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FUNCTIONS BELONGING TO A DIRICHLET SUBALGEBRA OF THE DISK ALGEBRA

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Browder and Wermer in [2] give a method for constructing Dirichlet subalgebras of the disk algebra. In this note we show that these Dirichlet algebras do not contain any non-constant functions which satisfy a Lipschitz-one condition on a subinterval of the unit circle.

Let A be a uniform algebra on a compact Hausdorff space X (see [6] for facts about uniform algebras). We say that A is a *Dirichlet algebra on X* if $\operatorname{Re} A = \{\operatorname{Re}(f): f \in A\}$ is uniformly dense in $C_R(X)$, the real-valued continuous functions on X. Let $T=\{z:|z|=1\}$ and $U=\{z:|z|<1\}$ be the unit circle and the open unit disk in the complex plane. The *disk algebra*, $A(T)=\{f \in C(T): f$ extends analytically to $U\}$ is a Dirichlet algebra on T (see, for example, [4], p. 43). Browder and Wermer in [2] give a method for constructing subalgebras of $A(T)$ which are still Dirichlet algebras on T. Their method goes as follows: Let $p(e^{it})$ be a homeomorphism of T such that $dp(e^{it})/dt=0$ a.e. with respect to Lebesgue measure on T. Define

$$A_p = A(T) \cap \{f \in C(T): f \circ p \in A(T)\}.$$

Browder and Wermer show that A_p is a Dirichlet algebra on T. In [1] Blumenthal shows that A_p is a maximal uniform algebra in $A(T)$.

The method of showing that A_p is a Dirichlet algebra is indirect in that the result is obtained by showing that there are no non-zero real annihilating measures for A_p. No work appears to have been done on describing what types of functions may belong to A_p. Our theorem below gives a result in this direction.

THEOREM. *If $f \in A_p$ and if f satisfies a Lipschitz-one condition on some subinterval of T, then f is a constant function.*

This theorem is a consequence of the following lemma which is of some independent interest.

LEMMA. *Suppose $F \in A(T)$ and suppose there is an interval $I=\{\exp(it): a \le t \le b\}$ so that F is of bounded variation on I. Assume also that*

$$\frac{dF(e^{it})}{dt} = 0 \quad \text{a.e.} \quad \text{for} \quad t \in [a, b].$$

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Then F is a constant function.

Proof. We first note that if $I=T$, then the Lemma follows directly from the following two classical results which may be found in [3], p. 42.

(1) If $f \in A(T)$ and if $f(e^{it})$ is of bounded variation on T, then $f(e^{it})$ is absolutely continuous on T.

(2) If $f \in A(T)$, then $f(e^{it})$ is absolutely continuous on T if and only if $f'(z) \in H^1(U)$. Moreover, if $f'(z) \in H^1(U)$, then

$$\frac{df(e^{it})}{dt} = ie^{it} \lim_{r\to 1} f'(re^{it}) \quad \text{a.e.}$$

For the case $I \neq T$, the idea of the proof is as follows. We multiply F by a suitable polynomial in order to form a new function G which is of bounded variation along a simple closed piecewise analytic curve in $\bar{U}$ which contains I. Statements (1) and (2) can be applied to the function $G \circ \phi(z)$ where $\phi(z)$ is a conformal map of U onto the region lying inside the curve. The final result then follows by a simple argument.

Let $F(z)$ denote the analytic extension of $F(e^{it})$ to U and let $M=\max\{|F(z)|: |z|\leq 1\}$. An elementary computation involving the Cauchy integral formula gives

$$|F'(re^{it})| \leq \frac{M}{1-r} \quad \text{for} \quad 0 \leq r < 1 \tag{3}$$

Set

$$G(z) = F(z)[(1-e^{-ia}z)(1-e^{-ib}z)]^2$$

Since we have $G'(\mathrm{re}^{it})=(\partial G(\mathrm{re}^{it})/\partial r)e^{-it}$ for $0\leq r<1$, we may use (3) to conclude that $(\partial G(\mathrm{re}^{ia})/\partial r)$ and $(\partial G(\mathrm{re}^{ib})/\partial r)$ are continuous for $0\leq r\leq 1$.

We let $V=\{\mathrm{re}^{it}: a<t<b \text{ and } 0<r<1\}$. Suppose $\phi(z)$ is a conformal map from U onto V. Then ϕ extends to be a homeomorphism of $\bar{U}$ onto $\bar{V}$, and $\phi(e^{it})$ is absolutely continuous ([3], p. 44). We will suppose that $\phi(1)=e^{ia}$, $\phi(e^{it_1})=e^{ib}$, and $\phi(e^{it_2})=0$.

The hypothesis that $F(e^{it})$ is of bounded variation along I implies that $G \circ \phi(e^{it})$ is of bounded variation on $[0, t_1]$. Moreover, the continuity of $(\partial G(\mathrm{re}^{ia})/\partial r)$ and $(\partial G(\mathrm{re}^{ib})/\partial r)$ for $0\leq r\leq 1$ implies that $G \circ \phi(e^{it})$ is of bounded variation on $[t_1, 2\pi]$. Consequently, $G \circ \phi(e^{it})$ is of bounded variation on T. Since $G \circ \phi(z) \in A(T)$, we can conclude by (1) and (2) that $(d(G \circ \phi(e^{it}))/dt)$ gives the boundary values of a function in $H^1(U)$.

By hypothesis $(dF(e^{it})/dt)=0$ a.e. for $t \in [a, b]$ and from this we obtain $d(F \circ \phi(e^{it})/dt)=0$ a.e. for $t \in [0, t_1]$ ([5], Corollary 2). If we let

$$P(z) = [(1-e^{-ia}z)(1-e^{-ib}z)]^2,$$

then

$$\frac{d(G \circ \phi(e^{it}))}{dt} = \frac{d(F \circ \phi(e^{it}))}{dt} P \circ \phi(e^{it}) + F \circ \phi(e^{it}) \frac{d(P \circ \phi(e^{it}))}{dt}$$

a.e. for $t \in [0, 2\pi]$. But then

$$\frac{d(G \circ \phi(e^{it}))}{dt} = F \circ \phi(e^{it}) \frac{d(P \circ \phi(e^{it}))}{dt} \quad \text{a.e.} \quad \text{for} \quad t \in [0, t_1].$$

Since $F \circ \phi(e^{it})(d(P \circ \phi(e^{it}))/dt)$ gives the boundary values for a function in $H^1(U)$, we therefore conclude that $(d(F \circ \phi(e^{it}))/dt)=0$ a.e. for $t \in [0, 2\pi]$. Since $F \circ \phi(z)$ extends analytically across $\{e^{it}: t_1 < t < t_2\}$, it follows that $F(z)$ is a constant function. This completes the proof.

Proof of Theorem. If $f(e^{it})$ satisfies the hypothesis of the theorem, then $F(e^{it}) = f \circ p(e^{it}) \in A(T)$. If $f(e^{it})$ is Lip-1 on $\{e^{it}: A \le t \le B\}$, then $F(e^{it})$ is of bounded variation on $p^{-1}(\{e^{it}: A \le t \le B\}) = \{e^{it}: a \le t \le b\}$. The singularity of $p(e^{it})$ and the Lip-1 condition on $f(e^{it})$ imply that $dF(e^{it})/dt) = 0$ a.e. for $t \in [a, b]$. We now apply the Lemma to conclude that F, and hence f, is identically constant on T.

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