Proceedings of the Edinburgh Mathematical Society (1990) **33**, 11–22 ©

ON THE DECOMPOSITION OF THE s-RADICAL OF A NEAR-RING

by J. F. T. HARTNEY

(Received 27th October 1987)

This paper concerns a Jacobson-type radical for the near-ring N. This radical, denoted by $J_s(N)$ has an external representation on a type-0 N-group of a very special kind. Such N-groups are said to be of type-s. The main objective of this paper is to decompose $J_s(N)$ as a sum $J_s(N)=J_{1/2}(N)+A+B$ for N satisfying the descending chain condition for N-subgroups. In this decomposition $J_{1/2}(N)$ is nilpotent and A is the unique minimal ideal modulo which $J_s(N)$ is nilpotent.

1980 *Mathematics subject classification* (1985 Revision): 16-XX.

1. Introduction

Throughout this article our near-rings will be assumed to be zero symmetric and right distributive. If the near-ring N satisfies the descending chain condition for left ideals (N-subgroups) we say that N satifies the *DCCL* (respectively, *DCCN*). Our direct sums will always be direct sums of N-kernels (kernels of N-homomorphisms). Hence, $\Omega=\bigoplus_{j\in J}\Omega_j$ for the N-group Ω will always mean that Ω_j is an N-kernel of Ω for each $j\in J$. The following theorem which is referred to as the Main Theorem is proved.

Let N be a near-ring which satisfies the DCCN. If the s-radical $J_s(N)$ is not zero, then there exists an ideal A of N contained in $J_s(N)$ such that A is uniquely minimal amongst all ideals B of N for which $J_s(N/B)$ is non-zero and nilpotent. Moreover $J_s(N/A)=J_s(N)/A$.

The main theorem is then used to decompose the s-radical as $J_s(N)=J_{1/2}(N)+A+B$, where A is the unique minimal ideal of the main theorem and B is an ideal which is nilpotent modulo an ideal C to be defined in the sequel.

This generalises the work initiated in [**5**]. If the near-ring N is itself nilpotent modulo some ideal C, then $J_{1/2}(N/C)=J_0(N/C)=J_s(N/C)$. To avoid this trivial situation we will assume throughout that the near-ring N is not nilpotent modulo any ideal $C\neq N$. That is, we assume throughout that $N^k\nsubseteq C$ for every positive integer k and every ideal $C\neq N$. Also, $J_s(N)\neq N$ throughout this paper. We use as our main tool the nil-rigid series of a near-ring N first defined and discussed by S. D. Scott [**9**]. Scott's generalized version of his original work appears in [**10**]. The length of the nil-rigid series throws some light on the manner in which the s-radical decomposes with $J_{1/2}(N)$ as a summand (not necessarily a direct summand) but many problems remain. We refer the reader to Meldrum [6] and Pilz [8] for the basic definitions and results used in this paper.

2. Preliminaries

The following results are of great importance in the development of the theory in subsequent sections. Proofs of these reults can be found in [4, 6] and also in [1].

Theorem 2.1. *Let N be a near-ring which satisfies the DCCL. Then the factor N-group $N-J_{1/2}(N)=\bigoplus_{i=1}^{k}\Omega_i$, where Ω_i is an N-group of type-0 for $i=1,\ldots,k$.*

Lemma 2.2. *Let $\Omega=\bigoplus_{i\in I}\Omega_i$ be an N-group, where each Ω_i is an N-group of type-0. Then any N-kernel Δ of Ω is a direct summand of Ω with a direct sum of some of the Ω_i as a co-summand. That is $\Omega=\bigoplus_{j\in J\leqq I}\Omega_j\oplus\Delta$.*

Details of the theory involving the s-radical, $J_s(N)$ of a near-ring N can be found in [2] and [6]. We give a summary of some of the results which we will require later.

An N-group Ω of type-0 is said to be of type-s if for all $\omega\in\Omega$ for which $N\omega\neq(0)$ we have

(i) $N\omega=\bigoplus_{i\in I}\Omega_i$, where each Ω_i is of type-0

and

(ii) there exists $\omega'\in N\omega$ such that $N\omega=N\omega'$ and the annihilating left ideal $(0:\omega)$ is equal to the annihilating left ideal $(0:\omega')$.

An ideal A of N is said to be s-primitive if it is the annihilator of some N-group of type-s. $J_s(N)$ is the intersection of all s-primitive ideals of N. If N has no s-primitive ideals, we define $J_s(N)$ to be N itself. A left ideal L of N is said to be s-modular if the factor N-group $N-L$ is of type-s.

We have the inclusions $J_0(N)\leqq J_{1/2}(N)\leqq J_s(N)\leqq J_1(N)$. The following theorem can be proved using Theorem 2.1 and Lemma 2.2.

Theorem 2.3. *If N is a near-ring satisfying the DCCL, then any ideal containing $J_{1/2}(N)$ is an intersection of s-modular left ideals.*

From Theorem 2.3 it follows that $J_s(N)$ is the smallest two-sided ideal containing $J_{1/2}(N)$ in the *DCCL* case. Indeed if N satisfies the *DCCL*, then $J_s(N)$ may be defined as the smallest two-sided ideal which is an interesection of 0-modular left ideals (cf. [2]).

We now give a brief description of an ideal of N which is in some sense dual to $J_s(N)$. For further details we refer the reader to [3].

Let $\mathscr{F}$ be the collection of all ideals A of N which are of the form $A=\bigoplus_{i\in I}Ne_i$, where

(i) Ne_i is of type-0 and $e_i\in Ne_i$ for each $i\in I$

https://doi.org/10.1017/S0013091500028844 Published online by Cambridge University Press

and

(ii) $e_i^2 = e_i$ and $e_i e_j = 0$ if $i > j$ for some ordering on the index set I.

An element of $\mathscr{F}$ is said to be *$\mathscr{F}$-decomposable* or to have an *$\mathscr{F}$-decomposition.*

Using Zorn's lemma one can show that if $\mathscr{F} \neq \emptyset$, then it possesses a unique maximal element which we call the socle-ideal of N and which we denote by $\mathrm{Soi}(N)$. If $\mathscr{F} = \emptyset$ we define $\mathrm{Soi}(N)$ to be the zero ideal. If N satisfies the *DCCL*, then $\mathrm{Soi}(N)$ has an $\mathscr{F}$-decomposition $\mathrm{Soi}(N) = \bigoplus_{i=1}^k Ne_i$, with $\{e_1, \ldots, e_k\}$ an orthogonal indempotent set. Moreover we have

Theorem 2.4. [3] *Let N be a near-ring wich satisfies the DCCL. Then $N = \mathrm{Soi}(N) \oplus L$ where L is a left ideal containing $J_{1/2}(N)$.*

Theorem 2.5. [3] *Let N be a near-ring which satisfies the DCCL. Then $J_s(N)$ is the unique smallest ideal amongst all ideals B of N for which $\mathrm{Soi}(N/B) = N/B$. Moreover, $J_s(N) = (0)$ if and only if $\mathrm{Soi}(N) = N$.*

3. Proof of the main theorem

We begin by giving an outline of the concept of a *nil-rigid* series for the near-ring N.

Let $\mathrm{nil}(N)$ denote the nil radical of the near-ring N. An ideal A of N is said to be *rigid* if whenever B is an ideal of N contained in A, then

$$(A/B) \cap \mathrm{nil}(N/B) = (0).$$

In [9] Scott showed that there is a unique maximal rigid ideal in N which he calls the crux of N and which we will denote by $\mathrm{Crux}(N)$. In [10], $\mathrm{Crux}(N)$ is arrived at via a more general route. It emerges as a semi-simple part corresponding to the Baer lower radical.

Definition. For the near-ring N satisfying the ascending chain condition for ideals, let $L_1 = \mathrm{nil}(N)$ and let C_1 be the ideal containing L_1 such that

$$C_1/L_1 = \mathrm{Crux}(N/L_1).$$

Further, let L_2 be the ideal containing C_1 such that

$$L_2/C_1 = \mathrm{nil}(N/C_1).$$

If α is a non-limit ordinal define L_α to be the ideal of N containing $C_{\alpha-1}$ such that

$$L_\alpha/C_{\alpha-1} = \mathrm{nil}(N/C_{\alpha-1})$$

and C_α to be the ideal containing L_α such that

$$C_\alpha/L_\alpha = \mathrm{Crux}(N/L_\alpha).$$

If α is a limit ordinal, define

$$C_\alpha = \bigcup_{\beta<\alpha} C_\beta$$

and

$$L_\alpha = \bigcup_{\beta<\alpha} L_\beta.$$

The transfinite ascending sequence of ideals.

$$\{0\}, L_1, C_1, L_2, C_2, \ldots$$

is called the *nil-rigid* series of the near-ring N.

For the near-ring N we state the following facts concerning Crux(N) and nil-rigid series.

(a) Soi(N)$\leqq$Crux(N) and if N satisfies the *DCCN*, then Crux(N)$-$Soi(N), [**3**].

(b) For near-rings N satisfying the *DCCN*˙ Soi(N)$=$Crux$N$ and nil(N)$=J_0(N)$. In this case Crux(N) and nil(N) cannot be simultaneously zero. Thus in this case nil-rigid series are strictly ascending [**10**].

(c) If N satisfies the *DCCN*, then the nil-rigid series of N is finite and there exists a positive integer α such that $C_\alpha = N$. Recall that N is not nilpotent modulo any proper ideal; hence the series cannot stop at an L_β for any β.

The positive integer α is called the *nil-rigid length* or the *nil-rigid class number* of N if $C_\alpha = N$.

Lemma 3.1. *Let N be a near-ring satisfying the DCCN and let α be its nil-rigid length. The s-radical $J_s(N)$ is nilpotent if and only if $\alpha = 1$.*

Proof. $J_s(N)$ nilpotent implies that $J_s(N) = J_0(N)$ and hence by Theorem 2.5 Crux$(N/J_0)$$=Soi(N/J_s)$$=N/J_s$. It follows that the nil-rigid series for N is

$$L_1 = J_0(N) = J_s(N), \qquad N = C_1.$$

Conversely, $\alpha = 1$ implies $N/J_0 = C_1/J_0 =$ Soi(N/J_0) and again by Theorem 2.5 we have $J_0(N) = J_s(N)$. Hence $J_s(N)$ is nilpotent.

We note that if N has nil-rigid length $\alpha = 1$ and $J_s(N) \neq (0)$, then the main theorem is true for N. We need only prove the main theorem for $\alpha > 1$. For such cases $J_s(N) \neq (0)$ by Lemma 3.1.

Lemma 3.2. *Let N be a near-ring which satisfies the DCCL. For any ideal A of N, $J_s(N/A) = (J_s(N) + A)/A$. Moreover, $J_s(N/A)$ is nilpotent if and only if $(J_s(N))^m \leqq A$ for some positive integer m, depending on A.*

Proof. By Theorem 2.3 $J_s(N)+A$ is an intersection of s-modular left ideals of N and hence $(J_s(N)+A)/A$ is an intersection of s-modular left ideals of N/A. Thus $(J_s(N)+A)/A \geqq J_s(N/A)$. On the other hand, the intersection of all s-modular left ideals of N containing A also contains $J_s(N)+A$ and it follows that $J_s(N/A) \geqq (J_s(N)+A)/A$.

Hence we have equality

$$J_s(N/A)=(J_s(N)+A)/A.$$

Lemma 3.3. *Let N be a near-ring satisfying the DCCN. If in the nil-rigid series of N we have $L_\alpha < C_\alpha = N$ and $\alpha > 1$, then $J_s(N/C_{\alpha-1})$ is non-zero and nilpotent. Moreover, $(J_s(N))^m \leqq C_{\alpha-1}$ for some positive integer m.*

Proof. $C_\alpha = N$ implies that $\mathrm{Soi}(N/L_\alpha)=N/L_\alpha$. Hence by Theorem 2.5, $L_\alpha \geqq J_s(N)$ and we have

$$J_0(N/C_{\alpha-1})=L_\alpha/C_{\alpha-1} \geqq (J_s(N)+C_{\alpha-1})/C_{\alpha-1}=J_s(N/C_{\alpha-1}).$$

Thus $J_s(N/C_{\alpha-1})=J_0(N/C_{\alpha-1})$ is nilpotent.

Since the nil-rigid series of N is strictly ascending $L_\alpha \neq C_{\alpha-1}$ and we have that $J_s(N/C_{\alpha-1}) \neq (0)$.

Lemma 3.4. *Let N be a near-ring which satisfies the DCCN. Suppose that in the nil-rigid series for N we have $L_\alpha < C_\alpha = N$ and $\alpha > 1$. Then there exists an ideal A of N such that A is uniquely minimal amongst all ideals B of N for which $J_s(N/B)$ is non-zero and nilpotent. Moreover,*

$$A \leqq J_s(N) \cap C_{\alpha-1} \quad \text{and} \quad J_s(N/A)=J_s(N)/A.$$

Proof. Let $\xi=\{B: B \text{ an ideal of } N, J_s(N/B) \text{ is nilpotent}\}$. By Theorem 3.3 $\xi \neq \emptyset$. Also, if $B \in \xi$ put $C=J_s(N) \cap B$. Then since $(J_s(N))^m \leqq B$ for some positive integer m (Lemma 3.2), it follows that $(J_s(N))^r \leqq C$ for some positive integer r and hence $C \in \xi$. Since N satisfies the $DCCN$, ξ has a minimal element A, say. By the above observation we may assume that A is contained in $J_s(N)$. If $D \in \xi$, then by the above $D \cap A \in \xi$ so that $A \leqq D$ by the minimality of A. Thus the ideal A is uniquely minimal in ξ. Clearly, by Lemmas 3.2 and 3.3 $J_s(N/A)=(J_s(N))/A$ and we see that $A \leqq J_s(N) \cap C_{\alpha-1}$. Finally, we show that $J_s(N/A) \neq (0)$, that is $J_s(N) \neq A$. If $J_s(N)=A$, then from the proof of Lemma 3.3, we have

$$L_\alpha/C_{\alpha-1}=(J_s(N)+C_{\alpha-1})/C_{\alpha-1}=(A+C_{\alpha-1})/C_{\alpha-1}=C_{\alpha-1}/C_{\alpha-1}=0.$$

This contradicts the fact that the nil-rigid series for N is strictly ascending. Thus $J_s(N) \neq A$.

Corollary. *If $J_s(N) \neq (0)$, then it cannot be idempotent; that is $(J_s(N))^2 \neq J_s(N)$.*

We remark that Lemma 3.4 is false in the case of the radicals $J_1(N)$ and $J_2(N)$. For suppose N is a near-ring which satisfies the $DCCN$ and let $J_s(N)=(0)$. Then $N=\mathrm{Soi}(N)$ is idempotent and hence if $J_1(N)\neq(0)$, it is idempotent.

The main theorem, stated in the introduction, can now immediately be deduced from Lemmas 3.1 and 3.4.

Definition. The unique minimal ideal A of the main theorem will be called the *s-socle.*

We conclude this section by discussing some of the properties of the s-socle.

Theorem 3.5. *Let N be a near-ring with DCCN and suppose that in the nil-rigid series for N we have $L_\alpha < C_\alpha = N$ with $\alpha > 1$. Then the s-socle is not contained in $L_{\alpha-1}$.*

Proof. Let A be the s-socle of N. If $\alpha=2$, then $L_{\alpha-1}=L_1=J_0(N)$ and $A\leqq L_1=J_0(N)$ implies that $J_s(N)$ is nilpotent.

But by Lemma 3.1 $J_s(N)$ is nilpotent if and only if $\alpha=1$. Thus A is not contained in L_1 and the theorem is true for $\alpha=2$. Now suppose that $\alpha>2$. If $A\leqq L_{\alpha-1}$, then since $(J_s(N))^m\leqq A$ for some positive integer m and $L_{\alpha-1}/C_{\alpha-2})$ is a nilpotent ideal of $N/C_{\alpha-2}$ we have that $(J_s(N))^r\leqq C_{\alpha-2}$ for some positive integer r. Thus

$$J_0(N/C_{\alpha-2})=L_{\alpha-1}/C_{\alpha-2}\geqq(J_s(N)+C_{\alpha-2})/C_{\alpha-2}=J_s(N/C_{\alpha-2}).$$

It follows that $C_{\alpha-1}/L_{\alpha-1}=\mathrm{Soi}(N/L_{\alpha-1})$ so that $N=C_{\alpha-1}$, contradicting the strict ascendency of the nil-rigid series of N. This proves the theorem.

Now $J_0(N)\cap\mathrm{Soi}(N)=(0)$ for otherwise $J_0(N)$ would contain an idempotent $e\neq0$, appearing in the $\mathscr{F}$-decomposition of $\mathrm{Soi}(N)$. In fact, in [3] it is shown that $J_{1/2}(N)\cap\mathrm{Soi}(N)=(0)$. Clearly, $\mathrm{Soi}(N)\oplus J_0(N)$ has an $\mathscr{F}$-decomposition modulo $J_0(N)$. Thus $\mathrm{Soi}(N)\oplus J_0(N)\leqq C_1$, where C_1 is the ideal in the nil-rigid series of N such that $\mathrm{Soi}(N/J_0(N))=C_1/J_0(N)$. Consequently we have the following theorem which follows immediately from Theorem 3.5.

Theorem 3.6. *Let N be a near-ring with DCCN such that in the nil-rigid series for $N, L_\alpha < C_\alpha = N$ with $\alpha>2$. Then the s-socle is not contained in* $\mathrm{Soi}(N)$.

Thus Theorem 3.6 tells us that, in order to find examples of near-rings for which the s-socle is contained in $\mathrm{Soi}(N)$ we need look no further than near-rings with nil-rigid class number equal to 2. Indeed, from Theorem 3.5 and the fact that the s-socle is contained in $C_{\alpha-1}$ we can immediately deduce the following theorem.

Theorem 3.7. *Let N be a near-ring which satisfies the DCCN and suppose that in the nil-rigid series for N we have $L_\alpha < C_\alpha = N$. Then the s-socle A of N is contained in C_1 if and only if $\alpha=2$.*

The following theorem provides us with a class of near-rings in which the s-socle is always contained in $\mathrm{Soi}(N)$.

Theorem 3.8. *Let N be a near-ring with DCCN and let the nil-rigid length of N be 2. If $\mathrm{Soi}(N)\oplus J_0(N)=C_1$, then the s-socle, A of N is contained in $\mathrm{Soi}(N)$.*

Proof. Since $C_{\alpha-1}=C_1$ we have $A\leqq C_1=\mathrm{Soi}(N)\oplus J_0(N)$. But $J_0(N)$ is a nilpotent ideal of N; hence $\mathrm{Soi}(N)\oplus J_0(N)$ is nilpotent modulo $\mathrm{Soi}(N)$. It follows that $A+\mathrm{Soi}(N)$ is nilpotent modulo $\mathrm{Soi}(N)$ and that $A^h\leqq\mathrm{Soi}(N)$ for some positive integer h. Thus $(J_s(N))^k\leqq\mathrm{Soi}(N)$. But A is uniquely minimal amongst all ideals B of N with the property that $(J_s(N))^r\leqq B$ for some positive integer r. It follows that $A\leqq\mathrm{Soi}(N)$.

We remark that the equality $C_1=\mathrm{Soi}(N)\oplus J_0(N)$ does not always hold in the case of near-rings N with nil-rigid class number equal to 2. We give the following example by way of illustration in which we use the representation theory developed in [5]. We note that, in the case of d.g. near-rings, with $DCCN$ the critical ideal, $\mathrm{Crit}(N)$ of [5] is precisely the socle ideal, $\mathrm{Soi}(N)$.

Example. Let V be a reduced free group on m generators whose laws are precisely the universal laws of S_5, the symmetric group on 5 symbols. Take m at least as great as the minimum number of generators for S_5. Let N be the Neumann d.g. near-ring [7] associated with V; N is finite with an identity. Now $V-K\cong S_5$ for some normal subgroup K of V and so there is a left ideal L such that $N-L$ is a faithful N-group. Moreover, every N-subgroup of $N-L$ is monogenic and there is a one–one lattice correspondence between the subgroups of S_5 and the N-subgroups of $N-L$. Under this correspondence the N-kernels of $N-L$ correspond to the normal subgroups of S_5. Now S_5 does not have a simple subgroup as a direct summand so that $N-L$ cannot have an N-group of type-0 as a direct summand. By the theory in [5] $\mathrm{Crit}(N)=\mathrm{Soi}(N)$ annihilates $N-L$ and $\mathrm{Soi}(N)=(0)$ because $N-L$ is faithful. We note that the only simple groups which are homomorphic images of subgroups of S_5 and A_5 are A_5 itself and groups of prime order. Let Ω be the subgroup of $N-L$ which corresponds to A_5. Then $\bar{N}=N/J_0(N)$ is 0-primitive with Ω a faithful $\bar{N}$-group. Also, $\mathrm{Soi}(\bar{N})=C_1/J_0(N)$ is idempotent and consists of a direct sum of copies of Ω. Hence it does not annihilate Ω. Now the $\bar{N}$-subgroups of Ω corresponding to subgroups of A_5 isomorphic to A_4 do not have $\bar{N}$-subgroups of type-0 as direct summands and hence they are annihilated by $\mathrm{Soi}(\bar{N})$. Let Δ be a faithful $\bar{\bar{N}}$-group, where $\bar{\bar{N}}=N/C_1$. Further, let Γ be any type-0 $\bar{\bar{N}}$-group which is a homomorphic image of an $\bar{\bar{N}}$-subgroup of Δ. Then by the above Γ is of type-2 and we have $J_s(\bar{\bar{N}})=J_2(\bar{\bar{N}})=J_0(\bar{\bar{N}})$.

Thus the nil-rigid series for N is

$$L_1=J_0(N),\,C_1,\,L_2,\,C_2=N$$

where $L_2/C_1=J_0(\bar{\bar{N}})$. We see that $\mathrm{Soi}(N)\oplus J_0(N)=J_0(N)\neq C_1$.

The following theorem is immediate.

https://doi.org/10.1017/S0013091500028844 Published online by Cambridge University Press

Theorem 3.9. *If N is a near-ring which satisfies the DCCN, then the s-socle A is nilpotent, if and only if $J_s(N)$ is nilpotent.*

Corollary 1. *The s-socle A is contained in $J_0(N)$ if and only if the nil-rigid class number of N is* 1.

Corollary 2. *If the s-socle of a near-ring is not zero, then it cannot be nilpotent.*

Theorem 3.10. *Let N be a near-ring which satisfies the DCCN. Then $J_{1/2}(N)$ is contained in the s-socle A if and only if $J_{1/2}(N)=(0)$.*

Proof. If $J_{1/2}(N)=(0)$, then certainly $J_{1/2}(N)\leqq A$. In fact, $J_{1/2}(N)=J_s(N)$ by Theorem 2.3 so that $A=J_{1/2}(N)=(0)$. Conversely, if $J_{1/2}(N)\leqq A$, then by the remark following Theorem 2.3, $J_s(N)\leqq A$. Hence $J_s(N)=A$ which is not possible if $J_s(N)\neq(0)$ by the main theorem. This proves the result.

Theorem 3.11. *Let N be a near-ring which satisfies the DCCN. Then the s-socle A is contained in $J_{1/2}(N)$ if and only if the nil-rigid class number α of N is* 1.

Proof. If $A\leqq J_{1/2}(N)$, then $J_s(N)$ is nilpotent so that $\alpha=1$ by Lemma 3.1, conversely, $\alpha=1$ implies $J_s(N)$ is nilpotent so that $A=(0)\leqq J_{1/2}(N)$.

4. A decomposition for $J_s(N)$

We begin by characterizing the left ideals of the near-ring N which contain $J_{1/2}(N)$. This will enable us to express $J_{1/2}(N/B)$, in terms of $J_{1/2}(N)$, where B is any ideal of N.

Lemma 4.1. *Let N be a near-ring which satisfies the DCCL. Any left ideal L of N which contains $J_{1/2}(N)$ is an intersection of* 0*-modular left ideals.*

Proof. Since $L-J_{1/2}(N)$ is an N-kernel of $N-J_{1/2}(N)$ from Theorem 2.1 and Lemma 2.2 we have $N-L=\bigoplus_{i=1}^{k}\Delta_i$, where Δ_i is an N-group of type-0 for $i=1,\dots,k$. Since L contains $J_{1/2}(N)$ it is a modular left ideal [**2, 8**]. That is, there exists $e\in N$ such that $ne-n\in L$ for all $n\in N$. Now the coset $e+L$ has a unique expression of the form

$$e+L=\bar{e}_1+\bar{e}_2+\cdots+\bar{e}_k,\ \bar{e}_i\in\Delta_i\qquad i=1,\dots,k.$$

Putting $\bar{e}_i=e_i+L$, $i=1,\dots,k$ we have for any $n\in N$

$$\begin{aligned}n+L=ne+L&=n(\bar{e}_1+\bar{e}_2+\cdots+\bar{e}_k)\\&=n\bar{e}_1+n\bar{e}_2+\cdots+n\bar{e}_k,\end{aligned}$$

because of the validity of left distribution over a direct sum of N-kernels. Thus we have

$$n+L=(ne_1+L)+(ne_2+L)+\cdots+(ne_k+L).$$

It is clear that Δ_i is monogenic by e_i+L for $i=1,\ldots,k$ and hence the annihilating left ideal $(0:\bar{e}_i)$ is 0-modular. It follows readily that $L=\bigcap_{i=1}^{k}(0:\bar{e}_i)$.

Theorem 4.2. *Let N be a near-ring satisfying the DCCL and B any ideal of N. Then $J_{1/2}(N/B)=(J_{1/2}(N)+B)/B$.*

Proof. Clearly, $J_{1/2}(N/B)\geqq(J_{1/2}(N)+B)/B$. On the other hand, $J_{1/2}(N)+B$ is a left ideal containing $J_{1/2}(N)$ and hence it is an intersection of 0-modular left ideals of N, by Lemma 4.1.

Consequently, $(J_{1/2}(N)+B)/B$ is an intersection of 0-modular left-ideals of the factor near-ring N/B and hence must contain $J_{1/2}(N/B)$. The equality follows.

For the remainder of this section we will assume that the near-ring N satisfies the *DCCN* and has nil-rigid length $\alpha>1$. We note that $\mathrm{Soi}(N/L_{\alpha-1})=\mathrm{Crux}(N/L_{\alpha-1})=C_{\alpha-1}/L_{\alpha-1}$. Also, A will denote the s-socle throughout.

Now Theorem 2.4 tells us that

$$N/L_{\alpha-1}=C_{\alpha-1}/L_{\alpha-1}\oplus L/L_{\alpha-1},\quad \text{where}\quad L_{\alpha-1}/C_{\alpha-2}=J_0(N/C_{\alpha-2})$$

if $\alpha>2$, $L_1=J_0(N)$ and $L/L_{\alpha-1}$ is a left ideal of $N/L_{\alpha-1}$ containing $J_{1/2}(N/L_{\alpha-1})$. By Theorem 4.2 we have $J_{1/2}(N/L_{\alpha-1})=(J_{1/2}(N)+L_{\alpha-1})/L_{\alpha-1}$. From Lemma 2.2 we deduce that the factor near-ring $N/L_{\alpha-1}$ has the decomposition

$$N/L_{\alpha-1}=(A+L_{\alpha-1})/L_{\alpha-1}\oplus L'/L_{\alpha-1}\oplus L/L_{\alpha-1},$$

because $(A+L_{\alpha-1})/L_{\alpha-1}$ is an N-kernel of $C_{\alpha-1}/L_{\alpha-1}$. Using the modular law we obtain the decomposition

$$\begin{aligned}J_s(N/L_{\alpha-1})&=(J_s(N)+L_{\alpha-1})/L_{\alpha-1}\\&=[(A+L_{\alpha-1})/L_{\alpha-1}]\oplus[(J_s(N)+L_{\alpha-1})/L_{\alpha-1}\cap(L'/L_{\alpha-1}\oplus L/L_{\alpha-1})].\end{aligned}$$

Since the sum on the right is direct and $(J_s(N))^r\leqq A$ for some positive integer r it follows that

$$(J_s(N)+L_{\alpha-1})/L_{\alpha-1}\cap(L'/L_{\alpha-1}\oplus L/L_{\alpha-1})\tag{1}$$

is a nilpotent left ideal of $N/L_{\alpha-1}$ and hence is contained in $J_{1/2}(N/L_{\alpha-1})$. But $J_{1/2}(N/L_{\alpha-1})\leqq L/L_{\alpha-1}$ therefore the intersection (1) above is precisely $J_{1/2}(N/L_{\alpha-1})$. We have shown that the following decomposition of $J_s(N/L_{\alpha-1})$ occurs.

$$J_s(N/L_{\alpha-1})=(A+L_{\alpha-1})/L_{\alpha-1}\oplus J_{1/2}(N/L_{\alpha-1}).$$

Thus we have proved the following theorem.

Theorem 4.3. *Let N be a near-ring which satisfies the DCCN and such that N has nil-rigid class number $\alpha > 1$. Then the s-radical $J_s(N)$ decomposes as*

$$J_s(N) = J_{1/2}(N) + A + B$$

where A is the s-socle of N and $B = J_s(N) \cap L_{\alpha-1}$.

Corollary. *If the near-ring N has nil-rigid length $\alpha = 2$, then*

$$J_s(N) = J_{1/2}(N) + A.$$

Proof. This follows immediately from

$$L_{\alpha-1} = L_1 = J_0(N) \leqq J_{1/2}(N).$$

Whether $B = J_s(N) \cap L_{\alpha-1} \leqq A + J_{1/2}(N)$ always implies that $\alpha = 2$ is an open question. For $\alpha > 1$ the equality $B = A + J_{1/2}(N)$ cannot hold for otherwise we would have a contradiction to Theorem 3.5.

Since $\mathrm{Soi}(N)$ is contained in C_1 for any near-ring N, from Theorem 3.7 we must have $\alpha \leqq 2$ in order that the s-socle be contained in $\mathrm{Soi}(N)$. By Theorem 2.7 of [3] an ideal is contained in $\mathrm{Soi}(N)$ if and only if its intersection with $J_{1/2}(N)$ is the zero ideal. From this it is easy to see that if $A \leqq \mathrm{Soi}(N)$, then $J_s(N) = A \oplus J_{1/2}(N)$. Thus we see that the decomposition given in Section 4 of [3] occurs in a relatively small class of near-rings.

We have seen that if N is a near-ring with $DCCN$ and with nil-rigid class number $\alpha > 1$, then $A \leqq J_s(N) \cap C_{\alpha-1}$. We now investigate this relationship further.

Suppose C is an ideal of N with an $\mathscr{F}$-decomposition $C = \bigoplus_{i=1}^{s} Ne_i$. Let $B \neq 0$ be an ideal of N contained in C. Then any $b \in B$ has a unique expression of the form $b = n_1e_1 + \cdots + n_se_s$. Since the e_i, $i = 1, \ldots, s$ form an orthogonal idempotent set we have that $be_i = n_ie_i$ and hence $b = be_1 + \cdots + be_s$. But B is an ideal so tht each component be_i is in B and $B = Be_1 \oplus \cdots \oplus Be_s$. Now each Be_i is an N-kernel of Ne_i and hence re-indexing if necessary, we have

$$B = Ne_1 \oplus \cdots \oplus Ne_r, \quad \text{for} \quad r \leqq s$$

because each Ne_i is of type-0.

Thus we see that B has an $\mathscr{F}$-decomposition with components from the summands in C.

Now let N be a near-ring which satisfies the $DCCN$ and having nil-rigid class number $\alpha > 1$. We have that $C_{\alpha-1}$ has an $\mathscr{F}$-decomposition modulo $L_{\alpha-1}$, that is $C_{\alpha-1}/L_{\alpha-1} = \bigoplus_{i=1}^{m} \Delta_i$, where Δ_i is an $(N/L_{\alpha-1})$-group of type-0 and $\Delta_i = N/L_{\alpha-1}(\bar{e}_i)$ with $\{\bar{e}_1, \ldots, \bar{e}_m\}$ an orthogonal idempotent set, $\bar{e}_i = e_i + L_{\alpha-1}$, $i = 1, \ldots, m$.

By the previous discussion $((J_s(N) \cap C_{\alpha-1} + L_{\alpha+1})/L_{\alpha-1}$ is $\mathscr{F}$-decomposable. Also

$e_i+L_{\alpha-1}\in[(J_s(N)\cap C_{\alpha-1})+L_{\alpha-1}]/L_{\alpha-1}$ implies that $e_i+L_{\alpha-1}=x+L_{\alpha-1}$, where $x\in J_s(N)\cap C_{\alpha-1}$.

We have that $e_i+L_{\alpha-1}=e_i^m+L_{\alpha-1}=x^m+L_{\alpha-1}$ for any positive integer m and since $J_s(N)^k\leqq A$, where A is the s-socle and k some positive integer, it follows that $e_i+L_{\alpha-1}\in(A+L_{\alpha-1})/L_{\alpha-1}$. Thus

$$((J_s(N)\cap C_{\alpha-1})+L_{\alpha-1})/L_{\alpha-1}=(A+L_{\alpha-1})/L_{\alpha-1}.$$

Thus we have proved the following theorem.

Theorem 4.4. *Let N be a near-ring satisfying the DCCN and having nil-rigid class number equal to $\alpha>1$. Then $J_s(N)\cap C_{\alpha-1}=A+(J_s(N)\cap L_{\alpha-1})$, where A is the s-socle of N.*

Corollary 1. *If $\alpha=2$, then $J_s(N)\cap C_{\alpha-1}=J_s(N)\cap C_1=A+J_0(N)$.*

Proof. $L_{\alpha-1}=J_0(N)$ and $J_s(N)\cap J_0(N)=J_0(N)$.

Corollary 2. *If $\alpha=2$ and $A\leqq\mathrm{Soi}(N)$, then $J_s(N)\cap\mathrm{Soi}(N)=A$.*

Acknowledgements. This research was completed whilst I was a visitor at the Universities of Edinburgh and Nottingham. I thank the Anglo-American Corporation, the British Council and the Edinburgh Mathematical Society for every assistance given. I also wish to express my appreciation to Drs R. R. Laxton and J. D. P. Meldrum for their valuable comments.

REFERENCES

1. J. F. T. Hartney, Radicals and antiradicals in near-rings (Ph.D. thesis, University of Nottingham, 1979).

2. J. F. T. Hartney, A radical for near-rings, *Proc. Roy. Soc. Edinburgh* **93A** (1982), 105–110.

3. J. F. T. Hartney, An antiradical for near-rings, *Proc. Roy. Soc. Edinburgh* **96A** (1984), 185–191.

4. R. R. Laxton, Prime ideals and the ideal radical of a distributively generated near-ring, *Math. Z.* **83** (1964), 8–17.

5. R. L. Laxton and A. W. Machin, On the decomposition of near-rings, *Abh. Math. Sem. Univ. Hamburg* **38** (1972), 221–230.

6. J. D. P. Meldrum, *Near-rings and their Links with Groups* (Research Notes Math. **134,** Pitman, 1985).

7. Hanna Neumann, On varieties of groups and their associated near-rings, *Math. Z.* **65** (1956), 36–69.

8. Gunter Pilz, *Near-rings* (North-Holland Math. Studies **23,** 1977).

9. S. D. Scott, *Near-rings and Near-ring Modules* (Ph.D. thesis, Australian National University, 1970).

10. S. D. Scott, Formation radicals for near-rings, *Proc. London Math. Soc.*. (*3*) **25** (1972), 441–464.

Department of Mathematics
University of the Witwatersrand
Johannesburg, South Africa
2050