AN OPTIMUM METHOD FOR CALCULATING RESTRICTED THREE-BODY ORBITS

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ABSTRACT. The restricted three-body problem (RTBP) has in the past played an essential role in many different areas of dynamical astronomy, and indications are that this will continue. As the state of the art in computing becomes more advanced, larger numbers of integrations and longer durations are attempted. Thus, computational efficiency and accuracy are becoming more important. Also, the use of the RTBP in many different areas leads to the desire for a general integration method.

In order to maximize the efficiency of orbit calculations, comparisons are made of different methods of integration. The results can be summarized as follows: 1. The Bulirsch-Stoer extrapolation method is extremely fast and accurate, and is the method of choice. 2. Regularization of the equations of motion is essential. 3. When applicable, a manifold correction algorithm, originally due to Nacozy (1971), reduces numerical errors to the limits of machine accuracy, and at a cost of only 1 to 3 percent in cpu time.

1. Qualities of B-S Integration, Regularization, and Manifold Correction

There are three ways to improve speed and accuracy in integrating the restricted three-body equations. First, use a fast and efficient algorithm. The Bulirsch-Stoer (B-S) extrapolation method (see Press *et al.*, 1986) is a very fast and efficient algorithm for solving ordinary differential equations. Second, regularize the equations of motion. In the following, two-body regularization was incorporated (see, *e.g.*, Stiefel and Scheifele 1970, Bettis and Szebehely 1971, Murison 1988). Third, take advantage of integrals of the motion for error checking and correction.

Nacozy (1971) found a clever way to use constants of the motion to monitor and correct errors in position and velocity. When an integral exists, the motion is confined to a surface in the phase space. When a numerical error occurs, the motion jumps to a different surface. Nacozy's correction scheme uses Lagrange multipliers to find the least-squares shortest path back to the original manifold. Murison (1988) has applied this manifold correction to both the unregularized and regularized restricted three-body equations.

To illustrate the usefulness of these techniques, the orbit shown in figure 1 was integrated several different ways. This orbit is difficult,

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in the sense that it has many close approaches to the primary m2. For this illustration, the integration continued for 10 orbital periods of the primaries. During the integration, the error in the Jacobi constant C was monitored.

Figure 2 shows the error in C during an integration using an efficient, fourth-order Runga-Kutta method with variable step size (the curve labeled "RK"). Notice the steady increase of error with time. The CPU time needed by a VAX 11/780 was 617 seconds. This figure also shows the error when the B-S method was used. Discrete jumps, caused by close approaches, are larger. But the CPU time was a factor of 10 less (61.6 sec).

Figure 3 shows the effect of manifold correction. The B-S method was used again. Notice the drastic collapse of the errors. Yet the increased cost in CPU time was only 2.8 percent (to 63.3 sec)!

To illustrate the effect of regularization, figure 4 shows the result of using the B-S method with regularized equations of motion (no manifold correction). This is the curve labeled "BS+R". Note that *the vertical scale has been magnified by a factor of 1000.* The errors are incredibly small, compared to the previous figures. This is a result of trying to keep the amount of machine effort about the same (CPU time was 57.5 sec).

Finally, figure 4 also shows the result of combining the B-S method, regularization, and manifold correction (the curve labeled "BS+MC+R"). The CPU time was 59.0 sec. The errors have been reduced to effectively nothing (!), with a net *savings* of CPU time, compared to the B-S method alone (figure 2).

Thus, we conclude that the B-S method of integration is very fast, and, when combined with regularization and especially manifold correction, extremely accurate. We also note the both importance of regularization and the small cost of manifold correction.

2. Backwards Integration Check

As a check on the B-S integrator, an orbit very similar to the one of figure 1, but with even closer approaches to m2, was integrated forward and then backward in time. The errors in the final values of x, y, x′, and y′ were noted as a function of length and accuracy of the integration. Both normal and regularized equations were used. Figure 5 shows the distance from m2 as a function of time for this orbit (escape occurs at about $T = 22.8$ primary periods). One can see that close approaches are frequent. When regularization was turned on, the integration switched to regularized equations when $r2 \leq 0.11$. This limit is shown in figure 5. Thus, most of the orbit was regularized when regularization was turned on.

The difference between the initial and final x coordinate is a good indicator of how well the integrator is able to retrace the orbit backwards. The log of Δx is plotted as a function of forward integration time in figure 6. The B-S accuracy parameter (see Press *et al.*, 1986) is denoted by ε. One sees that, for this particular orbit, (1) regularization improves the accuracy by a factor of from 30 to over

1000, (2) for moderate integration times $T \simeq 50$ the orbit is believable *in detail* (not just in a statistical sense, as is commonly the case in these kinds of calculations), and (3) values of ε smaller than $\sim 10^{-12}$ are required.

3. Application to Satellite Capture

The ability to integrate the restricted three-body equations quickly and accurately enables us to explore numerically the restricted three-body problem to a much further and more general extent than has previously been possible. Murison (1988) contains examples of such extensive calculations, in the framework of satellite capture.

4. Conclusions

A. The B-S extrapolation method by itself is very fast, and fairly accurate.
B. Regularization of the equations of motion is essential.
C. Manifold correction, when applicable, reduces the errors to effectively zero and is very inexpensive.
D. The combination of the above three allows one to make extensive numerical explorations which were previously not feasible.

These methods should find application to systems other than the restricted three-body equations as well.

References

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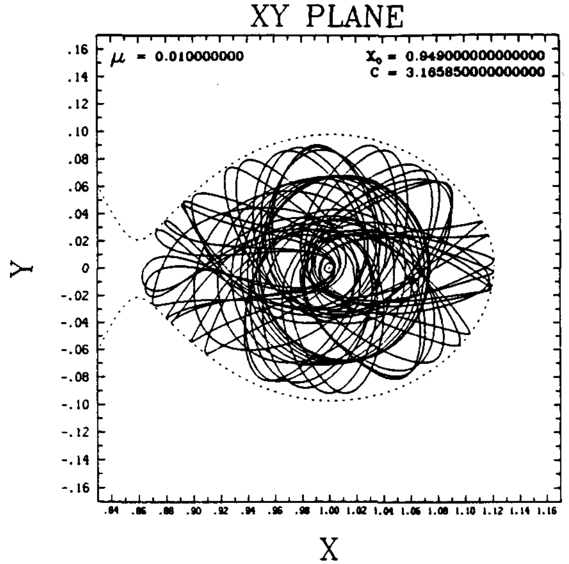

Figure 1. Orbit used for comparison of integration methods. The smaller primary m2 is located at (x,y) = (1.0,0.0). The dashed curve is the zero-velocity curve for this orbit. Notice several close approaches to m2.

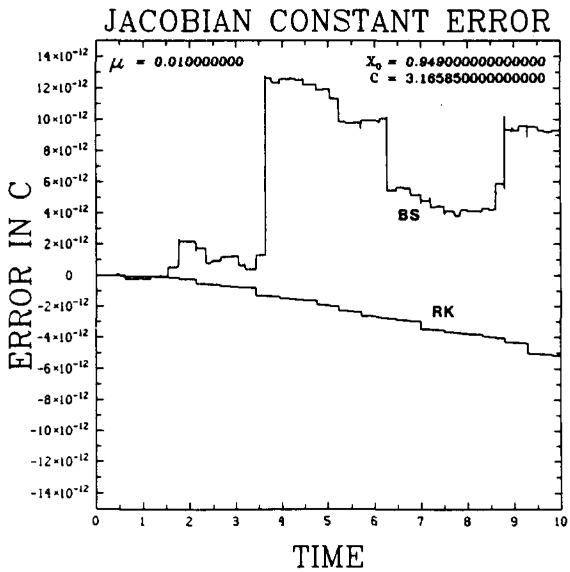

Figure 2. Error in Jacobi constant C vs. time in units of primary orbital periods. BS = Bulirsch-Stoer, RK = fourth-order Runga-Kutta. The RK calculation took ten times more cpu time than the BS calculation.

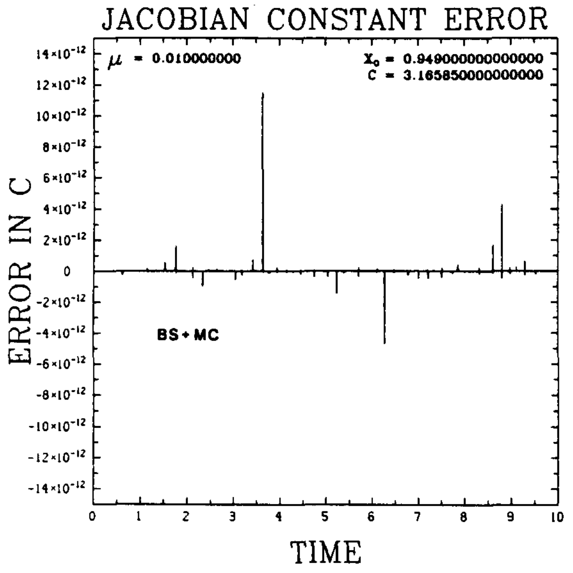

Figure 3. Error in Jacobi constant C vs. time. Bulirsch-Stoer, with manifold correction. Note drastic reduction of errors (compare with BS curve of figure 2).

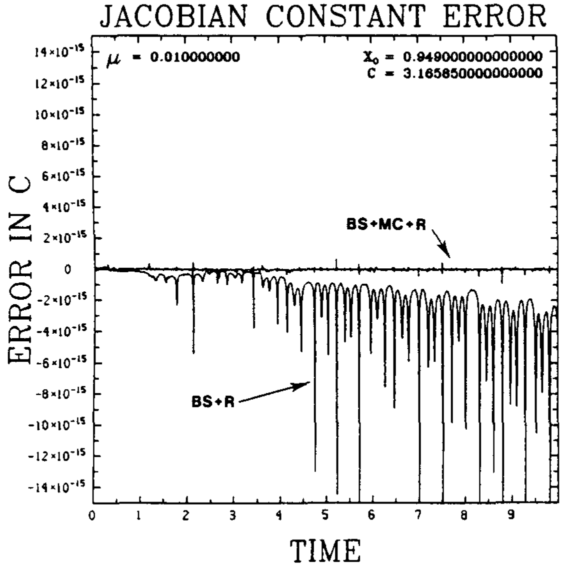

Figure 4. Error in Jacobi constant C vs. time. Curve labeled BS+R is Bulirsch-Stoer and regularized equations. BS+MC+R is combination of Bulirsch-Stoer, manifold correction, and regularization. Notice factor of 1000 change in scale.

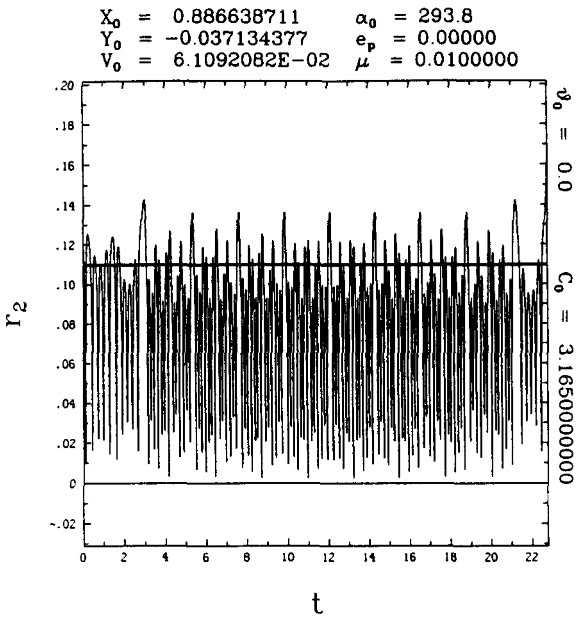

Figure 5. Distance from m2 vs. time for orbit used in backward integration check. Solid line at r2 = 0.11 shows limit below which regularization was in effect during the "regularized" calculation (see text). Note the several very close approaches to m2.

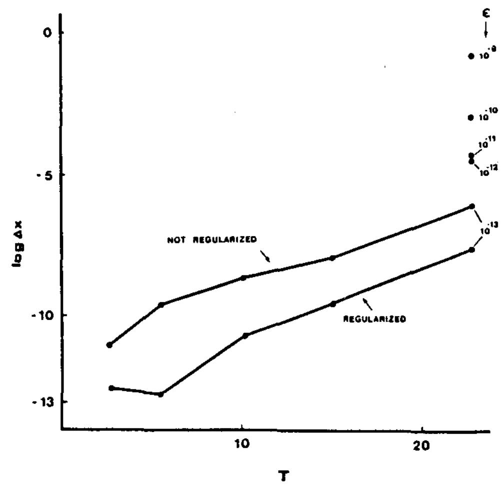

Figure 6. Log of difference in x position (Log Δx) between start and end of backwards integration check, as a function of forward integration time. Curves are shown for regularized and unregularized calculations with $\varepsilon = 10^{-13}$. Other points for different ε, at T = 22.8, represent unregularized calculations.