**], the authors defined the notion of 2-semiparallel submanifold and they classify normally flat surfaces in the space form $N^n(c)$. In [**7**], Lumiste investigated non-normally flat 2-semiparallel surfaces satisfying the condition $\tilde{R} \cdot \overline{\nabla} h = 0$.

In the present section our aim is to find the characterisation of normally flat submanifolds in the Euclidean space $\mathbb{E}^{n+d}$ satisfying the condition $\mathcal{C} \cdot \overline{\nabla} h = 0$ and $\mathcal{C} \cdot \overline{\nabla} h = LQ(g, \overline{\nabla} h)$. Firstly we give the following definition:

Definition 3.1. Let M be an n-dimensional, $(n \geqslant 4)$, normally flat submanifold in an $(n+d)$-dimensional Riemannian manifold $\widetilde{M}$. We define

$$\begin{aligned}(\mathcal{C}(X,Y)\cdot\overline{\nabla}h)(U,V,W) = -(\overline{\nabla}h)(\mathcal{C}(X,Y)U,V,W) - (\overline{\nabla}h)(U,\mathcal{C}(X,Y)V,W) \\ -(\overline{\nabla}h)(U,V,\mathcal{C}(X,Y)W),\end{aligned} \tag{3.1}$$

for $X, Y, U, V, W \in TM$. If for all point $p \in M$, the tensor $\mathcal{C}(X,Y) \cdot \overline{\nabla} h = 0$ then M is called *Weyl 2-semiparallel.*

Theorem 3.1. *Let M be an n-dimensional, $(n \geqslant 4)$, normally flat Weyl 2-semiparallel submanifold in $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. Then locally either*

(i) *M has parallel second fundamental form or*

(ii) *M is conformally flat.*

Proof: Suppose that M is an n-dimensional, $(n \geqslant 4)$, normally flat submanifold in $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. Choose orthonormal bases $\{\xi_1, \dots, \xi_d\}$ and $\{e_1, \dots, e_n\}$ of the normal space $T_p^\perp M$ and the tangent space T_pM respectively at a point $p \in M$. So using (1.6), (3.1) and Codazzi equations (1.1) we obtain

$$(\mathcal{C}(e_i,e_j)\cdot\overline{\nabla}h)(e_i,e_i,e_i) = -3A(\overline{\nabla}h)(e_j,e_i,e_i), \tag{3.2}$$

$$(\mathcal{C}(e_i,e_j)\cdot\overline{\nabla}h)(e_i,e_i,e_j) = A\big[-2(\overline{\nabla}h)(e_j,e_i,e_j) + (\overline{\nabla}h)(e_i,e_i,e_i)\big], \tag{3.3}$$

$$(\mathcal{C}(e_i,e_j)\cdot\overline{\nabla}h)(e_i,e_j,e_j) = A\big[-(\overline{\nabla}h)(e_j,e_j,e_j) + 2(\overline{\nabla}h)(e_i,e_i,e_j)\big] \tag{3.4}$$

and

$$(\mathcal{C}(e_i,e_j)\cdot\overline{\nabla}h)(e_j,e_j,e_j) = 3A(\overline{\nabla}h)(e_i,e_j,e_j). \tag{3.5}$$

for $e_i, e_j \in T_pM$, where $A = \Big[-K_{ij} + \big(S_{ii} + S_{jj} - \kappa(n-1)\big)/(n-2)\Big]$.

By assumption, since M is Weyl 2-semiparallel, from (3.2)–(3.5), we get

$$A(\overline{\nabla}h)(e_j,e_i,e_i) = 0, \tag{3.6}$$

$$A\big[-2(\overline{\nabla}h)(e_j,e_i,e_j) + (\overline{\nabla}h)(e_i,e_i,e_i)\big] = 0, \tag{3.7}$$

$$A\big[-(\overline{\nabla}h)(e_j,e_j,e_j) + 2(\overline{\nabla}h)(e_i,e_i,e_j)\big] = 0 \tag{3.8}$$

and

$$A(\overline{\nabla}h)(e_i, e_j, e_j) = 0, \tag{3.9}$$

at $p \in M$. Suppose $(\overline{\nabla}h)(e_i, e_j, e_j) = 0$, $(\overline{\nabla}h)(e_j, e_i, e_i) = 0$ and $A \neq 0$. Using the Codazzi equations (1.1), we can substitute the last equalities into (3.7) and (3.8) respectively. So we obtain $(\overline{\nabla}h)(e_i, e_i, e_i) = (\overline{\nabla}h)(e_j, e_j, e_j) = 0$ which gives us $\overline{\nabla}h = 0$ at p.

Now suppose $\overline{\nabla}h \neq 0$ at p. Therefore from (3.6)–(3.9) we obtain

$$A = \left[-K_{ij} + \frac{1}{n-2}\left(S_{ii} + S_{jj} - \frac{\kappa}{n-1}\right)\right] = 0.$$

By a similar discussion in the proof of Theorem 2.2 we obtain $C = 0$ at p. Our theorem is thus proved. ∎

Now we give an extension of Theorem 3.1.

Theorem 3.2. *Let M be an n-dimensional, $(n \geqslant 4)$, normally flat submanifold in $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. If M satisfies the condition $\mathcal{C}\cdot\overline{\nabla}h = LQ(g, \overline{\nabla}h)$, where $L : M \to \mathbb{R}$ is a function, then locally the relation $\mathcal{C}(e_i, e_j) = L(e_i \wedge e_j)$ holds on M for $e_i, e_j \in T_pM$.*

Proof: Suppose that M is an n-dimensional, $(n \geqslant 4)$, normally flat submanifold in $\mathbb{E}^{n+d}$. Choose orthonormal bases $\{\xi_1, \dots, \xi_d\}$ and $\{e_1, \dots, e_n\}$ of the normal space $T_p^{\perp}M$ and the tangent space T_pM respectively at a point $p \in M$. So using (1.7), (1.2) and the Codazzi equations (1.1) we obtain

$$Q(g, \overline{\nabla}h)(e_i, e_i, e_i; e_i, e_j) = 3(\overline{\nabla}h)(e_j, e_i, e_i), \tag{3.10}$$

$$Q(g, \overline{\nabla}h)(e_i, e_i, e_j; e_i, e_j) = 2(\overline{\nabla}h)(e_j, e_i, e_j) - (\overline{\nabla}h)(e_i, e_i, e_i), \tag{3.11}$$

$$Q(g, \overline{\nabla}h)(e_i, e_j, e_j; e_i, e_j) = (\overline{\nabla}h)(e_j, e_j, e_j) - 2(\overline{\nabla}h)(e_i, e_i, e_j), \tag{3.12}$$

and

$$Q(g, \overline{\nabla}h)(e_j, e_j, e_j; e_i, e_j) = -3(\overline{\nabla}h)(e_i, e_j, e_j), \tag{3.13}$$

for $e_i, e_j \in T_pM$. Assume that $\mathcal{C} \cdot \overline{\nabla}h = LQ(g, \overline{\nabla}h)$ holds at $p \in M$. So combining (3.2)–(3.5) and (3.10)–(3.13), we have

$$[A + L](\overline{\nabla}h)(e_j, e_i, e_i) = 0, \tag{3.14}$$

$$[A + L]\left[-2(\overline{\nabla}h)(e_j, e_i, e_j) + (\overline{\nabla}h)(e_i, e_i, e_i)\right] = 0, \tag{3.15}$$

$$[A + L]\left[-(\overline{\nabla}h)(e_j, e_j, e_j) + 2(\overline{\nabla}h)(e_i, e_i, e_j)\right] = 0, \tag{3.16}$$

and

$$[A + L](\overline{\nabla}h)(e_i, e_j, e_j) = 0, \tag{3.17}$$

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at $p \in M$. If M is Weyl 2-semiparallel then the condition $\mathcal{C} \cdot \overline{\nabla} h = LQ(g, \overline{\nabla} h)$ is trivially realised at p. Now suppose $\mathcal{C} \cdot \overline{\nabla} h \neq 0$ at p. Then from (3.14)–(3.17) we get $A + L = -K_{ij} + 1/(n-2)\big(S_{ii} + S_{jj} - (\kappa/n - 1)\big) + L = 0$ at p. By a similar discussion in the proof of Theorem 2.2 we obtain $\mathcal{C}(e_i, e_j) = L(e_i \wedge e_j)$ at p. Hence we get the result as required. ∎

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