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GOING DOWN AND OPEN EXTENSIONS

STEPHEN McADAM

Introduction. We call an extension of commutative rings, $R \subset T$, open if the spec mapping from spec (T) to spec (R), which sends the prime Q of T to $Q \cap R$, is an open mapping. It is easy to show, as for example in [**1**], that if $R \subset T$ is open then it satisfies going down. In general, the converse is false, as is shown by $Z \subset Z_{(2Z)}$ with Z the integers. To the best of this author's knowledge, it is an open question whether for an integral extension, going down and open are equivalent. The purpose of this paper is to prove the following two results:

(i) Let R be such that for any ideal J of R there are only finitely many primes of R minimal over J. If T is either a finitely generated or an integral extension of R, then if $R \subset T$ has going down, it is open.

(ii) For $R \subset T$ integral, the extension has going down if and only if $R \subset R[t]$ has going down for all $t \in T$. ((ii) Remains true if "going down" is replaced by open, as the techniques in [**1**] easily allow one to see.)

Notation and definitions. Throughout this paper $R \subset T$ will denote commutative rings with common identity. Going up, going down, incomparability and lying over are as defined in [**2**, section 1–6]. If I is an ideal of T, then $D(I) = \{Q \text{ prime in } T \mid I \not\subset Q\}$ is open in spec (T).

If f denotes the spec map, then let

$$\begin{aligned} C(I) &= \text{spec } (R) - f(D(I)) \\ &= \{P \text{ prime in } R \mid \text{every prime of } T \text{ lying over } P \text{ contains } I\}. \end{aligned}$$

Thus $R \subset T$ is open if and only if $C(I)$ is closed in spec (R) for all ideals $I \subset T$.

Let W be a subset of spec (R). We will say that W is weakly closed if the following is true: if $P \in$ spec (R) and P equals an intersection of primes in W, then P is in W. Clearly if W is closed in the spec topology then W is weakly closed. However the converse trivially fails; for instance let $W =$ spec $(Z) - \{2Z\}$.

We develop our main result.

LEMMA 1. *Let the domain T be a finitely generated extension of the domain R. Suppose that U is a set of non-zero primes of R such that $0 = \bigcap\{P \in U\}$. Then there is a set of primes of T, V, such that $0 = \bigcap\{Q \in V\}$ and $Q \cap R \in U$ for all $Q \in V$.*

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Proof. It is certainly enough to assume that T is generated by a single element over R. If $T = R[x]$ with x an indeterminate, then the set $V = \{PR[x] \mid P \in U\}$ satisfies the lemma. Therefore we assume that $T = R[t]$ with t algebraic over R, satisfying the polynomial $r_nt^n + r_{n-1}t^{n-1} + \ldots + r_0 = 0$, $r_n \neq 0$. Let $U' = \{P \in U \mid r_n \in P\}$ and $U'' = \{P \in U \mid r_n \notin P\}$. Then $0 = (\bigcap\{P \in U'\}) \cap (\bigcap\{P \in U''\})$. However R is a domain and $0 \neq r_n \in \bigcap\{P \in U'\}$ so that $\bigcap\{P \in U''\} = 0$.

We claim that for each $P \in U''$, there is a prime Q of T with $Q \cap R = P$. Let $P \in U''$. Then $r_n \notin P$ and there is a prime of $R[1/r_n]$ lying over P. However

$$R \subset R\left[\frac{r_0}{r_n}, \frac{r_1}{r_n}, \ldots \frac{r_{n-1}}{r_n}\right] \subset R\left[\frac{1}{r_n}\right]$$

and so there is a prime of $R[r_0/r_n, \ldots r_{n-1}/r_n]$ lying over P. Now

$$R\left[\frac{r_0}{r_n}, \ldots \frac{r_{n-1}}{r_n}\right] \subset R\left[\frac{r_0}{r_n}, \ldots \frac{r_{n-1}}{r_n}\right][t]$$

is an integral extension, so that this last ring has a prime lying over P. Finally $R \subset R[t] = T \subset R[r_0/r_n, \ldots r_{n-1}/r_n]\ [t]$ showing that T contains a prime lying over P.

To complete the proof, let $V = \{Q \text{ prime in } T \mid Q \cap R \in U''\}$. Clearly for $Q \in V$, $Q \cap R \in U$. Also since $\bigcap\{P \in U''\} = 0$ and every prime in U'' is the contraction of a prime in V, we have $(\bigcap\{Q \in V\}) \cap R = 0$. However $R \subset T$ is algebraic so that we must have $\bigcap \{Q \in V\} = 0$.

LEMMA 2. *Let $R \subset T$ be domains satisfying going up and incomparability. If U is a set of non-zero primes of R such that $0 = \bigcap\{P \in U\}$, then there is a set of primes of T, V, such that $0 = \bigcap\{Q \in V\}$ and $Q \cap R \in U$ for all $Q \in V$.*

Proof. By going up, each prime of U is the contraction of a prime of T. Let $V = \{Q \text{ prime in } T \mid Q \cap R \in U\}$. Clearly $Q \cap R \in U$ for $Q \in V$ and $(\bigcap\{Q \in V\}) \cap R = 0$. Suppose that $\bigcap\{Q \in V\} \neq 0$. Then that intersection could be expanded to a prime Q' of T maximal with respect to being disjoint from the multiplicativity closed set $R - \{0\}$. We would then have $0 \cap R = 0 = Q' \cap R$ and $0 \subset Q'$ contradicting incomparability.

PROPOSITION 1. *Let $R \subset T$ be rings satisfying either*

(a) *T is finitely generated over R, or*

(b) *$R \subset T$ satisfies going up and incomparability.*

Let I be an ideal of T. Then $C(I)$ is weakly closed.

Proof. Suppose that $U_0 \subset C(I)$ and that $\bigcap\{P \in U_0\}$ is a prime ideal, say P_0. Let Q_0 be a prime of T with $Q_0 \cap R = P_0$. To show that $P_0 \in C(I)$, we must show that $I \subset Q_0$. (If no such Q_0 exists, we are done.)

Consider $R/P_0 \subset T/Q_0$ and the set of primes $U = \{P/P_0 \mid P \in U_0\}$ of R/P_0. The intersection of that set of primes is 0. In case condition (a) holds, $R/P_0 \subset T/Q_0$ is finitely generated. If condition (b) holds then $R/P_0 \subset T/Q_0$ satisfies going up and incomparability. Thus in either case, by Lemmas 1 and 2, we see that there is a set V of primes of T/Q_0 whose intersection is 0 and each of whose contractions to R/P_0 is in U. That is, there is a set, V_0, of primes of T such that $\bigcap\{Q \in V_0\} = Q_0$ and $Q \cap R \in U_0$ for all $Q \in V_0$. However $Q \in V_0$ implies that $Q \cap R \in U_0 \subset C(I)$ so that we must have $I \subset Q$ for all $Q \in V_0$. Thus $I \subset \bigcap\{Q \in V_0\} = Q_0$.

THEOREM 1. *Suppose that for each ideal J of R, there are only finitely many primes of R minimal over J. Let $R \subset T$ satisfy going down and suppose that either T is finitely generated over R or $R \subset T$ satisfies going up and incomparability. Then $R \subset T$ is open.*

Proof. Let I be an ideal of T. We must show that $C(I)$ is closed in spec (R). Let $J = \bigcap\{P \in C(I)\}$ and suppose that P_1 is prime in R with $J \subset P_1$. We must show that $P_1 \in C(I)$. Let us first assume that P_1 is in fact minimal over J. Let $P_1, P_2, \ldots P_n$ be all the primes minimal over J. For $i = 1, \ldots n$ let $J_i = \bigcap\{P \in C(I) | P_i \subset P\}$. Since each $P \in C(I)$ contains one of $P_1, \ldots P_n$, we have $J_1 \cap \ldots \cap J_n = \bigcap\{P \in C(I)\} = J \subset P_1$. However for $k = 2, \ldots n$ we have $P_k \subset J_k$ but $P_k \not\subset P_1$. Thus $J_k \not\subset P_1$ for $k = 2, \ldots, n$, and so $J_1 \subset P_1$. However $P_1 \subset J_1$, so that $P_1 = J_1$ is an intersection of primes of $C(I)$. Because $C(I)$ is weakly closed by Proposition 1, $P_1 \in C(I)$.

We now assume that P_1 is an arbitrary prime containing J. Then there is a prime $P_0 \subset P_1$ with P_0 minimal over J. By the argument just given, $P_0 \in C(I)$. To show that $P_1 \in C(I)$, we consider a prime Q_1 of T with $Q_1 \cap R = P_1$ (if any such exist) and must show that $I \subset Q_1$. By going down, since $P_0 \subset P_1 = Q_1 \cap R$, there is a prime Q_0 of T with $Q_0 \subset Q_1$ and $Q_0 \cap R = P_0$. However $P_0 \in C(I)$ implies that $I \subset Q_0$, and so $I \subset Q_0 \subset Q_1$.

Remark. This is stronger and simpler than [**3**, p. 48].

COROLLARY. *Let $R \subset T$ be an integral extension with going down. If every ideal of R has only finitely many primes minimal over it, then $R \subset T$ is open.*

Proof. $R \subset T$ being integral, has going up and incomparability.

LEMMA 3. *Let $R \subset T$ be rings. Suppose that for any ring S with $R \subset S \subset T$ and S finitely generated over R, $R \subset S$ has going down. Then $R \subset T$ has going down.*

Proof. This is an easy exercise using [**2**, Exercise 37(iii), p. 44].

PROPOSITION 2. *Let $R \subset T$ be an integral extension. Then $R \subset T$ has going down if and only if $R \subset R[t]$ has going down for all $t \in T$.*

Proof. Suppose that $R \subset T$ has going down. For $t \in T$, since lying over holds in $R[t] \subset T$, it is a triviality that $R \subset R[t]$ has going down.

Conversely suppose that going down holds between R and every simple extension of R contained in T. To show that $R \subset T$ satisfies going down we may assume by Lemma 3 that T is finitely generated over R. Suppose that $R \subset T$ fails going down, and let $P \subset P'$ be primes of R such that there is a prime Q' of T with $Q' \cap R = P'$, and such that no prime contained in Q' lies over P. Since $R \subset T$ is a finitely generated integral extension, there are only finitely many primes of T which lie over P, say $Q_1, \ldots Q_n$. By assumption $Q_i \not\subset Q'$ for $i = 1, 2, \ldots n$ and so we may choose $t \in Q_1 \cap \ldots \cap Q_n - Q'$. In $R[t]$ the prime $Q' \cap R[t]$ lies over P'. By going down in $R \subset R[t]$ there is a prime Q_0 of $R[t]$ with $Q_0 \subset Q' \cap R[t]$ and $Q_0 \cap R = P$. Since $R[t] \subset T$ is integral, for some $i = 1, \ldots n$ we have $Q_i \cap R[t] = Q_0$. Thus $t \in Q_i \cap R[t] = Q_0 \subset Q' \cap R[t] \subset Q'$, a contradiction.

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University of Texas,
Austin, Texas