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CHARACTERIZING HERMITIAN VARIETIES IN THREE- AND FOUR-DIMENSIONAL PROJECTIVE SPACES

ANGELA AGUGLIA

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Abstract

We characterize Hermitian cones among the surfaces of degree $q + 1$ of $\mathrm{PG}(3, q^2)$ by their intersection numbers with planes. We then use this result and provide a characterization of nonsingular Hermitian varieties of $\mathrm{PG}(4, q^2)$ among quasi-Hermitian ones.

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1. Introduction

An *m-character set* in the projective space $\mathrm{PG}(n, q)$, q any prime power, is a set of points of $\mathrm{PG}(n, q)$ with the property that the intersection number with any hyperplane only takes m values, where m is a positive integer.

A nonsingular Hermitian variety $\mathcal{H}(n, q^2)$ of $\mathrm{PG}(n, q^2)$ is a remarkable example of a two-character set, precisely a set of $(q^{n+1} + (-1)^n)(q^n - (-1)^n)/(q^2 - 1)$ points of $\mathrm{PG}(n, q)$ with the property that a hyperplane Π meets it in either

$$(q^n + (-1)^{n-1})(q^{(n-1)} - (-1)^{(n-1)})/(q^2 - 1)$$

points, in case Π is a nontangent hyperplane to $\mathcal{H}(n, q^2)$, or

$$1 + q^2(q^{n-1} + (-1)^n)(q^{(n-2)} - (-1)^n)/(q^2 - 1)$$

points, in case Π is a tangent hyperplane to $\mathcal{H}(n, q^2)$; see [21].

Quasi-Hermitian varieties are generalizations of nonsingular Hermitian varieties such that they have the same size and the same intersection numbers with respect to hyperplanes.

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Actually, a point set $\mathcal{S}$ of $\mathrm{PG}(n, q^2)$, $n > 2$, having the same intersection numbers with respect to hyperplanes as a nonsingular Hermitian variety $\mathcal{H}(n, q^2)$ has also the same number of points of $\mathcal{H}(n, q^2)$; for $n = 2$, the size of $\mathcal{S}$ can be either $q^3 + 1$, that is, the size of a Hermitian curve also called a classical unital, or $q^2 + q + 1$, which is the number of points of a Baer subplane of $\mathrm{PG}(2, q^2)$; see [7].

As far as we know, the only quasi-Hermitian varieties of $\mathrm{PG}(n, q^2)$ which are not isomorphic to Hermitian varieties were constructed in the series of papers [1, 3, 5, 6, 17, 18].

The definition of a quasi-Hermitian variety can be extended to that of a singular quasi-Hermitian variety, that is, point sets which have the same number of points and the same intersection numbers with respect to hyperplanes as singular Hermitian varieties. Each cone over a quasi-Hermitian variety is a singular quasi-Hermitian variety; thus, a natural question is also whether such a cone is isomorphic to a singular Hermitian variety.

Various characterizations of a nonsingular Hermitian variety among the quasi-Hermitian ones in $\mathrm{PG}(n, q^2)$, with $n \in \{2, 3\}$, have been given, but very few in higher dimensional cases; see [2, 7, 15, 19]. In [2], singular Hermitian varieties were also characterized among singular quasi-Hermitian ones.

Here we first consider point sets of $\mathrm{PG}(3, q^2)$ such that their intersection numbers with respect to planes take three values as well as the Hermitian cone with one singular point, that is, $q^2 + 1$, $q^3 + 1$, or $q^3 + q^2 + 1$.

Combining geometric and combinatorial arguments with algebraic geometry, we prove the following result.

THEOREM 1.1. *Let $\mathcal{S}$ be a surface of $\mathrm{PG}(3, q^2)$ of degree $q + 1$. If every plane meets $\mathcal{S}$ in $q^2 + 1$, $q^3 + 1$, or $q^3 + q^2 + 1$ points of $\mathrm{PG}(3, q^2)$, then $\mathcal{S}$ is a cone projecting a Hermitian curve in a plane π from a point V not in π.*

Next, we also provide the following characterization of nonsingular Hermitian varieties of $\mathrm{PG}(4, q^2)$.

THEOREM 1.2. *Let $\mathcal{S}$ be a quasi-Hermitian variety of $PG(4, q^2)$. If $\mathcal{S}$ is a hypersurface of degree $q + 1$, then $\mathcal{S}$ is a nonsingular Hermitian variety.*

2. Preliminaries

Let $\Sigma = \mathrm{PG}(n, q^2)$ be the Desarguesian projective space of dimension n over $\mathrm{GF}(q^2)$ and denote by $X = (x_1, x_2, \ldots, x_{n+1})$ homogeneous coordinates for its points.

We use σ to write the involutory automorphism of $\mathrm{GF}(q^2)$ which leaves all the elements of the subfield $\mathrm{GF}(q)$ invariant. A Hermitian variety $\mathcal{H}(n, q^2)$ is the set of all points X of Σ which are self conjugate under a Hermitian polarity h. If H is the Hermitian $(n + 1) \times (n + 1)$ matrix associated with h, then the Hermitian variety $\mathcal{H}(n, q^2)$ has equation

$$XH(X^\sigma)^T = 0.$$

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When H is nonsingular, the corresponding Hermitian variety is nonsingular, whereas if H has rank $r+1$, with $r<n$, the related Hermitian variety is singular and it is a cone $\Pi_{n-r-1}\mathcal{H}(r,q^2)$ with vertex an $(n-r-1)$-space Π_{n-r-1} and basis a nonsingular Hermitian variety $\mathcal{H}(r,q^2)$ of an r-space disjoint from Π_{n-r-1}.

A *d-singular quasi-Hermitian variety* is a subset of points of $\mathrm{PG}(n,q^2)$ having the same number of points and the same intersection sizes with hyperplanes as a singular Hermitian variety with a singular space of dimension d.

Nonsingular Hermitian varieties of $\mathrm{PG}(n,q^2)$ are in particular hypersurfaces. We recall that a projective *hypersurface* $\mathcal{S}$ of degree d is a set of points of $\mathrm{PG}(n,q^2)$ whose homogenous coordinates satisfy

$$F(X_0,X_1,\ldots,X_n)=0,$$

where F is a form of degree d over $\mathrm{GF}(q^2)$.

However, to understand the geometry of the hypersurface $\mathcal{S}$, the zeros of F over $\mathrm{GF}(q^2)$ and over any extension of $\mathrm{GF}(q^2)$ are required. Thus, $\mathcal{S}$ is viewed as a hypersurface over the algebraic closure of $\mathrm{GF}(q^2)$ and a point of $\mathrm{PG}(n,q^2)$ in $\mathcal{S}$ is called a $\mathrm{GF}(q^2)$-*point* or a *rational point* of $\mathcal{S}$; in general, a $\mathrm{GF}(q^{2i})$-*point* of $\mathcal{S}$ is a point $P(a_0,\ldots,a_n)$ in $\mathrm{PG}(n,q^{2i})$ such that $F(a_0,\ldots,a_n)=0$. The number of $\mathrm{GF}(q^{2i})$-points of $\mathcal{S}$ is denoted by $N_{q^{2i}}(\mathcal{S})$. When $n=2$, a projective hypersurface $\mathcal{S}$ is called a *projective plane curve*, whereas when $n=3$, $\mathcal{S}$ is called a *projective surface*.

The following results will be crucial to our proof.

Lemma 2.1 [20]. *Let d be an integer with $1\le d\le q+1$ and C be a curve of degree d in $\mathrm{PG}(2,q)$ defined over $\mathrm{GF}(q)$, which may have $\mathrm{GF}(q)$-linear components. Then the number $N_{q^2}(C)$ of rational points of C is at most $dq+1$ and $N_q(C)=dq+1$ if and only if C is a pencil of d lines of $\mathrm{PG}(2,q)$.*

Lemma 2.2 [12, 13, 16]. *Let d be an integer with $2\le d\le q+2$ and C be a curve of degree d in $\mathrm{PG}(2,q)$ without $\mathrm{GF}(q)$-line components. Then the number of rational points of C is at most $(d-1)q+1$ except for a class of plane curves of degree four over $\mathrm{GF}(4)$ having 14 points.*

Lemma 2.3 [10]. *Suppose that $q\ne 2$. Let C be a plane curve over $\mathrm{GF}(q^2)$ of degree $q+1$ without $\mathrm{GF}(q^2)$-line components. If C has q^3+1 points over $\mathrm{GF}(q^2)$, then C is a Hermitian curve.*

Lemma 2.4 [14]. *Let $\mathcal{S}$ be a surface in $\mathrm{PG}(3,q^2)$ without $\mathrm{GF}(q^2)$-plane components. If the degree of $\mathcal{S}$ is $q+1$ and the number of its rational points is $(q^3+1)(q^2+1)$, then $\mathcal{S}$ is a nonsingular Hermitian surface.*

Finally, a hyperplane of $\mathrm{PG}(n,q^2)$ intersecting a point set $\mathcal{S}$ of the projective space in i points will be called an *i-hyperplane*, whereas a line meeting $\mathcal{S}$ in s points will be called an *s-secant line* if $s\ge 1$ or an *external line* to $\mathcal{S}$ if $s=0$.

Lemma 2.5 [19]. *If each intersection number with planes and hyperplanes of a point set $\mathcal{H}$ in $\mathrm{PG}(4,q^2)$ is also an intersection number with planes and hyperplanes of $\mathcal{H}(4,q^2)$, then $\mathcal{H}$ is a nonsingular Hermitian variety $\mathcal{H}(4,q^2)$.*

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3. Hermitian cones of $\mathrm{PG}(3, q^2)$

THEOREM 3.1. *Let $\mathcal{S}$ be a surface of $\mathrm{PG}(3, q^2)$ of degree $q + 1$, q any prime power. If every plane meets $\mathcal{S}$ in $q^2 + 1$, $q^3 + 1$, or $q^3 + q^2 + 1$ points of $\mathrm{PG}(3, q^2)$, then $\mathcal{S}$ is a cone projecting a Hermitian curve in a plane π from a point V not in π.*

PROOF. Let π be a $(q^3 + q^2 + 1)$-plane. As $\mathcal{S}$ is a surface of degree $q + 1$, then $C = \mathcal{S} \cap \pi$ is a plane curve of degree $q + 1$. By Lemma 2.2, C must have some $\mathrm{GF}(q^2)$-line component and thus, by Lemma 2.1, C turns out to be a pencil of $q + 1$ lines of π. Furthermore, each line of π has to meet $\mathcal{S}$ in 1, $q + 1$, or $q^2 + 1$ rational points and, in particular, the surface $\mathcal{S}$ contains lines of $\mathrm{PG}(3, q^2)$.

Now assume that the plane π is a $(q^3 + 1)$-plane which does not have any $\mathrm{GF}(q^2)$-line of $\mathcal{S}$. In this case $C = \pi \cap \mathcal{S}$ is a plane curve of degree $q + 1$ without $\mathrm{GF}(q^2)$-line components and it has $q^3 + 1$ $\mathrm{GF}(q^2)$-points; thus, by Lemma 2.3, C is a nonsingular Hermitian curve for $q \neq 2$.

We are going to prove that $\mathcal{S}$ meets every line of $\mathrm{PG}(3, q^2)$, that is, $\mathcal{S}$ is a blocking set with respect to lines of the projective space. First, we assume that $q \neq 2$ and consider a line r of $\mathrm{PG}(3, q^2)$. If r is on a $(q^3 + q^2 + 1)$-plane, then r is at least a 1-secant line of $\mathcal{S}$. In the case in which r lies on a $(q^3 + 1)$-plane, say π, either π contains some line of $\mathcal{S}$ or $\pi \cap \mathcal{S}$ is a Hermitian unital of π; in both cases r turns out to be at least a 1-secant line of $\mathcal{S}$.

Thus, if r is an external line to $\mathcal{S}$, all planes through r have to be $(q^2 + 1)$-planes and the number $N_{q^2}(\mathcal{S})$ of rational points of $\mathcal{S}$ is $(q^2 + 1)^2$. Let t be a 1-secant line of $\mathcal{S}$ lying in some $(q^3 + q^2 + 1)$-plane and let t_i denote the numbers of i-planes through t. Counting the number of $\mathrm{GF}(q^2)$-points of $\mathcal{S}$ by using all planes through t,

$$(q^2 + 1)^2 = t_{q^2+1} q^2 + t_{q^3+1} q^3 + t_{q^3+q^2+1}(q^3 + q^2) + 1,$$

that gives

$$1 = (q - 1)t_{q^3+1} + q t_{q^3+q^2+1},$$

namely, $t_{q^3+1} = 0$ and $t_{q^3+q^2+1} = 1/q$, which is a contradiction.

Now we assume that $q = 2$. An algebraic plane curve of degree three in $\mathrm{PG}(2, 4)$, with nine rational points, without $\mathrm{GF}(4)$-line components is a unital or is projectively equivalent to the curve $C' : X_0^3 + wX_1^2 + w^2X_2^3 = 0$, which meets each line in zero, two or three rational points; see [9, §11]. Therefore, if r is an external line to $\mathcal{S}$, then r could be contained either in 5-planes or in 9-planes. Suppose that there is at least a planar section of $\mathcal{S}$ which consists of five rational points on a line. In this case, a 9-plane never can intersect $\mathcal{S}$ in an algebraic plane curve which is projectively equivalent to C'; therefore, only 5-planes can pass through an external line r of $\mathcal{S}$. Arguing as in the case $q \neq 2$, we get a contradiction.

Hence, each planar section of $\mathcal{S}$ with five points has to be an absolutely irreducible cubic curve with a cusp or a nonsingular cubic with one rational inflexion; see [9, §11]. Thus, a line of $\mathcal{S}$ lies either on a 9-plane or on a 13-plane, whereas a 2-secant line lies either on a 5-plane or on a 9-plane. Let m be a 2-secant line of a 5-plane, which we

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know to exist, and denote by x_m the number of 5-planes through m. Next, take a line ℓ of $\mathcal{S}$ and denote by x_ℓ the number of 9-planes through ℓ. Counting the number of GF(4)-points of $\mathcal{S}$ by using all planes through ℓ and all planes through m,

$$x_\ell(9-5) + (5-x_\ell)(13-5) + 5 = x_m(5-2) + (5-x_m)(9-2) + 2,$$

that gives $x_\ell = x_m + 2$. As $x_m \geq 1$, we obtain $x_\ell \in \{3, 4, 5\}$. Consequently, the number of rational points $N_4(\mathcal{S}) \in \{33, 29, 25\}$. In order to prove that none of the previous possibilities can occur for $N_4(\mathcal{S})$, we count in a double way the number of planes, the number of pairs (P, π), where $P \in \mathrm{PG}(3, 4)$ and π is a plane through P, and the number of pairs $((P, Q), \pi)$, where $P, Q \in \mathrm{PG}(3, 4)$ and π is a plane through P and Q. Let x, y, z denote the numbers of 5-, 9-, and 13-planes, respectively, we get the following equations:

$$\begin{cases} x + y + z = 85, \\ 5x + 9y + 13z = 21N_4(S), \\ 20x + 72y + 156z = 5N_4(S)(N_4(S) - 1). \end{cases} \tag{3.1}$$

For $N_4(\mathcal{S}) = 25$ or $N_4(\mathcal{S}) = 29$, (3.1) provides $z = 0$ or $z = -1$, respectively; in both cases we have a contradiction. When $N_4(\mathcal{S}) = 33$, (3.1) gives $z = 3$, that is, there are three 13-planes, each of which meets $\mathcal{S}$ in three concurrent lines. On the other hand, exactly two 13-planes have to pass through each line of $\mathcal{S}$ and hence we get a contradiction. Thus, $\mathcal{S}$ is a blocking set with respect to lines of $\mathrm{PG}(3, q^2)$ for all prime power q.

We recall that a blocking set with respect to lines of $\mathrm{PG}(2, q^2)$ which consists of $q^2 + 1$ points is a line; see [4]. Thus, if π is a $(q^2 + 1)$-plane, then $\pi \cap \mathcal{S}$ consists of $q^2 + 1$ points on a line.

Furthermore, each line which is not contained in $\mathcal{S}$ meets $\mathcal{S}$ in i points with $1 \leq i \leq q + 1$ as $\mathcal{S}$ is a surface of degree $q + 1$ over $\mathrm{GF}(q^2)$.

The next step is to prove that each line meets $\mathcal{S}$ in one, $q + 1$, or $q^2 + 1$ $\mathrm{GF}(q^2)$-points. By way of contradiction, assume that there is an i-secant line to $\mathcal{S}$, say m, with $2 \leq i \leq q$. Then each plane through m has to be a $(q^3 + 1)$-plane containing some line of $\mathcal{S}$. Counting the number of $\mathrm{GF}(q^2)$-points of $\mathcal{S}$ by using all planes through m, we obtain $N_{q^2}(\mathcal{S}) = (q^2 + 1)(q^3 + 1 - i) + i = q^5 + q^3 + q^2 - iq^2 + 1$. Let us consider a $(q^2 + 1)$-plane, say α, and let ℓ be the line $\alpha \cap \mathcal{S}$. By considering that each plane through ℓ has at most $q^3 + q^2 + 1$ $\mathrm{GF}(q^2)$-points of $\mathcal{S}$, we get that $N_{q^2}(\mathcal{S}) \leq q^2 q^3 + q^2 + 1 = q^5 + q^2 + 1$. Then $i \geq q$ and hence $i = q$, which gives $N_{q^2}(\mathcal{S}) = q^5 + q^2 + 1$. In particular, each line of $\mathcal{S}$ is contained in at most one $(q^2 + 1)$-plane. Now let x_i denote the numbers of i-planes with respect to $\mathcal{S}$. In this case double counting arguments give

$$\begin{cases} \sum_i x_i = (q^4 + 1)(q^2 + 1), \\ \sum_i i x_i = (q^5 + q^2 + 1)(q^4 + q^2 + 1), \\ \sum_{i=1} i(i-1)x_i = (q^5 + q^2 + 1)(q^5 + q^2)(q^2 + 1). \end{cases} \tag{3.2}$$

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By solving (3.2), we obtain in particular that the number x_{q^2+1} of (q^2+1)-planes is q^3+1.

Denote by $\Sigma = \{\alpha_1, \ldots, \alpha_{q^3+1}\}$ the set of all (q^2+1)-planes to $\mathcal{S}$ and set $\ell_i = \alpha_i \cap \mathcal{S}$ for all $\alpha_i \in \Sigma$. We observe that any two lines ℓ_i and ℓ_j, with $i \neq j$, intersect in a point and three of these lines never form a triangle. In fact, a triangle PQR of such lines would be contained in a (q^3+1)-plane π; since every line of π would meet $\mathcal{S}$ in at least two points, we would obtain in particular that every line of π through P would be at least a q-secant to $\mathcal{S}$ and hence we would get $|\pi \cap \mathcal{S}| \geq (q^2-1)(q-1) + 2q^2 + 1 > q^3+1$, which is a contradiction.

This means that the q^3+1 lines contained in $\mathcal{S}$ are concurrent at a point V. Since $\mathcal{S}$ has exactly $q^2(q^3+1)+1$ rational points, each other line contained in $\mathcal{S}$ cannot pass through V and has to meet q^2+1 lines among the lines ℓ_i, with $1 \leq i \leq q^3+1$. Thus, we find a GF(q^2)-planar component of $\mathcal{S}$ which is excluded. Hence, $\mathcal{S}$ contains exactly q^3+1 lines and, for each line ℓ contained in $\mathcal{S}$, exactly one (q^2+1)-plane through it exists whereas no plane through ℓ is a (q^3+1)-plane. But, then, there are no (q^3+1)-planes containing some line of $\mathcal{S}$, which is a contradiction.

Thus, each line which is not contained in $\mathcal{S}$ meets $\mathcal{S}$ in either one or $q+1$ rational points. For $q \neq 2$, from [11, Theorem 23.5.1], $\mathcal{S}$ has to be a cone $\Pi_0\mathcal{S}'$ with $\mathcal{S}'$ of type:

(I) a unital;
(II) a subplane PG$(2,q)$;
(III) a set of type $(0,q)$ plus an external line;
(IV) the complement of a set of type $(0,q^2-q)$.

As the possible intersection sizes with planes of PG$(3,q^2)$ are q^2+1, q^3+1, q^3+q^2+1, possibilities II, III, and IV must be excluded, since their sizes cannot be possible. This implies that $\mathcal{S} = \Pi_0\mathcal{S}'$, where $\mathcal{S}'$ is a unital. On the other hand, $\mathcal{S}'$ turns out to be an algebraic curve of degree $q+1$ without linear components and with q^3+1 points over GF(q^2). Thus, for $q \neq 2$, Lemma 2.3 applies and $\mathcal{S}'$ has to be a Hermitian curve.

For $q = 2$, there is just one point set in PG$(3,4)$ up to equivalence, meeting each line in one, three, or five points and each plane in five, nine, or 13 points, that is, the Hermitian cone; see [8, Theorem 19.6.8]. Thus also for $q = 2$ our theorem follows. □

As an easy consequence of Theorem 3.1, we get the following result.

Corollary 3.2. *Let $\mathcal{S}$ be a surface of* PG$(3,q^2)$ *of degree d. If every plane meets $\mathcal{S}$ in q^2+1, q^3+1, or q^3+q^2+1 points over* GF(q^2), *then $d \geq q+1$. If $d = q+1$, then $\mathcal{S}$ is a cone over a Hermitian curve.*

4. A characterization of $\mathcal{H}(4,q^2)$

Theorem 4.1. *Let $\mathcal{S}$ be a quasi-Hermitian variety of PG$(4,q^2)$. If $\mathcal{S}$ is a hypersurface of degree $q+1$, then $\mathcal{S}$ is a nonsingular Hermitian variety.*

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PROOF. We recall that $\mathcal{S}$ has $q^7 + q^5 + q^2 + 1$ rational points and its intersection numbers with respect to hyperplanes over $\mathrm{GF}(q^2)$ are $q^5 + q^2 + 1$ or $q^5 + q^3 + q^2 + 1$.

First, we prove that $\mathcal{S}$ does not contain any plane of $\mathrm{PG}(4, q^2)$. Suppose on the contrary that there is a plane α which is contained in $\mathcal{S}$. Let us denote by x the number of hyperplanes through α meeting $\mathcal{S}$ in $q^5 + q^2 + 1$ $\mathrm{GF}(q^2)$-points.

Counting the number $N_{q^2}(\mathcal{S})$ of $\mathrm{GF}(q^2)$-points of $\mathcal{S}$ by using all hyperplanes through α,

$$q^7 + q^5 + q^2 + 1 = N_{q^2}(\mathcal{S}) = (q^2 + 1 - x)(q^5 + q^3 - q^4) + x(q^5 - q^4) + q^4 + q^2 + 1,$$

that is,

$$xq^3 = -q^6 + q^5 + q^2,$$

which is a contradiction. This implies that $\Sigma = \mathcal{S} \cap \Pi$ is an algebraic surface of degree $q + 1$ without $\mathrm{GF}(q^2)$-plane components. In the case in which $N_{q^2}(\Sigma) = q^5 + q^3 + q^2 + 1$, by Lemma 2.4, Σ is a nonsingular Hermitian surface.

Now let Π' be a hyperplane of $\mathrm{PG}(4, q^2)$ meeting $\mathcal{S}$ in $q^5 + q^2 + 1$ rational points and set $\Sigma' = \Pi' \cap \mathcal{S}$. We are going to study the planar sections of Σ'. Thus, let us denote by α a plane contained in Π'. If at least one $(q^5 + q^3 + q^2 + 1)$-hyperplane passes through α, then $\alpha \cap \mathcal{S}$ is either a Hermitian curve or a pencil of $q + 1$ concurrent lines and, hence, $N_{q^2}(\alpha \cap \Sigma') = q^3 + 1$ or $N_{q^2}(\alpha \cap \Sigma') = q^3 + q^2 + 1$.

Suppose that all hyperplanes containing α meet $\mathcal{S}$ in $q^5 + q^2 + 1$ rational points and set $y = N_{q^2}(\alpha \cap \mathcal{S})$. Then

$$q^7 + q^5 + q^2 + 1 = N_{q^2}(\mathcal{S}) = (q^2 + 1)(q^5 + q^2 + 1 - y) + y,$$

namely, $y = q^2 + 1$ and, thus, $N_{q^2}(\alpha \cap \Sigma') = q^2 + 1$.

Theorem 3.1 applies and Σ' turns out to be a cone over a Hermitian curve. Then each intersection number over $\mathrm{GF}(q^2)$ with planes and hyperplanes of $\mathcal{S}$ is also an intersection number with planes and hyperplanes of $\mathcal{H}(4, q^2)$. By Lemma 2.5, $\mathcal{S}$ has to be a nonsingular Hermitian variety of $\mathrm{PG}(4, q^2)$. □

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ANGELA AGUGLIA, Dipartimento di Meccanica,
Matematica e Management, Politecnico di Bari, Via Orabona 4,
I-70126 Bari, Italy
e-mail: angela.aguglia@poliba.it

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