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THICK GROUPS AND ESSENTIALLY FINITELY INDECOMPOSABLE GROUPS

KHALID BENABDALLAH AND ROBERT WILSON

In this article, we examine the relationship between the notions of thick groups and that of essentially finitely indecomposable (e.f.i.) groups. These groups are indecomposable objects in the quotient category of the category of abelian groups induced by the Serre-class of bounded groups.

John M. Irwin conjectured that these two concepts are equivalent. We show here that this is true provided e.f.i. groups have the purification property for certain subsocles. We define a seemingly new concept: the notion of quasi-large subgroup of a group by dropping off the condition of full invariance in R. S. Pierce's definition (see [**6**]). We show that thickness defined in terms of large subgroup is equivalent to thickness defined through quasi-large subgroups. This point of view introduces some simplifications in many aspects of problems dealing with thickness. As an example, we use our results to construct a somewhat more conceptual proof of a result on small homomorphisms due to C. Meggiben [**5**].

All groups considered are primary abelian groups. The notation is that of [**4**]. The symbol $\bigoplus_c$ denotes direct sums of cyclic groups.

1. Preliminaries. In [**2**] it was shown that all K-high subgroups of G are bounded if and only if there exists $n \in Z^+$ such that $(K + p^n G[p])/K$ is finite and K contains the socle of the divisible subgroup of G. For our purposes here, we need the following characterization of such subgroups:

THEOREM 1.1. *Let K be a subgroup of a reduced group G. Then all K-high subgroups are bounded if and only if for every pure subgroup H containing K, G/H is the direct sum of a bounded group and a finite rank divisible group.*

Proof. If H is a pure subgroup containing K, then G/H contains no unbounded direct sum of cyclic groups; otherwise, G would contain a copy of it disjoint from H and thus from K. Therefore G/H is the direct sum of a bounded group and a finite rank divisible group.

Conversely if some K-high subgroup M of G is unbounded, then since M is reduced there exists a basic subgroup B of M of infinite corank. Now, $G/B = (M/B) \oplus (R/B)$ where R can be chosen to contain K. It turns out that R is a pure subgroup and G/R is divisible of infinite rank.

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We recall that a subsocle $S \subset G[p]$ is said to be *strongly purifiable* if every pure subgroup K such that $K[p] \subset S$ can be extended to a pure subgroup H such that $H[p] = S$.

THEOREM 1.2. *Let K be a subgroup of G such that all K-high subgroups are bounded. Then $K[p]$ is strongly purifiable.*

Proof. Let $K[p] = S$ and let H be a maximal pure subgroup such that $H[p] \subset S$. Then by [**1**, Lemma 2.3.], $(H + S)/H \subset (G/H)^1$ and is disjoint from the divisible part of G/H. But one can see easily that G/H has a bounded basic subgroup; therefore $(G/H)^1$ is divisible and $(H + S)/H = 0$, so that $H[p] = S$.

Along similar lines we have the following results:

THEOREM 1.3. *Let H be a subgroup of G. The following properties are equivalent.*
(1) *K pure and $K \supset H$ implies G/K is bounded.*
(2) *K pure and $K \supset H$ implies G/K is reduced.*
(3) *There exists K pure such that $K[p] = H[p]$ and G/K is bounded.*
(4) *$H \supset p^nG[p]$ for some $n \in Z^+$.*

Proof. We show only (2) implies (3), the other implications being easily verified. We apply the same reasoning as that used at the end of the proof of Theorem 1.1 to see that all H-high subgroups of G are bounded. Then, by Theorem 1.2 there exists K pure such that $K[p] = H[p]$. By Theorem 1.1, $G/K = B/K \oplus D/K$ where B/K is bounded and D/K is divisible. Now B is pure and G/B is divisible, therefore $B[p]$ is dense in $G[p]$ in the p-adic topology of G. Since $B[p] \supset H[p]$, there exists a subgroup M of G maximal with respect to $M \supset H$ and $M[p] = B[p]$. By [**4**, Theorem 66.3], M is pure and G/M is divisible. Therefore, $M = G$ and since $B[p] = M[p] = G[p]$, $B = G$.

2. Quasi-large subgroups. We say that a subgroup H of G is *quasi-large* (abbreviated q.l.) if $G = H + B$ for every basic subgroup of G. This is a generalization of the notion of large subgroups first introduced in [**6**] where it is required further that H be fully-invariant in G. It is easy to see that if H is q.l. in G then p^nH is q.l. for every $n \in Z^+$.

We have the following comprehensive characterizations.

THEOREM 2.1. *Let H be a subgroup of G. The following properties are equivalent.*
(1) *H is* q.l. *in G.*
(2) *For every $n \in Z^+$, K pure and $K \supset p^nH$ implies G/K is reduced.*
(3) *There exists a monotone increasing sequence $n_1, n_2, \ldots$ in Z^+ such that $(p^{n_i}G)[p] \subset p^iH$, for all $i \in Z^+$.*
(4) *There exists a monotone increasing sequence $m_1, m_2, \ldots$ in Z^+ such that $(p^{m_i}G)[p^i] \subset H$ for all $i \in Z^+$.*
(5) *H contains a large subgroup of G.*
(6) *$G[p] \subset p^nH + B$ for every basic subgroup B of G and $n \in Z^+$.*

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Proof. (1) $\Rightarrow$ (2). Clearly, since H q.l. implies $p^n H$ q.l. in G, we need only show this for H. Let K be a pure subgroup of G containing H. Then $G/H = (D/H) \oplus (R/H)$ where D/H is divisible and R/H is reduced. Now, R is pure in G and G/R is divisible; therefore if B is basic in R it is also basic in G and $G = B + H \subset R$. Thus $G/H = R/H$ is reduced.

(2) $\Rightarrow$ (3). We apply Theorem 1.3 to $p^n H$ for every n.

(3) $\Rightarrow$ (4). Let $m_1 = n_1$ and let $k = m_i + i + 1$ and define $m_{i+1} = n_k$. Then it is easy to verify that $(p^{m_i}G)[p^i] \subset H$.

(4) $\Rightarrow$ (5). $\sum (p^{m_i}G)[p^i]$ is a large subgroup of G contained in H.

(5) $\Rightarrow$ (6). Obvious.

(6) $\Rightarrow$ (1). A routine induction argument shows that $G[p^i] \subset H + B$ for all $i \in Z^+$. Therefore $H + B = G$ and H is q.l.

We remark that the equivalence of (4) and (5) was established in [**6**] and property (4) is called in [**4**] the *Pierce-condition* for a subgroup. Many results in [**6**] can be also immediately proved for q.l. subgroups. In particular, in a group G we have the following.

(1) The finite intersections of q.l. subgroups are q.l.
(2) The inverse image of q.l. subgroups by homomorphisms are q.l.
(3) A subgroup containing a q.l. is q.l.
(4) Pure q.l. subgroups are cobounded summands.
(5) K pure in G and H q.l. in G implies $H \cap K$ q.l. in K.
(6) H q.l. in K and K q.l. in G implies H q.l. in G.
(7) H q.l. implies $H[p]$ supports a pure subgroup of G.

3. Thick groups and e.f.i. groups. A group G is *thick* if $G/K = \bigoplus_c$ implies K is q.l. in G. A group G is *e.f.i.* if it has no unbounded $\bigoplus_c$ summand. (e.f.i. = essentially finitely indecomposable) J. M. Irwin conjectured that e.f.i. and thick are equivalent properties. We prove here a weaker form of this conjecture. We need first a few lemmas.

LEMMA 3.1. *Let K be a subgroup of G such that $G/K = \bigoplus_c$. If $K[p]$ supports a pure subgroup M of G, then $G/p(M \cap K) = \bigoplus_c$.*

Proof. By [**3**, Theorem 2.2.] $G/M = \bigoplus_c$ and $G = M \oplus R$, where $R = \bigoplus_c$. Now since $(M \cap K)[p] = M[p]$, we have

$$M/(M \cap K) \cong pM/p(M \cap K) = p(M/p(M \cap K)).$$

However, $M/M \cap K \cong (K + M)/K = \bigoplus_c$, since $G/K = \bigoplus_c$, thus by [**4**, Proposition 18.3], $M/p(M \cap K) = \bigoplus_c$, and $G/p(M \cap K) \cong (M/p(M \cap K)) \oplus R$, is also $\bigoplus_c$.

THEOREM 3.2. *Let G be a group. The following properties are equivalent.*

(1) *G is thick.*
(2) *G is* e.f.i. *and $K[p]$ is purifiable for every subgroup K such that $G/K = \bigoplus_c$.*

(3) *$K \supset p^nG[p]$ for some $n \in Z^+$ for every subgroup K of G such that $G/K = \bigoplus_c$.*

Proof. (1) $\Rightarrow$ (2). If G is thick it is clearly e.f.i. and if $G/K = \bigoplus_c$ then K is q.l. so that $K[p]$ is purifiable.

(2) $\Rightarrow$ (3). Let $G/K = \bigoplus_c$ and let M be a pure subgroup of G supported by $K[p]$. Then $G/M = \bigoplus_c$ and $G = M \oplus R$ where $R = \bigoplus_c$. Now, since G is e.f.i. there exists $n \in Z^+$ such that $p^nR = 0$. Therefore, $p^nG[p] \subset M[p] = K[p] \subset K$.

(3) $\Rightarrow$ (2). In view of Theorem 2.1 (3) we need only show that if $G/K = \bigoplus_c$ then pK contains a subgroup H such that $G/H = \bigoplus_c$. Let $G/K = \bigoplus_c$; then $K \supset p^nG[p]$, thus by Theorem 1.3 $K[p]$ supports a pure subgroup M. Let $H = p(M \cap K)$; then $H \subset pK$ and $G/H = \bigoplus_c$ by Lemma 3.1.

As an application of this result we obtain immediately the following well known facts.

COROLLARY 3.3. (J. M. Irwin, unpublished). *Pure complete groups are* e.f.i. *if and only if they are thick.*

COROLLARY 3.4. *Torsion complete groups are thick.*

4. Some properties of torsion complete groups. We establish in this section two interesting properties of torsion complete groups and then use them to derive a result due to C. Meggiben [**5**]. We say that a group G is *separable* if $G^1 = 0$.

THEOREM 4.1. *Let K be a closed bounded subgroup of a separable group G. Then G is torsion complete if and only if G/K is torsion complete.*

Proof. Let G be torsion complete and suppose $K \subset G[p]$. Then K supports a pure subgroup M of G, and since K is closed M is closed and therefore $G = M \oplus L$ for some subgroup L of G. Now $G/K \cong (M/K) \oplus L$, where L is torsion complete and since $M/K \cong pM$ it is also torsion complete; therefore G/K is torsion complete. By induction we suppose that the result if true for every closed $K \subset G[p^n]$ for every torsion complete group G. Let K be a closed subgroup of G contained in $G[p^{n+1}]$. Then $K/K[p]$ is a closed subgroup of $G/K[p]$ contained in $(G/K[p])[p^n]$. Now $K[p]$ is closed in G; thus $G/K[p]$ is torsion complete and by induction $(G/K[p])/(K/K[p])$ is torsion complete and isomorphic to G/K. Reciprocally suppose G/K is torsion complete with $K \subset G[p^n]$ for some $n \in Z^+$. Then $p^nG \cong G/G[p^n] \cong (G/K)/(G[p^n]/K)$ and since G is separable $G[p^n]/K$ is a closed subgroup of G/K. Therefore p^nG is torsion complete and so is G.

The next result is a technical lemma.

LEMMA 4.2. *Let K be a closed subgroup of a torsion complete group G which is not quasi-large. Then there exists a closed subgroup H of G such that for some $m, n \in Z^+, m \geqq n$,*

$$p^mK \subset H \subset p^nK \quad \textit{and} \quad p^kG[p] \not\subset H \quad \textit{for every } k \in Z^+.$$

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Proof. Let n be the least positive integer such that $p^kG[p] \not\subset p^nK$ for every $k \in Z^+$. Such an n exists, otherwise K would be q.l. by Theorem 2.1. If $n = 0$ we set $H = K$. If $n > 0$ then there exists $s \in Z^+$ such that $p^sG[p] \subset K$. Now $K[p]$ supports a pure subgroup M and $G = M \oplus B$ where $p^sB = 0$. Let $H_1 = p(M \cap K)$; then $G/H_1 \cong (M/H_1) \oplus B$ is separable and therefore H_1 is closed in G. Furthermore $p^{s+1}K \subset H_1 \subset pK$. We use now H_1 and repeat the procedure. After at most n steps, we find the required subgroup H.

As an application we have the following.

THEOREM 4.3. [**5**, Theorem 3.1]. *Let G be a torsion complete group, A a separable group,* $\mathrm{Hom}_s(G, A) = \{\varphi \in \mathrm{Hom}(G, A) | \ker \varphi$ *is* q.l.$\}$. *Then* $\mathrm{Hom}_s(G, A) = \mathrm{Hom}(G, A)$ *if and only if A contains no unbounded torsion complete group.*

Proof. Suppose $\mathrm{Hom}_s(G, A) \neq \mathrm{Hom}(G, A)$. Then there exists a subgroup K of G which is not q.l. and G/K isomorphic to a subgroup of A. Now K is closed and by the preceding lemma there exists a closed subgroup H of G such that $p^mK \subseteq H \subseteq p^nK$ and $p^KG[p] \not\subset H$ for all $K \in Z^+$. Let M be a pure subgroup of G such that $M[p] = H[p]$; then $G = M \oplus L$ where L is an unbounded torsion complete subgroup of G. Furthermore $L \cap K$ is bounded, indeed

$$0 = L \cap M = L \cap H \supset L \cap p^mK \supset p^m(L \cap K).$$

Now by Theorem 4.1, $L/L \cap K \cong (L + K)/K$ is an unbounded torsion complete group in A.

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Université de Montréal,
Montréal, P.Q.