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ON THE MAXIMAL CIRCUMRADIUS OF A PLANAR CONVEX SET CONTAINING ONE LATTICE POINT

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We obtain a result about the maximal circumradius of a planar compact convex set having circumcentre O and containing no non-zero lattice points in its interior. In addition, we show that under certain conditions, the set with maximal circumradius is a triangle with an edge containing two lattice points.

1. Introduction

Let K be a compact convex set in the plane and let Λ be the integral lattice. We say that K is Λ-*admissible* if the interior of K contains the origin O, but no non-zero points of Λ. (This usage differs from the more conventional 'Λ is K-admissible', but is more convenient here where Λ is fixed and K varies.)

A number of results concerning the circumradius of a general convex set are known (see for example [**1, 2, 4, 5, 6, 8**]). However, there are relatively few results on the circumradius of a convex set constrained by lattice points (see for example [**7**]). In this paper we obtain a result on the circumradius of a planar, Λ-admissible, compact convex set having circumcentre O.

Let S be the square with vertices $(\pm 1, \pm 1)$. We label the eight lattice points $(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1),(1,0)$ on S by $L_1, L_2, \ldots, L_8$ respectively. We also label the half edges, $L_1L_2, L_2L_3, \ldots, L_8L_1$ of S, h_1, h_2, $\ldots$, h_8 respectively. Let O be the circumcentre of K and let C be the corresponding circumcircle. Suppose that K crosses the half edge h_i. Then h_i partitions K into two regions, one of which does not contain O. Let K' denote this region. Then if K' intercepts C, we say that K intercepts C beyond the half edge h_i. We prove here the following result concerning the circumradius of K.

Theorem. *Let K be a compact Λ-admissible convex set in the plane with circumcentre O, and circumradius $R(K)$. Then $R(K) \leqslant \alpha$ where $\alpha \approx 1.685$, unless K intercepts C beyond exactly two opposite half edges of S, in which case no upper bound can be found for $R(K)$. The bound α is best possible.*

A set for which $R(K) = \alpha$ is attained is shown in Figure 1. The exact value for α is $\alpha = \sqrt{5y^2 + 4y + 1}$ where y is the positive root of $25y^3 + 20y^2 - 3 = 0$.

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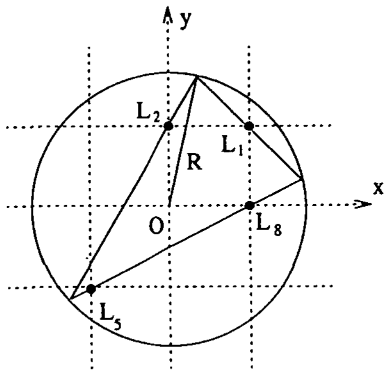

Figure 1: A set K for which $R(K) = \alpha$.

2. Narrowing the search

If K lies within the square S, then $R(K) \leqslant \sqrt{2} < \alpha$. Hence we may assume that $R(K) > \sqrt{2}$, and we need only consider sets which extend beyond the boundary of S. Convexity arguments show that the convex set K may cross at most four of the eight half edges of S, with no two of the crossed half edges comprising an edge of S. For a given set K, we call the set of half edges beyond which K intercepts C the *intercept set* of K.

It may be proved (see for example, [**9**, p. 59]) that either C contains two points of the boundary of K which are the ends of a diameter of C, or C contains three points of the boundary of K which form an acute-angled triangle containing O. Henceforth, we shall refer to this property of C as the 'circumcircle property'. Clearly by the circumcircle property, K must intercept C beyond at least two half edges of S.

If K intercepts C beyond exactly two half edges of S, using rotations about O and reflections in lines through O to discard equivalent cases, we need only consider the cases where the intercept set of K is

$$\text{(a)}\ \{h_1, h_3\} \quad \text{(b)}\ \{h_1, h_4\} \quad \text{(c)}\ \{h_1, h_5\} \quad \text{(d)}\ \{h_1, h_6\} \quad \text{(e)}\ \{h_1, h_8\}.$$

In cases (a) (b), (d) and (e), since O is interior to K, all intercepts of K and C lie in the half planes $y > 0$, $y > x$, $x > 0$ and $y > 0$ respectively. By the circumcircle property we can discard these cases.

In case (c) a Λ-admissible set K can be constructed having circumcentre O and diameter making a very small angle with the y-axis. It is easily seen that for such a set K, $R(K)$ may be made arbitrarily large. Therefore if K intercepts C beyond exactly two opposite half edges, no upper bound may be found for $R(K)$.

We may now assume that K intercepts C beyond three or four half edges of S. By the circumcircle property, C contains three points of the boundary of K which form the vertices of an acute-angled triangle T containing O. In the rest of the paper, we shall use *'triangle'* to mean a Λ-admissible closed set bounded by the edges of a triangle with circumcentre O. As $R(T) = R(K)$, it is sufficient to establish the theorem for the class of triangles. Since T crosses exactly three half edges of S, using rotations and reflections as before, we find that it suffices to consider the cases where the intercept set of T is

$$\text{(a) } \{h_1, h_3, h_5\} \quad \text{(b) } \{h_1, h_3, h_8\} \quad \text{(c) } \{h_1, h_4, h_7\} \quad \text{(d) } \{h_1, h_4, h_8\}.$$

In case (b), since O is interior to K, all intercepts of K and C lie in the half plane $y > 0$. By the circumcircle property, this case may be eliminated.

We now let F denote the family of triangles with circumcentre O and having intercept set (a), (c) or (d). It will be shown in each of these cases that F is contained in a closed disk. Therefore $R(T)$ is bounded and we let

$$m = \sup_{T \in F} R(T).$$

Let $\{T_i\}$ be a sequence in F such that $R(T_i)$ converges to m. Since T_i is contained in a closed disk, by Blaschke's selection theorem (see [**1**, p. 64]), there is a subsequence $\{T_j\}$ which converges to a triangle $\mathcal{T}$ in the Hausdorff metric. Since the circumradius function is continuous on F, we have

$$R(\mathcal{T}) = R\left(\lim_{j \to \infty} T_j\right) = \lim_{j \to \infty} R(T_j) = m.$$

Furthermore since F is compact, $\mathcal{T} \in F$. For a given intercept set, a set K with $R(K) = R(\mathcal{T})$ is referred to as a *maximal* set; in particular, a triangle T with $R(T) = R(\mathcal{T})$ is referred to as a *maximal* triangle.

In Section 3 we shall establish some properties of a maximal triangle. In Section 4, we prove two lemmas which will further narrow our search for a maximal set. In Section 5, Section 6 and Section 7, we shall prove the existence of a maximal triangle in cases (a), (c) and (d) respectively. For each case we shall also establish the uniqueness of the maximal triangle by employing the results in Section 4 to eliminate all other triangles in F as possible solutions.

Unless otherwise specified, the vertices of a triangle will be described in an anti-clockwise order.

3. Properties of a maximal triangle and a maximal set

The following lemmas establish some properties of a maximal triangle and a maximal set for intercept sets (a), (c) or (d) in Section 2.

Lemma 1. *If T is a maximal triangle, then each edge of T must contain at least one non-zero lattice point in its interior.*

Proof: Let $T = \triangle XYZ$ be a maximal triangle with edges x, y and z lying opposite the vertices X, Y and Z respectively. We suppose that there is at least one edge of T which does not contain a non-zero lattice point in its interior. If edge x (say) contains no non-zero lattice point in its interior, we enlarge T about X until x first contains such a lattice point. Denote this enlarged triangle T' with vertices X', Y', Z' and edges x', y', z'. Let K' be the set bounded by the circumcircle of T and the edges x', y' and z'. Clearly K' is Λ-admissible and $R(K') = R(T)$. By construction, K' is bounded by three straight edges and two arc boundaries of C. If now y' contains no non-zero lattice point in its interior, a small enlargement of T' about Y' results in a triangle T'' containing no non-zero lattice point in its interior. By construction, the set bounded by the edges of T'' and the circumcircle of T has three arc boundaries. On the other hand, if y' contains a lattice point in its interior, a small clockwise rotation about the lattice point (choose the lattice point closest to X' if there is more than one lattice point in the interior of y') also results in a set with three arc boundaries. We may therefore assume that K' is a set with three arc boundaries. But now a small enlargement of C about O to C' will result in a Λ-admissible set K'' bounded by the lines containing the straight edges of K' and arcs of C'. Clearly $R(K'') > R(K') = R(T)$, contradicting our assumption that T is a maximal triangle. $\square$

Lemma 2. *If K is a maximal set, then K is a triangle.*

Proof: We suppose that K is not a triangle. Then by the circumcircle property, K intercepts C in three points which form an acute-angled triangle T containing O. Since K is a maximal set, T is a maximal triangle. As $K \neq T$, there is an edge e of T whose interior lies in the interior of K. Hence e contains no non-zero lattice point in its interior. By Lemma 1, T is not a maximal triangle. Hence K is a triangle. $\square$

If each edge of a triangle contains exactly one non-zero lattice point in its interior, we call the join of a vertex to the opposite lattice point a *VL-line*.

Lemma 3. *If T is a maximal triangle then either*

(i) *its VL-lines are concurrent or*
(ii) *at least one of the edges of T contains two lattice points in its interior.*

Proof: Let $T = \triangle XYZ$ be a maximal triangle. By Lemma 1, each edge of T

contains at least one lattice point in its interior. Suppose there is exactly one lattice point, L_x, L_y and L_z lying in the interior of each of the edges x, y and z respectively and suppose that the VL-lines, XL_x, YL_y and ZL_z are not concurrent. We transform the edges of the triangle T by first rotating the edge XY (sliding the endpoints on C) about L_z through a small angle θ to $X'Y'$. This is followed by a rotation in the same direction of the edge YZ about L_x to the edge $Y'Z'$. Finally ZX is rotated in the same direction about L_y to $Z'X''$. We denote the described transformation on the edges of T by $f_c(T,\theta)$ if the edges of T are rotated in a clockwise manner, and by $f_a(T,\theta)$, if the edges of T are rotated in an anticlockwise direction. We note that $\angle YXX' = \angle X'Y'Y$ since these two angles are in the same segment of C subtended by chord $X'Y$. Similarly, $\angle Y'YX = \angle Y'X'X$. We therefore deduce that $\triangle YL_zY'$ and $\triangle X'L_zX$ are similar. Hence

$$\frac{XX'}{YY'} = \frac{X'L_z}{L_zY}.$$

Since X, Y and Z are oriented anticlockwise and θ is small and $\triangle XYZ$ contains the circumcentre in its interior, $f_a(T,\theta)$ gives $X'L_z = XL_z - \varepsilon$, where ε is small and positive. Therefore

$$\frac{XX'}{YY'} = \frac{XL_z - \varepsilon}{L_zY} < \frac{XL_z}{L_zY}.$$

Similarly

$$\frac{YY'}{ZZ'} < \frac{YL_x}{L_xZ}$$

and

$$\frac{ZZ'}{XX''} < \frac{ZL_y}{L_yX}.$$

Multiplying, we obtain

$$\frac{XX'}{XX''} < \frac{XL_z}{L_zY}.\frac{YL_x}{L_xZ}.\frac{ZL_y}{L_yX} = p, \quad \text{say},$$

where XX'/XX'' differs from p by a small amount ε_p. Similarly, if $f_c(T,\theta)$ is applied,

$$\frac{XX'}{XX''} > p,$$

where XX'/XX'' differs from p by a small amount ε_p'. We note that by Ceva's theorem (see [**3**, p. 90]), $p = 1$ if and only if the VL-lines XL_x, YL_y and ZL_z are concurrent. By assumption, $p \neq 1$.

If $p > 1$, $f_a(T,\theta)$ yields $XX'/XX'' = p - \varepsilon_p$. We choose θ so that ε_p is sufficiently small to give $XX'/XX'' > 1$. Since now $XX' > XX''$, the transformation results in a

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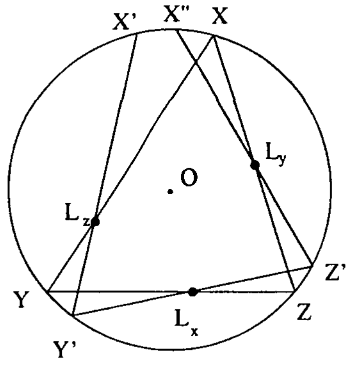

Figure 2: The effect of $f_a(T,\theta)$ on T for $p > 1$.

set K bounded by edges $X'Y'$, $Y'Z'$, $Z'X''$ and the arc $X''X'$ as shown in Figure 2. By Lemma 2, K is not a maximal set. Since $R(K) = R(T)$, T is therefore not a maximal triangle, contradicting our assumption.

If $p < 1$, $f_c(T,\theta)$ yields $XX'/XX'' = p+\varepsilon'_p$. We choose θ so that ε'_p is sufficiently small to give $XX'/XX'' < 1$. Since now $XX' < XX''$, the transformation results in a set K with an arc boundary. Arguing as before, T is not a maximal triangle.

Therefore, the maximal triangle is such that either its VL-lines are concurrent, or it has at least one edge containing two lattice points in its interior. $\square$

For future easy reference, we summarise our findings thus far in the following lemma.

Lemma 4. *A maximal set K is a triangle having a non-zero lattice point interior to each of its edges and such that either*

(i) *its VL-lines are concurrent or*
(ii) *at least one of its edges contains two lattice points in its interior.*

Proof: By Lemma 2, a maximal set K is a triangle. By Lemma 1, a maximal triangle has a non-zero lattice point interior to each of its edges and by Lemma 3, a maximal triangle has concurrent VL-lines or has at least one edge containing two lattice points. $\square$

4. Narrowing the search further

Henceforth we may restrict the members of F to those triangles with a given intercept set and with edges each containing a non-zero lattice point. For each intercept set we list the possible lattice points contained in each of the three edges of a triangle in F.

Intercept set	Edge 1	Edge 2	Edge 3
$\{h_1, h_3, h_5\}$	L_2	L_4	L_6 or L_1 or both
$\{h_1, h_4, h_7\}$	L_8	L_2 or L_4 or both	L_5 or L_7 or both
$\{h_1, h_4, h_8\}$	L_1	L_2 or L_4 or both	L_5 or L_8 or both

To each $T = \triangle XYZ$, we associate the ordered set (ℓ_1, ℓ_2, ℓ_3) called the *lattice-point set* where ℓ_i, $i = 1, 2, 3$ is a listing of the non-zero lattice points in the interior of the edges YZ, ZX and XY respectively. If T has an edge XY say, containing in its interior two non-zero lattice points L_z and L_z' in the order X, L_z, L_z', Y, then we write $\ell_3 = L_z L_z'$.

We shall now prove two lemmas to help us narrow down the possibilities for a maximal triangle. Lemma 5 establishes the uniqueness of a triangle $T^* = \triangle X^*Y^*Z^*$ with a given lattice-point set $(L_x, L_y, L_z L_z')$. Therefore from the above table, it may be deduced that there are at most five such triangles satisfying condition (ii) of Lemma 4 (see also Figure 3).

Lemma 5. *Let $T = \triangle XYZ$ and $T' = \triangle X'Y'Z'$ be two triangles with the same intercept set and lattice-point set $(L_x, L_y, L_z L_z')$. Then $T = T'$.*

Proof: We first suppose that $R(T') > R(T)$. Then X' lies on the side of X remote from Y, and Y' lies on the side of Y remote from X. The edge $X'Z'$ is therefore oriented clockwise about L_y from XZ and the edge $Y'Z'$ is oriented anticlockwise about L_x from YZ. The vertex Z' therefore lies in the interior of T and so O is not the circumcentre of T', contradicting our assumption on T'. Hence $R(T') \not> R(T)$. A similar argument shows that $R(T') \not< R(T)$. Therefore $R(T') = R(T)$. It follows that $X' = X$, $Y' = Y$ so $Z' = Z$ and therefore $T' = T$. ∎

The next lemma helps us to eliminate those triangles in F which do not satisfy conditions (i) and (ii) of Lemma 4. We shall be comparing a triangle $T^* = \triangle X^*Y^*Z^*$ along with its given lattice-point set $(L_x, L_y, L_z L_z')$ with a related triangle $T = \triangle XYZ$ having the same intercept set as T.

Lemma 6. *Let $T^* = \triangle X^*Y^*Z^*$ denote the unique triangle with a given lattice-point set $(L_x, L_y, L_z L_z')$. We define P^* to be the intersection of X^*L_x and Y^*L_y, and Q^* to be the intersection of Z^*P^* produced with X^*Y^*. If L_z lies in the open line segment Q^*X^*, then any triangle with lattice-point set (L_x, L_y, L_z) is not maximal. If L_z' lies in the open line segment Q^*Y^*, then any triangle with lattice-point set (L_x, L_y, L_z') is not maximal.*

Proof: Let $T = \triangle XYZ$ be a triangle with the lattice-point set (L_x, L_y, L_z). We define P to be the intersection of XL_x and YL_y, and Q to be the intersection of ZP produced with XY. We show that ZL_z cannot pass through P. It will then follow that the VL-lines of T are not concurrent and by Lemma 3, T is not a maximal

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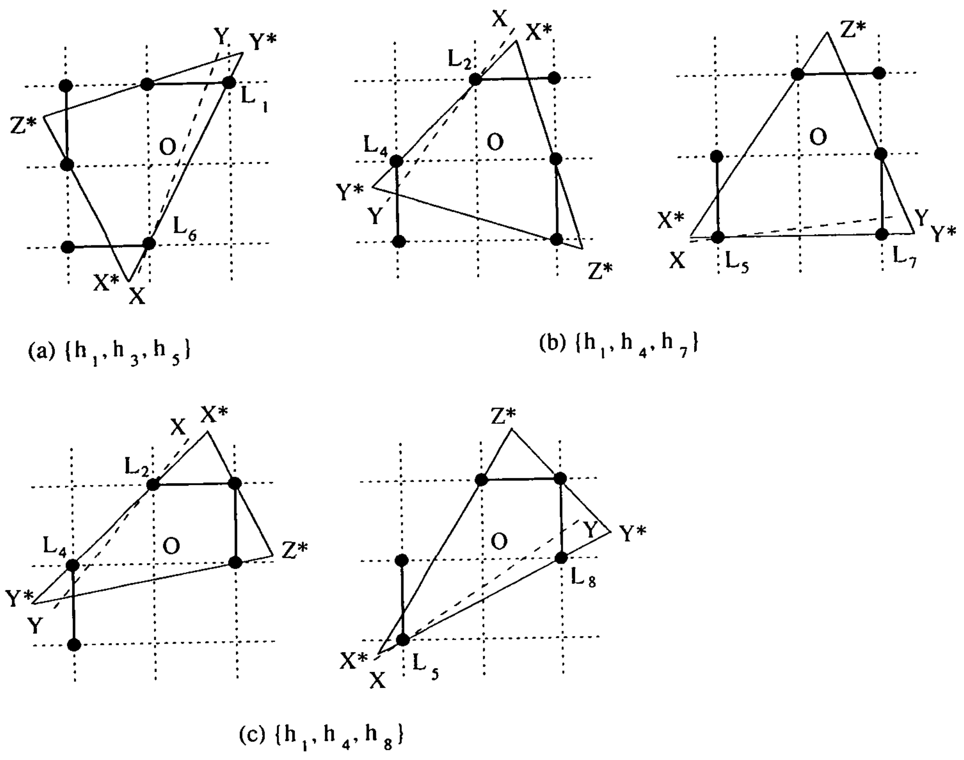

Figure 3: Triangles with edges containing two lattice points.

triangle. Let h_x and h_y be the open half planes bounded by the line Q^*Z^* containing X^* and Y^* respectively.

Figure 3 shows the five possible triangles T^*, and how the intercept set constrains the edge XY of any triangle $T = \triangle XYZ$ with lattice point set (L_x, L_y, L_z). Since T is Λ-admissible, T cannot contain L'_z in its interior. Thus in each case, since X^*, L_z, L'_z and Y^* are in the given order, the edge XY of T is oriented anticlockwise about L_z from the edge X^*Y^* of T^*. Similarly if T has lattice-point set (L_x, L_y, L'_z), then it cannot contain L_z in its interior and the edge XY of T is oriented clockwise about L'_z from X^*Y^*.

We first suppose that T is a maximal triangle. Therefore $R(T) \geqslant R(T^*)$ which implies that the vertices of T are exterior to T^*. As observed from Figure 3, the edge XY of T is oriented anticlockwise about L_z from the edge X^*Y^* of T^*, and since the vertices of T are exterior to T^*, the edges YZ and ZX of T are also oriented anticlockwise about L_x and L_y respectively from the corresponding edges of T^* (see

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Figure 4).

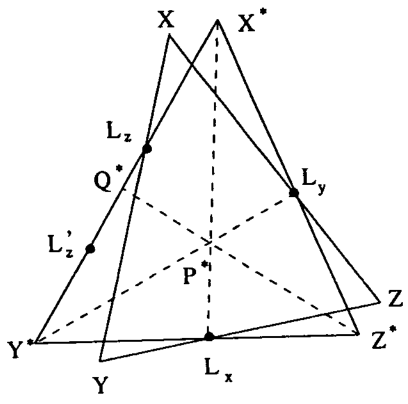

Figure 4: The case where $R(T) \geqslant R(T^*)$.

Therefore the VL-lines, XL_x and YL_y of T are oriented anticlockwise about L_x and L_y from X^*L_x and Y^*L_y respectively, placing the point P in the interior of $\triangle Y^*P^*L_x$. It follows that P lies in h_y. Since L_z lies in the open line segment Q^*X^*, the lattice point L_z lies in h_x. Also, since the edges YZ and ZX are oriented anticlockwise about L_x and L_y respectively from Y^*Z^* and Z^*X^*, the point Z necessarily lies in h_x. Hence the closed line segment ZL_z is contained in h_x. Therefore the closed line segment ZL_z and the point P are on opposite sides of the line Q^*Z^* and hence the VL-lines of T are not concurrent. Therefore T is not a maximal triangle.

We now let $T = \triangle XYZ$ be a triangle with the lattice-point set (L_x, L_y, L'_z). Arguing in a similar way as above, the edge XY of T is oriented clockwise about L'_z from X^*Y^*. Defining P and Q as above and using a similar argument, it may be shown that the closed line segment ZL_z and the point P again lie on opposite sides of the line Q^*Z^*. Hence the VL-lines of T are not concurrent and T is not a maximal triangle. ∎

In the subsequent sections, we shall employ Lemma 6 to show that a maximal triangle in F has an edge containing two lattice points in its interior. The maximal triangle may then be found by evaluating the circumradius for each triangle with an edge containing two interior lattice points.

We now list all lattice-point sets for a given intercept set. We shall employ the notation used in Lemma 6 throughout the rest of the paper. In addition, we shall denote by V_i, the vertex of T lying beyond h_i.

5. The $\{h_1, h_3, h_5\}$ case

Let F denote the family of triangles with circumcentre O and intercept set $\{h_1, h_3, h_5\}$. The following lattice-point sets occur for F:

$$\text{(a)}\ (L_2, L_4, L_6 L_1) \quad \text{(b)}\ (L_2, L_4, L_6) \quad \text{(c)}\ (L_2, L_4, L_1).$$

Let T be in F. Due to the constraints of the given intercept set, the edge of T containing L_2 has a slope between 0 and 1, and the edge containing L_4 has a slope less than -1. It follows that the vertex V_3 may not be further from the origin than the point $(-2, 1)$. Hence F is contained in the closed disk centred at O, radius $\sqrt{5}$ and by Blaschke's selection theorem, a maximal triangle $\mathcal{T}$ may be found in F.

In case (a) we let $T^* = \Delta V_5 V_1 V_3 \equiv \Delta X^* Y^* Z^*$ be the unique triangle with lattice-point set $(L_2, L_4, L_6 L_1)$. We assign the coordinates $(x, 2x-1)$ and $(y, 2y-1)$ to X^* and Y^* respectively. Since X^* and Y^* also lie on C,

$$x^2 + (2x-1)^2 = y^2 + (2y-1)^2$$

which gives $x + y = 4/5$. We let Z^* have coordinates (z_1, z_2). Since Y^*Z^* and Z^*X^* contain the lattice points $(0,1)$ and $(-1,0)$ respectively,

$$\frac{z_2 - 1}{z_1 - 0} = \frac{2y-2}{y}, \qquad \frac{z_2 - 0}{z_1 + 1} = \frac{2x-1}{x+1}.$$

Eliminating x and solving for z_1 and z_2, we obtain

$$z_1 = \frac{-y(6+5y)}{25y-18}, \qquad z_2 = -\frac{-23y + 10y^2 + 6}{25y - 18}.$$

As Z^* also lies on the circumcircle,

$$z_1^2 + z_2^2 = y^2 + (2y-1)^2.$$

Simplifying and factorising, we have

$$(5y-2)\left(25y^3 - 45y^2 + 25y - 6\right) = 0.$$

Since $y > 1$, we solve $25y^3 - 45y^2 + 25y - 6 = 0$ to obtain $y \approx 1.080$ and $R(T^*) \approx 1.582 < \alpha$.

We now proceed to show that any triangle T with lattice-point set (b) or (c) is not maximal. We consider the quadrangle $X^*Y^*L_2L_4$. Let K be the point of intersection of the lines X^*Y^* and L_2L_4, that is the point $(2,3)$. By the harmonic property of the quadrangle, the points X^*, Y^* separate Q^* and K harmonically. Therefore the cross-ratio $(X^*, Y^*; Q^*, K) = -1$. We now orthogonally project the four points X^*, Y^*, Q^* and K on the x-axis to obtain the points X_x^*, Y_x^*, Q_x^* and K_x respectively. Since the cross-ratio is unaltered under projection, we have $(X_x^*, Y_x^*; Q_x^*, K_x) = -1$. Letting the x-coordinate of Q^* be q, we have

$$\frac{q-x}{y-q} = -\frac{2-x}{y-2}.$$

which gives $q \approx 0.689$. Therefore L_6 lies in the open line segment Q^*X^* and L_1 lies in the open line segment Q^*Y^*. By Lemma 6, any triangle with lattice-point set (b) or (c) is not maximal.

Hence T is the triangle with lattice-point set (L_2, L_4, L_6L_1) and $R(T) \approx 1.582 < \alpha$.

6. The $\{h_1, h_4, h_7\}$ case

Let F denote the family of triangles with circumcentre O and intercept set $\{h_1, h_4, h_7\}$. The following lattice-point sets occur for F:

(a) (L_8, L_4, L_5L_7) (b) (L_5, L_8, L_2L_4) (c) (L_8, L_2L_4, L_5L_7)
(d) (L_7, L_8, L_2L_4) (e) (L_8, L_2, L_5L_7) (f) (L_4, L_5, L_8)
(g) (L_7, L_8, L_4) (h) (L_8, L_2, L_5) (i) (L_8, L_2, L_7).

Let T be in F. As before, given the intercept set of T, the edge of T containing L_2 has a slope which lies between $1/2$ and 1, and the edge of T containing L_8 has a slope greater than 1. It follows that the vertex V_1 may not be further from the origin than the point $(1,3)$. Hence F is contained in the closed disk centred at O, radius $\sqrt{10}$ and by Blaschke's selection theorem, a maximal triangle T may be found in F.

We first consider a triangle T with lattice-point set (a), (b), (c) or (f). We note that in these cases, $|OV_1| < \sqrt{5}$ and $|OV_4| \geqslant \sqrt{5}$. Hence T may not have circumcentre O. We may therefore eliminate cases (a), (b), (c) and (f). (In fact, any triangle with lattice-point set (f) contains the lattice point $(-2,-1)$ in its interior and is therefore not Λ-admissible.)

We now let $T^* = \Delta V_1V_4V_7 \equiv \Delta X^*Y^*Z^*$ be the triangle with lattice-point set (d). We assign the coordinates $(x, x+1)$ to X^*, $0 < x < 1$. Since X^* and Y^* lie on C, by symmetry Y^* has coordinates $(-x-1, -x)$. We let Z^* have coordinates (z_1, z_2). Since Z^*X^* and Y^*Z^* contain the lattice points $(1,0)$ and $(1,-1)$ respectively,

$$\frac{z_2 - 0}{z_1 - 1} = \frac{x+1}{x-1}, \quad \frac{z_2+1}{z_1-1} = \frac{-x+1}{-x-2}.$$

Solving for z_1 and z_2, we obtain

$$z_1 = -\frac{x^2 - 4x - 3}{5x + 1}, \quad z_2 = -\frac{x^2 + 3x + 2}{5x + 1}.$$

As Z^* also lies on C,

$$z_1^2 + z_2^2 = x^2 + (x + 1)^2.$$

Simplifying and factorising, we obtain

$$(2x + 1)(2x^3 + 2x^2 - 1) = 0.$$

Since $0 < x < 1$, we solve $2x^3 + 2x^2 - 1 = 0$ to obtain $x \approx 0.565$ and $R(T^*) \approx 1.664 < \alpha$.

We now show that any triangle T with lattice-point set (g) is not maximal. We consider the quadrangle formed by the points $X^*Y^*L_8L_7$. Letting the x-coordinate of Q^* be q and considering cross-ratios as in Section 4, we find that $q \approx 0.256$. Therefore L_4 lies in the open line segment Q^*Y^*. By Lemma 6, any triangle with lattice-point set (g) is not maximal.

We now let $T^* = \triangle V_4V_7V_1 \equiv \triangle X^*Y^*Z^*$ be the triangle with lattice-point set (e). We assign the coordinates $(1 + y, -1)$ to Y^*, $0 < y < 1$. Since X^* Y^* also lie on C, by symmetry, X^* has coordinates $(-1 - y, -1)$. We let Z^* have coordinates (z_1, z_2). Since Y^*Z^* and Z^*X^* contain the lattice points $(1, 0)$ and $(0, 1)$ respectively,

$$\frac{z_2 - 0}{z_1 - 1} = \frac{1}{-y}, \quad \frac{z_2 - 1}{z_1 - 0} = \frac{2}{1 + y}.$$

Solving for z_1 and z_2, we obtain

$$z_1 = \frac{-(y^2 - 1)}{1 + 3y}, \quad z_2 = \frac{3 + y}{1 + 3y}.$$

As Z^* also lies on C,

$$z_1^2 + z_2^2 = (1 + y)^2 + 1.$$

Simplifying and factorising, we obtain

$$(1 + y)(y^3 + 2y^2 + 2y - 1) = 0.$$

Since $0 < y < 1$, we solve $(y^3 + 2y^2 + 2y - 1) = 0$ for y to obtain $y \approx 0.353$ and $R(T^*) \approx 1.683 < \alpha$.

We now show that any triangle T with lattice-point set (h) or (i) is not maximal. We consider the quadrangle $X^*Y^*L_8L_2$. Letting the x-coordinate of Q^* be q and using cross-ratios as in Section 5, we find that $q \approx 0.915$. Therefore L_5 lies in the open line segment Q^*X^* and L_7 lies in the open line segment Q^*Y^*. By Lemma 6, any triangle with lattice-point set (h) or (i) is not maximal.

Hence T is the triangle with lattice-point set (L_8, L_2, L_5L_7) and $R(T) \approx 1.683 < \alpha$.

7. The $\{h_1, h_4, h_8\}$ case

Let F denote the family of triangles with circumcentre O and intercept set $\{h_1, h_4, h_8\}$. The following lattice-point sets occur for F:

(a) (L_1, L_4, L_5L_8) (b) (L_5, L_1, L_2L_4) (c) (L_1, L_2L_4, L_5L_8)
(d) (L_8, L_1, L_2L_4) (e) (L_1, L_2, L_5L_8) (f) (L_1, L_4, L_5)
(g) (L_8, L_1, L_4) (h) (L_1, L_2, L_8) (i) (L_1, L_2, L_5).

Let T be in F. As before, given the intercept set of T, the edge of T containing L_2 has a slope between $1/2$ and 1 and the edge containing L_1 has a slope less than 0. It follows that the vertex V_1 may not be further from the origin than the point $(1,3)$. Hence F is contained in the closed disk centred at O, radius $\sqrt{10}$ and by Blaschke's selection theorem, a maximal triangle $\mathcal{T}$ may be found in F.

We first consider a triangle with lattice-point set (a), (b), (c) or (f). We note that in all these cases, $|OV_1| < \sqrt{5}$ and $|OV_4| \geqslant \sqrt{5}$. Hence T may not have circumcentre O. Therefore we may disregard cases (a), (b), (c) and (f). (In fact, any triangle with lattice-point set (a) or (f) contains the lattice-point $(-2,-1)$ in its interior and is therefore not Λ-admissible.)

We now let $T^* = \triangle V_1V_4V_8 \equiv \triangle X^*Y^*Z^*$ be the triangle with lattice-point set (d). We assign the coordinates $(x, x+1)$ to X^*. Since X^* and Y^* also lie on C, by symmetry, Y^* has coordinates $(-x-1, -x)$. We let Z^* have coordinates (z_1, z_2). Since Z^*X^* and Y^*Z^* contain the lattice points $(1,1)$ and $(1,0)$ respectively,

$$\frac{z_2 - 1}{z_1 - 1} = \frac{x}{x-1}, \quad \frac{z_2 - 0}{z_1 - 1} = \frac{-x}{-x-2}.$$

Solving for z_1 and z_2, we obtain

$$z_1 = \frac{-x^2 + 2x + 2}{3x}, \quad z_2 = \frac{-x+1}{3}.$$

As Z^* also lies on C,

$$z_1^2 + z_2^2 = x^2 + (x+1)^2.$$

Simplifying and factorising, we obtain

$$(2x+1)(2x^3 + 2x^2 - 1) = 0.$$

Since $0 < x < 1$, we solve $2x^3 + 2x^2 - 1 = 0$ for x to obtain $x \approx 0.565$ and $R(T^*) \approx 1.664 < \alpha$.

We now proceed to show that any triangle T with lattice-point set (g) is not maximal. We consider the quadrangle $X^*Y^*L_8L_1$. Using the cross-ratio argument, we find that the x-coordinate q of Q^* is approximately 0.256 and therefore L_4 lies in the open line segment Q^*Y^*. By Lemma 6, any triangle with lattice-point set (g) is not maximal.

We now let $T^* = \Delta V_4V_8V_1 \equiv \Delta X^*Y^*Z^*$ be the triangle with lattice-point set (e). We assign the coordinates $(2x+1, x)$ and $(2y+1, y)$ to X^* and Y^* respectively. Since X^* and Y^* also lie on C,

$$(2x+1)^2 + x^2 = (2y+1)^2 + y^2$$

which gives $x + y = -\frac{4}{5}$. We let Z^* have coordinates (z_1, z_2). Since Y^*Z^* and Z^*X^* contain the lattice points $(1,1)$ and $(0,1)$ respectively,

$$\frac{z_2 - 1}{z_1 - 1} = \frac{y-1}{2y}, \quad \frac{z_2 - 1}{z_1 - 0} = \frac{x-1}{2x+1}.$$

Eliminating x and solving for z_1 and z_2 in terms of y, we obtain

$$z_1 = -\frac{-3 - 7y + 10y^2}{25y + 3}, \quad z_2 = -\frac{-12 - 21y + 5y^2}{25y + 3}.$$

As Z^* also lies on C,

$$z_1^2 + z_2^2 = (2y+1)^2 + y^2.$$

Simplifying and factorising we obtain

$$(5y+2)(25y^3 + 20y^2 - 3) = 0$$

Since $y > 0$, we solve $25y^3 + 20y^2 - 3 = 0$ for y to obtain $y \approx 0.326$ and $R(T^*) \approx 1.685 = \alpha$.

We now show that any triangle T with lattice-point set (h) or (i) is not maximal. We consider the quadrangle $X^*Y^*L_1L_2$ and using cross-ratios as before, we show that the x-coordinate q of Q^* is approximately 0.953. Therefore L_5 lies in the open line segment Q^*X^* and L_8 lies in the open line segment Q^*Y^*. By Lemma 6, any triangle with lattice-point set (h) or (i) is not maximal.

Hence T is the triangle with lattice-point set (L_1, L_2, L_5L_8) and $R(T) \approx 1.685 = \alpha$. We note that $\alpha = \sqrt{(2y+1)^2 + y^2}$, where $y \approx 0.326$.

Comparing the results in Section 5, Section 6 and Section 7. we conclude that the maximal set K is the triangle with lattice-point set (L_1, L_2, L_5L_8) with $R(K) \approx 1.685$. The theorem is therefore proved.

8. Comment

It is interesting to observe that the triangles with lattice-point sets $\{L_7, L_8, L_2L_4\}$ in Section 6 and $\{L_8, L_1, L_2L_4\}$ in Section 7 have the same circumradius. There does not appear to be any obvious algebraic connection, and we have been unable to find a simple geometrical proof.

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