1

The Minimal Model Program

This chapter outlines the general theory of the minimal model program. We shall study algebraic threefolds thoroughly in the subsequent chapters in alignment with the program. The reader who is not familiar with the program may grasp the basic notions at first and refer back later.

Blowing up a surface at a point is not an essential operation from the birational point of view. Its exceptional curve is characterised numerically as a (-1)-curve. As is the case in this observation, the intersection number is a basic linear tool in birational geometry. The minimal model program, or the MMP for short, outputs a representative of each birational class that is minimal with respect to the numerical class of the canonical divisor.

The MMP grew out of the surface theory with allowing mild singularities. For a given variety, it produces a minimal model or a Mori fibre space after finitely many birational transformations, which are divisorial contractions and flips. Now the program is formulated in the logarithmic framework where we treat a pair consisting of a variety and a divisor.

The MMP functions subject to the existence and termination of flips. Hacon and McKernan with Birkar and Cascini proved the existence of flips in an arbitrary dimension. Considering a flip to be the relative canonical model, they established the MMP with scaling in the birational setting. The termination of threefold flips follows from the decrease in the number of divisors with small log discrepancy. Shokurov reduced the termination in an arbitrary dimension to certain conjectural properties of the minimal log discrepancy.

It is also important to analyse the representative output by the MMP. The Sarkisov program decomposes a birational map of Mori fibre spaces into elementary ones. For a minimal model, we expect the abundance which claims the freedom of the linear system of a multiple of the canonical divisor. It defines a morphism to the projective variety associated with the canonical ring, which we know is finitely generated.

1.1 Preliminaries

We shall fix the notation and recall the fundamentals of algebraic geometry. The book [178] by Hartshorne is a standard reference.

The *natural numbers* begin with zero. The symbol $R_{\geq r}$ for $R = \mathbf{N}, \mathbf{Z}, \mathbf{Q}$ or $\mathbf{R}$ stands for the subset $\{x \in R \mid x \geq r\}$ and similarly $R_{>r} = \{x \in R \mid x > r\}$. For instance, $\mathbf{N} = \mathbf{Z}_{\geq 0}$. The quotient $\mathbf{Z}_r = \mathbf{Z}/r\mathbf{Z}$ is the *cyclic group* of order r. The *round-down* $\lfloor r \rfloor$ of a real number r is the greatest integer less than or equal to r, whilst the *round-up* $\lceil r \rceil$ is defined as $\lceil r \rceil = -\lfloor -r \rfloor$.

Schemes A *scheme* is always assumed to be separated. It is said to be *integral* if it is irreducible and reduced.

We work over the field $\mathbf{C}$ of complex numbers unless otherwise mentioned. An *algebraic scheme* is a scheme of finite type over $\operatorname{Spec} k$ for the algebraically closed ground field k, which is tacitly assumed to be $\mathbf{C}$. We call it a *complex scheme* when we emphasise that it is defined over $\mathbf{C}$. An algebraic scheme is said to be *complete* if it is proper over $\operatorname{Spec} k$. A *point* in an algebraic scheme usually means a closed point.

A *variety* is an integral algebraic scheme. A *complex variety* is a variety over $\mathbf{C}$. A *curve* is a variety of dimension one and a *surface* is a variety of dimension two. An *n-fold* is a variety of dimension n. The *affine space* $\mathbf{A}^n$ is $\operatorname{Spec} k[x_1, \ldots, x_n]$ and the *projective space* $\mathbf{P}^n$ is $\operatorname{Proj} k[x_0, \ldots, x_n]$. The origin of $\mathbf{A}^n$ is denoted by o.

The *germ* $x \in X$ of a scheme is considered at a closed point unless otherwise specified. It is an equivalence class of the pair (X, x) of a scheme X and a point x in X where (X, x) is equivalent to (X', x') if there exists an isomorphism $U \simeq U'$ of open neighbourhoods $x \in U \subset X$ and $x' \in U' \subset X'$ sending x to x'. By a *singularity*, we mean the germ at a singular point as a rule.

For a locally free coherent sheaf $\mathscr{E}$ on an algebraic scheme X, the *projective space bundle* $\mathbf{P}(\mathscr{E}) = \operatorname{Proj}_X S\mathscr{E}$ over X is defined by the symmetric $\mathscr{O}_X$-algebra $S\mathscr{E} = \bigoplus_{i \in \mathbf{N}} S^i\mathscr{E}$ of $\mathscr{E}$. It is a $\mathbf{P}^n$*-bundle* if $\mathscr{E}$ is of rank $n + 1$. In particular, the projective space $\mathbf{P}V = \operatorname{Proj} SV$ is defined for a finite dimensional vector space V. It is regarded as the quotient space $(V^\vee \setminus 0)/k^\times$ of the dual vector space $V^\vee$ minus zero by the action of the multiplicative group $k^\times = k \setminus \{0\}$ of the ground field k. As used above, the symbol $^\vee$ stands for the dual and $^\times$ for the group of units.

Morphisms For a morphism $\pi\colon X \to Y$ of schemes, the *image* $\pi(A)$ of a closed subset A of X and the *inverse image* $\pi^{-1}(B)$ of a closed subset B of Y are considered set-theoretically. When π is proper and A is a closed subscheme, we regard $\pi(A)$ as a reduced scheme. We also regard $\pi^{-1}(B)$ for a closed subscheme B as a reduced scheme and distinguish it from the scheme-theoretic fibre $X \times_Y B$.

A *rational map* $f \colon X \dashrightarrow Y$ of algebraic schemes is an equivalence class of a morphism $U \to Y$ defined on a dense open subset U of X. The *image* $f(X)$ of f is the image $p(\Gamma)$ of the graph Γ of f as a closed subscheme of $X \times Y$ by the projection $p \colon X \times Y \to Y$. We say that a morphism or a rational map is *birational* if it has an inverse as a rational map. Two algebraic schemes are *birational* if there exists a birational map between them. By definition, two varieties are birational if and only if they have the same function field.

Let $\pi \colon X \to Y$ be a morphism of algebraic schemes. We say that π is *projective* if it is isomorphic to $\operatorname{Proj}_Y \mathscr{R} \to Y$ by a graded $\mathscr{O}_Y$-algebra $\mathscr{R} = \bigoplus_{i \in \mathbf{N}} \mathscr{R}_i$ generated by coherent $\mathscr{R}_1$, with $\mathscr{R}_0 = \mathscr{O}_Y$. When Y is quasi-projective, the projectivity of π means that it is realised as a closed subscheme of a relative projective space $\mathbf{P}^n \times Y \to Y$. An invertible sheaf $\mathscr{L}$ on X is *relatively very ample* (or *very ample* over Y or π*-very ample*) if it is isomorphic to $\mathscr{O}(1)$ by an expression $X \simeq \operatorname{Proj}_Y \mathscr{R}$ as above. We say that $\mathscr{L}$ is *relatively ample* (π*-ample*) if $\mathscr{L}^{\otimes a}$ is relatively very ample for some positive integer a.

Suppose that $\pi \colon X \to Y$ is proper. We say that π *has connected fibres* if the natural map $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ is an isomorphism. This implies that the fibre $X \times_Y y$ at every $y \in Y$ is connected and non-empty [160, III corollaire 4.3.2]. The proof for a projective morphism is in [178, III corollary 11.3]. In general, π admits the *Stein factorisation* $\pi = g \circ f$ with $f \colon X \to Z$ and $g \colon Z \to Y$ defined by $Z = \operatorname{Spec}_Y \pi_*\mathscr{O}_X$, for which f is proper with connected fibres and g is finite. If π is a proper birational morphism from a variety to a normal variety, then the factor g in the Stein factorisation is an isomorphism and hence π has connected fibres. This is referred to as *Zariski's main theorem*.

Lemma 1.1.1 *Let $\pi \colon X \to Y$ and $\varphi \colon X \to Z$ be morphisms of algebraic schemes such that π is proper and has connected fibres. If every curve in X contracted to a point by π is also contracted by φ, then φ factors through π as $\varphi = f \circ \pi$ for a morphism $f \colon Y \to Z$.*

Proof Let Y^m and Z^m denote the sets of closed points in Y and Z respectively. For $y \in Y^m$, the inverse image $\pi^{-1}(y)$ is connected and $\varphi(\pi^{-1}(y))$ is one point. Define $f^m \colon Y^m \to Z^m$ by $f^m(y) = \varphi(\pi^{-1}(y))$. Since π is proper and surjective, for any closed subset B of Z, $\pi(\varphi^{-1}(B))$ is closed in Y and $(f^m)^{-1}(B|_{Z^m}) = \pi(\varphi^{-1}(B))|_{Y^m}$. Thus f^m extends to a continuous map $f \colon Y \to Z$, which is a morphism of schemes by the natural map $\mathscr{O}_Z \to \varphi_*\mathscr{O}_X = f_*\pi_*\mathscr{O}_X = f_*\mathscr{O}_Y$. □

Chow's lemma [160, II §5.6] replaces the proper morphism $\pi \colon X \to Y$ by a projective morphism. It asserts the existence of a projective birational morphism $\mu\colon X' \to X$ such that $\pi \circ \mu\colon X' \to Y$ is projective. The *projection formula* and the *Leray spectral sequence*, formulated for ringed spaces in [160, 0 §12.2], will be frequently used. The reference [198, section 3.6] explains spectral sequences from our perspective.

Theorem 1.1.2 (Projection formula) *Let $\pi\colon X \to Y$ be a morphism of ringed spaces. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and let $\mathscr{E}$ be a finite locally free $\mathscr{O}_Y$-module. Then there exists a natural isomorphism $R^i\pi_*\mathscr{F} \otimes \mathscr{E} \simeq R^i\pi_*(\mathscr{F} \otimes \pi^*\mathscr{E})$.*

Theorem 1.1.3 (Leray spectral sequence) *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of ringed spaces. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then there exists a spectral sequence*

$$E_2^{p,q} = R^p g_* R^q f_* \mathscr{F} \Rightarrow E^{p+q} = R^{p+q}(g \circ f)_*\mathscr{F}.$$

In practice for a spectral sequence $E_2^{p,q} \Rightarrow E^{p+q}$, we assume that $E_2^{p,q}$ is zero whenever p or q is negative. Then there exists an exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2.$$

If further $E_2^{p,q} = 0$ for all $p \geq 0$ and $q \geq 1$, then $E_2^{p,0} \simeq E^p$. Likewise if $E_2^{p,q} = 0$ for all $p \geq 1$ and $q \geq 0$, then $E_2^{0,q} \simeq E^q$.

Cohomologies We write $H^i(\mathscr{F})$ for the cohomology $H^i(X, \mathscr{F})$ of a sheaf $\mathscr{F}$ of abelian groups on a topological space X when there is no confusion. If X is noetherian, then $H^i(\mathscr{F})$ vanishes for all i greater than the dimension of X.

Let $\mathscr{F}$ be a coherent sheaf on an algebraic scheme X. If X is affine, then $H^i(\mathscr{F}) = 0$ for all $i \geq 1$. If $\pi\colon X \to Y$ is a proper morphism, then the higher direct image $R^i\pi_*\mathscr{F}$ is coherent [160, III théorème 3.2.1]. In particular if X is complete, then $H^i(\mathscr{F})$ is a finite dimensional vector space. The dimension of $H^i(\mathscr{F})$ is denoted by $h^i(\mathscr{F})$. The alternating sum $\chi(\mathscr{F}) = \sum_{i \in \mathbf{N}}(-1)^i h^i(\mathscr{F})$ is called the *Euler characteristic* of $\mathscr{F}$.

Let X be a complete scheme of dimension n. For a coherent sheaf $\mathscr{F}$ and an invertible sheaf $\mathscr{L}$ on X, the *asymptotic Riemann–Roch theorem* defines the *intersection number* $(\mathscr{L}^n \cdot \mathscr{F}) \in \mathbf{Z}$ by the expression

$$\chi(\mathscr{L}^{\otimes l} \otimes \mathscr{F}) = \frac{(\mathscr{L}^n \cdot \mathscr{F})}{n!} l^n + O(l^{n-1}),$$

where by *Landau's symbol* O, $f(l) = O(g(l))$ means the existence of a constant c such that $|f(l)| \leq c|g(l)|$ for any large l. By this, Grothendieck's *dévissage* yields the estimate $h^i(\mathscr{F} \otimes \mathscr{L}^{\otimes l}) = O(l^n)$ for all i [266, section VI.2].

If X is projective with a very ample sheaf $\mathscr{O}_X(1)$, then the Euler characteristic $\chi(\mathscr{F} \otimes \mathscr{O}_X(l))$ is described as a polynomial in $\mathbf{Q}[l]$, called the *Hilbert* *polynomial* of $\mathscr{F}$. The vanishing of $H^i(\mathscr{F} \otimes \mathscr{O}_X(l))$ below is known as *Serre vanishing*.

Theorem 1.1.4 (Serre) *Let $\mathscr{F}$ be a coherent sheaf on a projective scheme X. Then for any sufficiently large integer l, the twisted sheaf $\mathscr{F} \otimes \mathscr{O}_X(l)$ is generated by global sections and satisfies $H^i(\mathscr{F} \otimes \mathscr{O}_X(l)) = 0$ for all $i \geq 1$.*

We have the *cohomology and base change theorem* for flat families of coherent sheaves [160, III §§7.6–7.9], [361, section 5]. See also [178, section III.12].

Theorem 1.1.5 (Cohomology and base change) *Let $\pi\colon X \to T$ be a proper morphism of algebraic schemes. Let $\mathscr{F}$ be a coherent sheaf on X flat over T. Take the restriction $\mathscr{F}_t$ of $\mathscr{F}$ to the fibre $X_t = X \times_T t$ at a closed point t in T and consider the natural map*

$$\alpha_t^i\colon R^i\pi_*\mathscr{F} \otimes k(t) \to H^i(X_t, \mathscr{F}_t),$$

where $k(t)$ is the skyscraper sheaf of the residue field at t.

(i) *The dimension $h^i(\mathscr{F}_t)$ is upper semi-continuous on T and the Euler characteristic $\chi(\mathscr{F}_t)$ is locally constant on T.*

(ii) *Fix i and t and suppose that α_t^i is surjective. Then $\alpha_{t'}^i$ is an isomorphism for all t' in a neighbourhood at t. Further, $R^i\pi_*\mathscr{F}$ is locally free at t if and only if α_t^{i-1} is surjective.*

(iii) (Grauert) *Suppose that T is reduced. Fix i. If $h^i(\mathscr{F}_t)$ is locally constant, then $R^i\pi_*\mathscr{F}$ is locally free and α_t^i is an isomorphism.*

Divisors Let X be an algebraic scheme. We write $\mathscr{K}_X$ for the sheaf of total quotient rings of $\mathscr{O}_X$. If X is a variety, then it is the constant sheaf of the function field $K(X)$ of X. A *Cartier divisor* D on X is a global section of the quotient sheaf $\mathscr{K}_X^\times/\mathscr{O}_X^\times$ of multiplicative groups of units. It is associated with an invertible subsheaf $\mathscr{O}_X(D)$ of $\mathscr{K}_X$. If D is represented by local sections $f_i \in \mathscr{K}_{U_i}^\times$ with $f_i f_j^{-1} \in \mathscr{O}_{U_i \cap U_j}^\times$, then $\mathscr{O}_X(D)|_{U_i} = f_i^{-1}\mathscr{O}_{U_i}$. We say that D is *principal* if it is defined by a global section of $\mathscr{K}_X^\times$ or equivalently $\mathscr{O}_X(D) \simeq \mathscr{O}_X$. The principal divisor given by $f \in \Gamma(X, \mathscr{K}_X^\times)$ is denoted by $(f)_X$. If f_i belongs to $\mathscr{O}_{U_i} \cap \mathscr{K}_{U_i}^\times$ for all i, then D defines a closed subscheme of X and we say that D is *effective*.

The *Picard group* $\operatorname{Pic} X$ of X is the group of isomorphism classes of invertible sheaves on X. It has an isomorphism

$$\operatorname{Pic} X \simeq H^1(\mathscr{O}_X^\times).$$

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

In fact this holds for any ringed space via Čech cohomology. The proof is found in [440, section 5.4]. The isomorphism for a variety X is derived at once from the vanishing of $H^1(\mathscr{K}_X^\times)$ for the flasque sheaf $\mathscr{K}_X^\times$.

By *Serre's criterion*, an algebraic scheme X is normal if and only if it satisfies the conditions R_1 and S_2 defined as

(R_i) for any $\eta \in X$, $\mathscr{O}_{X,\eta}$ is regular if $\mathscr{O}_{X,\eta}$ is of dimension at most i and
(S_i) for any $\eta \in X$, $\mathscr{O}_{X,\eta}$ is Cohen–Macaulay if $\mathscr{O}_{X,\eta}$ is of depth less than i,

in which we consider scheme-theoretic points $\eta \in X$. Let X be a normal variety. A closed subvariety of codimension one in X is called a *prime divisor*. A *Weil divisor* D on X, or simply called a *divisor*, is an element in the free abelian group $Z^1(X)$ generated by prime divisors on X. A Cartier divisor on a normal variety is a Weil divisor. Every Weil divisor on a smooth variety is Cartier. The divisor D is expressed as a finite sum $D = \sum_i d_i D_i$ of prime divisors D_i with non-zero integers d_i. The *support* of D is the union of D_i. The divisor D is *effective* if all d_i are positive, and it is *reduced* if all d_i equal one. We write $D \le D'$ if $D' - D$ is effective. The *linear equivalence* $D \sim D'$ of divisors means that $D' - D$ is principal.

The divisor D is associated with a divisorial sheaf $\mathscr{O}_X(D)$ on X. A *divisorial* sheaf is a reflexive sheaf of rank one, where a coherent sheaf $\mathscr{F}$ is said to be *reflexive* if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ to the double dual is an isomorphism. The sheaf $\mathscr{O}_X(D)$ is the subsheaf of $\mathscr{K}_X$ defined by

$$\Gamma(U, \mathscr{O}_X(D)) = \{f \in K(X) \mid (f)_U + D|_U \ge 0\},$$

in which zero is contained in the set on the right by convention. The *divisor class group* $\operatorname{Cl} X$ is the quotient of the group $Z^1(X)$ of Weil divisors divided by the subgroup of principal divisors. It is regarded as the group of isomorphism classes of divisorial sheaves on X and has an injection $\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X$.

Linear systems Let X be a normal complete variety. Let D be a Weil divisor on X and let V be a vector subspace of global sections in $H^0(\mathscr{O}_X(D))$. The projective space $\Lambda = \mathbf{P}V^\vee = (V \setminus 0)/k^\times$ where k is the ground field is called a *linear system* on X. It defines a rational map $X \dashrightarrow \mathbf{P}V$. When $V = H^0(\mathscr{O}_X(D))$, we write $|D| = \mathbf{P}H^0(\mathscr{O}_X(D))^\vee$ and call it a *complete* linear system. By the inclusion $\mathscr{O}_X(D) \subset \mathscr{K}_X$, the linear system $|D|$ is regarded as the set of effective divisors D' linearly equivalent to D, and Λ is a subset of $|D|$. That is,

$$\Lambda \subset |D| = \{D' \ge 0 \mid D' \sim D\}.$$

The *base locus* of Λ means the scheme-theoretic intersection $B = \bigcap_{D' \in \Lambda} D'$ in X. We say that the linear system Λ is *free* if B is empty. We say that Λ is *mobile* if B is of codimension at least two. The divisor D is said to be *free* (resp.

mobile) if $|D|$ is free (resp. mobile). By definition, a free Weil divisor is Cartier. When $\emptyset \neq \Lambda \subset |D|$, Λ is decomposed as $\Lambda = \Lambda' + F$ with a mobile linear system $\Lambda' \subset |D - F|$ and the maximal effective divisor F such that $F \leq D_1$ for all $D_1 \in \Lambda$. The constituents Λ' and F are called the *mobile part* and the *fixed part* of Λ respectively. The rational map defined by Λ' is isomorphic to $X \dashrightarrow \mathbf{P}V$. The linear system Λ is mobile if and only if F is zero.

Even if X is not complete, the linear system $\Lambda = \mathbf{P}V^\vee$ is defined for a finite dimensional vector subspace V of $H^0(\mathscr{O}_X(D))$. We consider $|D|$ to be the direct limit $\varinjlim_V \Lambda$ of linear systems.

A *general* point in a variety Z means a point in a dense open subset U of Z. A *very general* point in Z means a point in the intersection $\bigcap_{i\in\mathbf{N}} U_i$ of countably many dense open subsets U_i. Thus by the general member of the linear system Λ, we mean a general point in Λ as a projective space. Bertini's theorem asserts that a free linear system on a smooth complex variety has a smooth member. The statement for the hyperplane section holds even in positive characteristic.

Theorem 1.1.6 (Bertini's theorem) *Let $\Lambda = \mathbf{P}V^\vee$ be a free linear system on a smooth variety X and let $\varphi\colon X \to \mathbf{P}V$ be the induced morphism. Suppose that φ is a closed embedding or the ground field is of characteristic zero. Then the general member H of Λ is a smooth divisor on X, and if the image $\varphi(X)$ is of dimension at least two, then H is a smooth prime divisor.*

The canonical divisor It is the canonical divisor that plays the most important role in birational geometry. The *sheaf of differentials* on an algebraic scheme X is denoted by Ω_X. When X is smooth, Ω_X^i denotes the i-th exterior power $\bigwedge^i \Omega_X$.

Definition 1.1.7 The *canonical divisor* K_X on a normal variety X is the divisor defined up to linear equivalence by the isomorphism $\mathscr{O}_X(K_X)|_U \simeq \Omega_U^n$ on the smooth locus U in X, where n is the dimension of X.

Example 1.1.8 The projective space $\mathbf{P}^n$ has the canonical divisor $K_{\mathbf{P}^n} \sim -(n+1)H$ for a hyperplane H. This follows from the *Euler sequence*

$$0 \to \Omega_{\mathbf{P}^n} \to \mathscr{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)} \to \mathscr{O}_{\mathbf{P}^n} \to 0.$$

One can describe $K_{\mathbf{P}^n}$ in an explicit way. Take homogeneous coordinates $x_0, \dots, x_n$ of $\mathbf{P}^n$. Let $U_i \simeq \mathbf{A}^n$ denote the complement of the hyperplane H_i defined by x_i. The chart U_0 admits a nowhere vanishing n-form $dy_1 \wedge \cdots \wedge dy_n$ with coordinates $y_1, \dots, y_n$ for $y_i = x_i x_0^{-1}$. It is expressed on the chart U_1 having coordinates $z_0, z_2, \dots, z_n$ for $z_i = x_i x_1^{-1}$ as the rational n-form $dz_0^{-1} \wedge d(z_2 z_0^{-1}) \wedge \cdots \wedge d(z_n z_0^{-1}) = -z_0^{-(n+1)} dz_0 \wedge dz_2 \wedge \cdots \wedge dz_n$, which has pole of order $n+1$ along H_0. Thus $K_{\mathbf{P}^n} \sim -(n+1)H_0$.

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

In spite of the ambiguity concerning linear equivalence, it is standard to treat the canonical divisor as if it were a specified divisor.

For a closed subscheme D of an algebraic scheme X, there exists an exact sequence $\mathscr{I}/\mathscr{I}^2 \to \Omega_X \otimes \mathscr{O}_D \to \Omega_D \to 0$, where $\mathscr{I}$ is the ideal sheaf in $\mathscr{O}_X$ defining D. This induces the *adjunction formula*, which connects the canonical divisor to that on a Cartier divisor.

Theorem 1.1.9 (Adjunction formula) *Let X be a normal variety and let D be a reduced Cartier divisor on X which is normal. Then $K_D = (K_X + D)|_D$ in the sense that $\mathscr{O}_D(K_D) \simeq \mathscr{O}_X(K_X + D) \otimes \mathscr{O}_D$.*

Duality Albeit Grothendieck's duality theory works in the derived category for proper morphisms [177], it is extremely hard to obtain the dualising complex and a trace map in a compatible manner. The theory becomes efficient if it is restricted to the Cohen–Macaulay projective case as explained in [178, section III.7] and [277, section 5.5]. For example, the dualising complex on a Cohen–Macaulay projective scheme X of pure dimension n is the shift $\omega_X[n]$ of the dualising sheaf ω_X.

Definition 1.1.10 Let X be a complete scheme of dimension n over an algebraically closed field k. The *dualising sheaf* ω_X for X is a coherent sheaf on X endowed with a *trace map* $t\colon H^n(\omega_X) \to k$ such that for any coherent sheaf $\mathscr{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathscr{F}, \omega_X) \times H^n(\mathscr{F}) \to H^n(\omega_X) \xrightarrow{t} k$$

induces an isomorphism $\mathrm{Hom}(\mathscr{F}, \omega_X) \simeq H^n(\mathscr{F})^\vee$.

The dualising sheaf is unique up to isomorphism if it exists. The projective space $\mathbf{P}^n$ has the dualising sheaf $\omega_{\mathbf{P}^n} \simeq \bigwedge^n \Omega_{\mathbf{P}^n}$. This with Lemma 1.1.11 yields the existence of ω_X for every projective scheme X by taking a finite morphism $X \to \mathbf{P}^n$ known as projective *Noether normalisation*. If X is embedded into a projective space P with codimension r, then $\omega_X \simeq \mathscr{E}xt^r_P(\mathscr{O}_X, \omega_P)$ [178, III proposition 7.5]. If X is a normal projective variety, then ω_X coincides with the sheaf $\mathscr{O}_X(K_X)$ associated with the canonical divisor.

For a finite morphism $\pi\colon X \to Y$ of algebraic schemes, the push-forward π_* defines an equivalence of categories from the category of coherent $\mathscr{O}_X$-modules to that of coherent $\pi_*\mathscr{O}_X$-modules. This associates every coherent sheaf $\mathscr{G}$ on Y functorially with a coherent sheaf $\pi^!\mathscr{G}$ on X satisfying $\pi_*\mathscr{H}om_X(\mathscr{F}, \pi^!\mathscr{G}) \simeq \mathscr{H}om_Y(\pi_*\mathscr{F}, \mathscr{G})$ for any coherent sheaf $\mathscr{F}$ on X.

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

Lemma 1.1.11 *Let $\pi\colon X \to Y$ be a finite morphism of complete schemes of the same dimension. If the dualising sheaf ω_Y for Y exists, then $\omega_X = \pi^!\omega_Y$ is the dualising sheaf for X.*

Proof Let n denote the common dimension of X and Y. For a coherent sheaf $\mathscr{F}$ on X, $\operatorname{Hom}_X(\mathscr{F}, \pi^!\omega_Y) = \operatorname{Hom}_Y(\pi_*\mathscr{F}, \omega_Y)$ is dual to $H^n(\mathscr{F}) = H^n(\pi_*\mathscr{F})$ by the property of ω_Y, where the latter equality follows from the Leray spectral sequence $H^p(R^q\pi_*\mathscr{F}) \Rightarrow H^{p+q}(\mathscr{F})$. □

The duality for Cohen–Macaulay sheaves on a projective scheme is derived from that on the projective space via projective Noether normalisation. See [277, theorem 5.71].

Theorem 1.1.12 (Serre duality) *Let X be a projective scheme of dimension n. Let $\mathscr{F}$ be a Cohen–Macaulay coherent sheaf on X with support of pure dimension n. Then $H^i(\mathscr{H}om_X(\mathscr{F}, \omega_X))$ is dual to $H^{n-i}(\mathscr{F})$ for all i.*

The *adjunction formula* $\omega_D \simeq \omega_X \otimes \mathscr{O}_X(D) \otimes \mathscr{O}_D$ holds for a Cohen–Macaulay projective scheme X of pure dimension and an effective Cartier divisor D on X. Compare it with Theorem 1.1.9.

Resolution of singularities A projective birational morphism is described as a blow-up. The *blow-up* of an algebraic scheme X along a coherent ideal sheaf $\mathscr{I}$ in $\mathscr{O}_X$, or along the closed subscheme defined by $\mathscr{I}$, is the projective morphism $\pi\colon B = \operatorname{Proj}_X \bigoplus_{i\in\mathbf{N}} \mathscr{I}^i \to X$. The pull-back $\mathscr{I}\mathscr{O}_B = \pi^{-1}\mathscr{I} \cdot \mathscr{O}_B$ in $\mathscr{O}_B$ is an invertible ideal sheaf. Notice that $\mathscr{I}\mathscr{O}_B$ is different from $\pi^*\mathscr{I}$. The blow-up π has the universal property that every morphism $\varphi\colon Y \to X$ that makes $\mathscr{I}\mathscr{O}_Y$ invertible factors through π as $\varphi = \pi \circ f$ for a morphism $f\colon Y \to B$.

Let $f\colon X \dashrightarrow Y$ be a birational map of varieties. The *exceptional locus* of f is the locus in X where f is not biregular. Let Z be a closed subvariety of X not contained in the exceptional locus of f. The *strict transform* f_*Z in Y of Z is the closure of the image of $Z \dashrightarrow Y$. When X and Y are normal, the *strict transform* f_*P in Y of an arbitrary prime divisor P on X is defined as a divisor in such a manner that f_*P is zero if P is in the exceptional locus of f. By linear extension, we define the strict transform f_*D in Y for any divisor D on X.

Resolution of singularities is a fundamental tool in complex birational geometry. We say that a reduced divisor D on a smooth variety X is *simple normal crossing*, or *snc* for short, if D is defined at every point x in X by the product $x_1 \cdots x_m$ of a part of a regular system $x_1, \ldots, x_n$ of parameters in $\mathscr{O}_{X,x}$.

Definition 1.1.13 A *resolution* of a variety X is a projective birational morphism $\mu\colon X' \to X$ from a smooth variety. The resolution μ is said to be *strong* if it is isomorphic on the smooth locus in X.

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

Definition 1.1.14 Let X be a normal variety, let Δ be a divisor on X and let $\mathscr{I}$ be a coherent ideal sheaf in $\mathscr{O}_X$. A *log resolution* of $(X, \Delta, \mathscr{I})$ is a resolution $\mu\colon X' \to X$ such that

- the exceptional locus E of μ is a divisor on X',
- the pull-back $\mathscr{I}\mathscr{O}_{X'}$ is invertible and hence defines a divisor D and
- $E + D + \mu_*^{-1}S$ has snc support for the support S of Δ.

The log resolution μ is said to be *strong* if it is isomorphic on the maximal locus U in X such that U is smooth, $\mathscr{I}|_U$ defines a divisor D_U and $D_U + S|_U$ has snc support. A (strong) log resolution of X means that of $(X, 0, \mathscr{O}_X)$, and those of (X, Δ) and $(X, \mathscr{I})$ are likewise defined.

The existence of these resolutions for complex varieties is due to Hironaka. The items (i) and (ii) below are derived from the main theorems I and II in [187] respectively.

Theorem 1.1.15 (Hironaka [187]) (i) *A strong resolution exists for every complex variety.*
(ii) *A strong log resolution exists for every pair $(X, \mathscr{I})$ of a smooth complex variety X and a coherent ideal sheaf $\mathscr{I}$ in $\mathscr{O}_X$.*

Hironaka's construction includes the existence of a strong log resolution $X' \to X$ equipped with an effective exceptional divisor E on X' such that $\mathscr{O}_{X'}(-E)$ is relatively ample.

Analytic spaces We shall occasionally consider a complex scheme to be an analytic space in the Euclidean topology. Whilst an algebraic scheme is obtained by gluing affine schemes in $\mathbf{A}^n$, an analytic space is constructed by gluing analytic models in a domain in $\mathbf{C}^n$. A reference is [151]. The ring of convergent complex power series is denoted by $\mathbf{C}\{x_1, \ldots, x_n\}$.

Let D be a domain in the complex manifold $\mathbf{C}^n$. Let $\mathscr{O}_D$ denote the sheaf of holomorphic functions on D. Let $\mathscr{I}$ be an ideal sheaf in $\mathscr{O}_D$ generated by a finite number of global sections. The locally $\mathbf{C}$-ringed space $(V, (\mathscr{O}_D/\mathscr{I})|_V)$ for the support V of the quotient sheaf $\mathscr{O}_D/\mathscr{I}$ is called an *analytic model*, where being $\mathbf{C}$-*ringed* means having the structure sheaf of $\mathbf{C}$-algebras. An *analytic space* is a locally $\mathbf{C}$-ringed Hausdorff space such that every point has an open neighbourhood isomorphic to an analytic model.

Every complex scheme X is associated with an analytic space X_h. This defines a functor h from the category of complex schemes to the category of analytic spaces. There exists a natural morphism $X_h \to X$ of locally $\mathbf{C}$-ringed spaces which maps X_h bijectively to the set of closed points in X. It pulls back a coherent sheaf $\mathscr{F}$ on X to a coherent sheaf $\mathscr{F}_h$ on X_h. When X is complete, it induces an equivalence of categories. This is known as the *GAGA principle*, which takes the acronym from the title of Serre's paper [414].

Theorem 1.1.16 (GAGA principle [163, exposé XII], [414]) *Let X be a complete complex scheme and let X_h be the analytic space associated with X. Then the functor h induces an equivalence of categories from the category of coherent sheaves on X to the category of coherent sheaves on X_h.*

For an analytic space V, the exponential function $\exp(2\pi\sqrt{-1}t)$ defines a group homomorphism $\mathscr{O}_V \to \mathscr{O}_V^\times$. The induced exact sequence

$$0 \to \mathbf{Z} \to \mathscr{O}_V \to \mathscr{O}_V^\times \to 0$$

is called the *exponential sequence*.

In principle, one can deal with analytic spaces analogously to complex schemes as in [29]. For an analytic space V, the *Oka–Cartan theorem* asserts the coherence of every ideal sheaf in $\mathscr{O}_V$ that defines an analytic subspace of V. For a proper map $\pi\colon V \to W$ of analytic spaces, the higher direct image $R^i\pi_*\mathscr{F}$ of a coherent sheaf $\mathscr{F}$ on V is coherent on W. In particular, the image $\pi(V)$ is the support of the analytic subspace of W defined by the kernel of the map $\mathscr{O}_W \to \pi_*\mathscr{O}_V$, which is referred to as the *proper mapping theorem*.

The canonical divisor on a normal analytic space may not be defined as a finite sum of prime divisors. Some notions such as projectivity of resolution of singularities only make sense on a small neighbourhood about a fixed compact subset of an analytic space. These will pose no obstacles as we mainly work on the germ at a point in the analytic category.

Notation 1.1.17 The symbol $\mathfrak{D}^n$ denotes a domain in the complex space $\mathbf{C}^n$ which contains the origin o. For example, we write $o \in \mathfrak{D}^n$ for a germ of a complex manifold.

1.2 Numerical Geometry

The intersection number is a basic linear tool in birational geometry. We shall define it in the relative setting of a proper morphism $X \to S$. This section works over an algebraically closed field k of any characteristic.

One encounters divisors with rational coefficients naturally. For example for a finite surjective morphism $X \to Y$ of smooth varieties tamely ramified along a smooth prime divisor D on Y, the ramification formula which will be proved in Theorem 2.2.20 expresses K_X as the pull-back of $K_Y + (1 - 1/m)D$ with the ramification index m along D. One also has divisors with real coefficients taking limits. We begin with formulation of these notions.

Let X be a normal variety. Let $Z^1(X)$ denote the group of Weil divisors on X. A **Q**-*divisor* is an element in the rational vector space $Z^1(X) \otimes \mathbf{Q}$. In like manner, an **R**-*divisor* is an element in $Z^1(X) \otimes \mathbf{R}$. An **R**-divisor D is expressed as a finite sum $D = \sum_i d_i D_i$ of prime divisors D_i with real coefficients d_i, and D is a **Q**-divisor if d_i are rational. It is *effective* if $d_i \geq 0$ for all i and $D \leq D'$ means that $D' - D$ is effective. The *round-down* $\lfloor D \rfloor$ and the *round-up* $\lceil D \rceil$ are defined as $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$ and $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$. We sometimes say that a usual divisor is *integral* to distinguish it from a **Q**-divisor and an **R**-divisor.

Let $C^1(X)$ denote the subgroup of $Z^1(X)$ generated by Cartier divisors on X. A **Q**-*Cartier* **Q**-divisor is an element in the rational vector space $C^1(X) \otimes \mathbf{Q}$. In other words, a **Q**-divisor D is **Q**-Cartier if and only if there exists a non-zero integer r such that rD is integral and Cartier. Likewise an **R**-*Cartier* **R**-divisor is an element in $C^1(X) \otimes \mathbf{R}$. An **R**-Cartier **Q**-divisor is always **Q**-Cartier but a **Q**-Cartier integral divisor is not necessarily Cartier.

Example 1.2.1 Consider the prime divisor $D = (x_1 = x_2 = 0)$ on the surface $X = (x_1^2 - x_2 x_3 = 0) \subset \mathbf{A}^3$ with coordinates x_1, x_2, x_3. Then $2D$ is the Cartier divisor defined by x_2 and the scheme-theoretic intersection $2D \cap l$ with the line $l = (x_1 = x_2 = x_3)$ in X is of length one. It follows that D is not Cartier.

Let $\pi \colon Y \to X$ be a morphism of normal varieties. The *pull-back* $\pi^* D$ of an **R**-Cartier **R**-divisor D on X is defined as an **R**-Cartier **R**-divisor on Y by the natural map $\pi^* \colon C^1(X) \otimes \mathbf{R} \to C^1(Y) \otimes \mathbf{R}$. If D is a **Q**-divisor, then so is $\pi^* D$.

Definition 1.2.2 Let X be a normal variety. We say that X is **Q**-*Gorenstein* if the canonical divisor K_X is **Q**-Cartier. We say that X is **Q**-*factorial* if all divisors on X are **Q**-Cartier, that is, $\operatorname{Cl} X / \operatorname{Pic} X$ is torsion. It is said to be *factorial* if all divisors are Cartier, that is, $\operatorname{Pic} X = \operatorname{Cl} X$.

The **Q**-factoriality is not an analytically local property.

Example 1.2.3 The algebraic germ $o \in X = (x_1 x_2 + x_3 x_4 = 0) \subset \mathbf{A}^4$ is not **Q**-factorial. The prime divisor $D = (x_1 = x_4 = 0)$ on X is not **Q**-Cartier and the divisor class group $\operatorname{Cl} X$ is $\mathbf{Z}[D] \simeq \mathbf{Z}$. Indeed, the blow-up B of X at o resolves the projection $X \dashrightarrow \mathbf{P}^3$ from o as a morphism $B \to \mathbf{P}^3$ and it yields a line bundle $B \to S$ over the surface $S = (x_1 x_2 + x_3 x_4 = 0) \simeq \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$. By this structure, $\operatorname{Pic} B$ is generated by the strict transforms D_B and E_B of D and $E = (x_2 = x_4 = 0)$. They satisfy the relation $D_B + E_B + F \sim 0$ for the exceptional divisor F of $B \to X$. Thus $\operatorname{Cl} X \simeq \operatorname{Pic}(B \setminus F) = \mathbf{Z}[D_B \setminus F]$.

On the other hand, the algebraic germ $o \in Y = (x_1 x_2 + x_3 x_4 + f = 0) \subset \mathbf{A}^4$ is factorial for a general cubic form f in $x_1, \ldots, x_4$. To see this, we compactify Y to $\bar{Y} = (x_0(x_1 x_2 + x_3 x_4) + f = 0) \subset \mathbf{P}^4$. The blow-up $\bar{B}$ of $\bar{Y}$ at o resolves

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

the projection from o as $\bar{B} \to \mathbf{P}^3$, and this is the blow-up of $\mathbf{P}^3$ along the sextic curve $(x_1x_2 + x_3x_4 = f = 0)$. By this structure, $\operatorname{Pic} \bar{B}$ is generated by the exceptional divisor $\bar{F}$ of $\bar{B} \to \bar{Y}$ and the strict transform $\bar{H}_B$ of $\bar{H} = \bar{Y} \setminus Y$. Thus $\operatorname{Cl} Y \simeq \operatorname{Pic}(\bar{B} \setminus (\bar{F} + \bar{H}_B)) = 0$.

The two germs $o \in X$ and $o \in Y$ become isomorphic in the analytic category as will be seen in Proposition 2.3.3. See Remark 3.1.11 for further discussion.

We shall fix the base scheme S and work *relatively* on a proper morphism $\pi\colon X \to S$ of algebraic schemes, which is frequently denoted by X/S. Every terminology is accompanied by the reference to the relative setting. The reference is omitted when we consider a complete scheme X with the structure morphism $X \to S = \operatorname{Spec} k$.

A *relative subvariety* Z of X/S means a closed subvariety of X such that $\pi(Z)$ is a point in S. A *relative m-cycle* on X/S is an element in the free abelian group $Z_m(X/S)$ generated by relative subvarieties of dimension m in X/S. For invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_m$ and a relative m-cycle Z on X, the *intersection number* $(\mathscr{L}_1 \cdots \mathscr{L}_m \cdot Z)$ is defined by the multilinear map

$$(\operatorname{Pic} X)^{\oplus m} \times Z_m(X/S) \to \mathbf{Z}$$

such that $(\mathscr{L}^m \cdot Z)$ for a relative subvariety Z coincides with $(\mathscr{L}^m \cdot \mathscr{O}_Z)$ in the asymptotic Riemann–Roch theorem $\chi(\mathscr{L}^{\otimes l} \otimes \mathscr{O}_Z) = (\mathscr{L}^m \cdot \mathscr{O}_Z)l^m/m! + O(l^{m-1})$. The *intersection number* $(D_1 \cdots D_m \cdot Z)$ with Cartier divisors D_i on X is defined as $(\mathscr{O}_X(D_1) \cdots \mathscr{O}_X(D_m) \cdot Z)$. If D_i are effective and intersect properly on a relative subvariety Z, then $(D_1 \cdots D_m \cdot Z)$ equals the length of the structure sheaf $\mathscr{O}_A$ of the artinian scheme $A = D_1 \cap \cdots \cap D_m \cap Z$. The length of $\mathscr{O}_{A,x}$ for $x \in A$ is referred to as the *local intersection number* at x and denoted by $(D_1 \cdots D_m \cdot Z)_x$. When X is a complete variety of dimension n with the structure morphism $X \to S = \operatorname{Spec} k$, we write $(\mathscr{L}_1 \cdots \mathscr{L}_n) = (\mathscr{L}_1 \cdots \mathscr{L}_n \cdot X)$ and $D_1 \cdots D_n = (D_1 \cdots D_n)_X = (D_1 \cdots D_n \cdot X)$.

By the extension $(\operatorname{Pic} X \otimes \mathbf{R}) \times (Z_1(X/S) \otimes \mathbf{R}) \to \mathbf{R}$, the *relative numerical equivalence* $\equiv_S$ is defined in both the real vector spaces $\operatorname{Pic} X \otimes \mathbf{R}$ and $Z_1(X/S) \otimes \mathbf{R}$ in such a way that it induces a perfect pairing

$$N^1(X/S) \times N_1(X/S) \to \mathbf{R}$$

of vector spaces on the quotients $N^1(X/S) = (\operatorname{Pic} X \otimes \mathbf{R})/\equiv_S$ and $N_1(X/S) = (Z_1(X/S) \otimes \mathbf{R})/\equiv_S$. When $S = \operatorname{Spec} k$, we just write $\equiv$, $N^1(X)$ and $N_1(X)$ without reference to S as remarked above.

Definition 1.2.4 The spaces $N^1(X/S)$ and $N_1(X/S)$ are finite dimensional [254, IV§4, proposition 3]. The equal dimension of $N^1(X/S)$ and $N_1(X/S)$ is called the *relative Picard number* of X/S and denoted by $\rho(X/S)$. When

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

$S = \operatorname{Spec} k$, this number is called the *Picard number* of the complete scheme X and denoted by $\rho(X)$.

Let $\varphi\colon Y \to X$ be a proper morphism. It induces the *pull-back* $\varphi^*\colon \operatorname{Pic} X \to \operatorname{Pic} Y$ and the *push-forward* $\varphi_*\colon Z_m(Y/S) \to Z_m(X/S)$ as group homomorphisms. The push-forward $\varphi_* Z$ of a relative subvariety Z of Y/S is $d\varphi(Z)$ if the morphism $Z \to \varphi(Z)$ is generically finite of degree d, and $\varphi_* Z$ is zero if $\varphi(Z)$ is of dimension less than that of Z. These satisfy the *projection formula*

$$(\varphi^*\mathscr{L}_1 \cdots \varphi^*\mathscr{L}_m \cdot Z) = (\mathscr{L}_1 \cdots \mathscr{L}_m \cdot \varphi_* Z)$$

for invertible sheaves $\mathscr{L}_i$ on X and a relative m-cycle Z on Y. They yield $\varphi^*\colon N^1(X/S) \to N^1(Y/S)$ and dually $\varphi_*\colon N_1(Y/S) \to N_1(X/S)$. One also has the natural surjection $N^1(Y/S) \twoheadrightarrow N^1(Y/X)$ and injection $N_1(Y/X) \hookrightarrow N_1(Y/S)$. If φ is surjective, then $\varphi^*\colon N^1(X/S) \to N^1(Y/S)$ is injective and $\varphi_*\colon N_1(Y/S) \to N_1(X/S)$ is surjective.

Henceforth we fix a proper morphism $\pi\colon X \to S$ from a normal variety to a variety and make basic definitions for an $\mathbf{R}$-Cartier $\mathbf{R}$-divisor D on X. We say that integral divisors D and D' on X are *relatively linearly equivalent* and write $D \sim_S D'$ if the difference $D - D'$ is linearly equivalent to the pull-back $\pi^* B$ of some Cartier divisor B on S. Namely, $D - D'$ is zero in the quotient $\operatorname{Cl} X/\pi^* \operatorname{Pic} S$. For $\mathbf{R}$-divisors D and D' on X, the *relative* $\mathbf{R}$*-linear equivalence* $D \sim_{\mathbf{R},S} D'$ means that $D - D'$ is zero in $(\operatorname{Cl} X/\pi^* \operatorname{Pic} S) \otimes \mathbf{R}$. When D and D' are $\mathbf{Q}$-divisors, this is referred to as the *relative* $\mathbf{Q}$*-linear equivalence* and denoted by $D \sim_{\mathbf{Q},S} D'$. The space $\operatorname{Pic} X \otimes \mathbf{R}$ is regarded as that of $\mathbf{R}$-linear equivalence classes of $\mathbf{R}$-Cartier $\mathbf{R}$-divisors on X. The intersection number $(D \cdot C)$ is defined for a pair of an $\mathbf{R}$-Cartier $\mathbf{R}$-divisor D on X and a relative one-cycle C on X/S. This makes the notion of *relative numerical equivalence* $D \equiv_S D'$ for $\mathbf{R}$-Cartier $\mathbf{R}$-divisors D and D' on X.

Definition 1.2.5 An $\mathbf{R}$-Cartier $\mathbf{R}$-divisor D on X/S is said to be *relatively nef* (or *nef* over S or π*-nef*) if $(D \cdot C) \ge 0$ for any relative curve C in X/S. When $S = \operatorname{Spec} k$, we just say that D is *nef* as usual.

A Cartier divisor D on X is said to be *relatively ample* (π*-ample*) if $\mathscr{O}_X(D)$ is a relatively ample invertible sheaf. It is said to be *relatively very ample* (π*-very ample*) if $\mathscr{O}_X(D)$ is a relatively very ample invertible sheaf. In spite of the geometric definition, the ampleness is characterised numerically.

Theorem 1.2.6 (Nakai's criterion) *Let $\pi\colon X \to S$ be a proper morphism of algebraic schemes. An invertible sheaf $\mathscr{L}$ on X is relatively ample if and only if $(\mathscr{L}^{\dim Z} \cdot Z) > 0$ for any relative subvariety Z of X/S.*

Kleiman's ampleness criterion rephrases Nakai's criterion in terms of the cones of divisors and curves. A *convex cone* C, or simply called a *cone*, in a finite dimensional real vector space V is a subset of V such that if $v, w \in C$ and $c \in \mathbf{R}_{>0}$, then $v + w \in C$ and $cv \in C$.

Definition 1.2.7 The *ample cone* $A(X/S)$ is the convex cone in $N^1(X/S)$ spanned by the classes of relatively ample Cartier divisors on X. The *closed cone* $\overline{\mathrm{NE}}(X/S)$ *of curves* is the closure of the convex cone in $N_1(X/S)$ spanned by the classes of relative curves in X/S.

Notice that $A(X/S)$ is an open cone since for a relatively ample divisor A and a Cartier divisor D, the sum $D + lA$ is relatively ample for large l.

Theorem 1.2.8 (Kleiman's ampleness criterion [254]) *Let $\pi\colon X \to S$ be a proper morphism of algebraic schemes. Then a Cartier divisor D on X is relatively ample if and only if the class of D belongs to the ample cone $A(X/S)$. If π is projective, then $A(X/S)$ and $\overline{\mathrm{NE}}(X/S)$ are dual with respect to the intersection pairing $N^1(X/S) \times N_1(X/S) \to \mathbf{R}$ in the sense that*

$$A(X/S) = \{y \in N^1(X/S) \mid (y, z) > 0 \text{ for all } z \in \overline{\mathrm{NE}}(X/S) \setminus 0\},$$
$$\overline{\mathrm{NE}}(X/S) \setminus 0 = \{z \in N_1(X/S) \mid (y, z) > 0 \text{ for all } y \in A(X/S)\}.$$

The theorem shows that if π is projective, then the closure of the ample cone $A(X/S)$ coincides with the *nef cone* $\mathrm{Nef}(X/S)$ in $N^1(X/S)$ spanned by relatively nef $\mathbf{R}$-Cartier $\mathbf{R}$-divisors. The duality of $A(X/S)$ and $\overline{\mathrm{NE}}(X/S)$ still holds for a $\mathbf{Q}$-factorial complete variety $X/S = \operatorname{Spec} k$ as studied in [254], but it fails for a proper morphism in general.

Example 1.2.9 Fujino [128] constructed an example of a non-projective complete toric threefold X with $\rho(X) = 1$ such that $\overline{\mathrm{NE}}(X)$ is a half-line $\mathbf{R}_{\ge 0}$. The book [140] by Fulton is a standard introduction to toric varieties. Let $v_1 = (1, 0, 1)$, $v_2 = (0, 1, 1)$, $v_3 = (-1, -1, 1)$ and $w_1 = (1, 0, -1)$, $w_2 = (0, 1, -1)$, $w_3 = (-1, -1, -1)$ in $N = \mathbf{Z}^3$. Take the fan Δ which consists of faces of the cones $\langle v_1, v_2, v_3\rangle$, $\langle w_1, w_2, w_3\rangle$, $\langle v_1, v_2, w_1\rangle$, $\langle v_2, w_1, w_2\rangle$, $\langle v_2, v_3, w_2, w_3\rangle$, $\langle v_3, v_1, w_3, w_1\rangle$. The toric variety X associated with (N, Δ) is the example.

The numerical nature extends the notion of ampleness to $\mathbf{R}$-divisors.

Definition 1.2.10 An $\mathbf{R}$-Cartier $\mathbf{R}$-divisor D on X/S is said to be *relatively ample* (*π-ample*) if the class of D belongs to the ample cone $A(X/S)$. In other words, D is expressed as a finite sum $D = \sum_i a_i A_i$ of relatively ample Cartier divisors A_i with $a_i \in \mathbf{R}_{>0}$.

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

We keep $\pi\colon X \to S$ being a proper morphism from a normal variety to a variety. For a Cartier divisor D on X, the natural map $\pi^*\pi_*\mathscr{O}_X(D) \to \mathscr{O}_X(D)$ defines a rational map

$$X \dashrightarrow \operatorname{Proj}_S S\pi_*\mathscr{O}_X(D)$$

over S for the symmetric $\mathscr{O}_S$-algebra $S\pi_*\mathscr{O}_X(D)$ of $\pi_*\mathscr{O}_X(D)$. The *relative base locus* of D is the closed subscheme B of X given by the ideal sheaf $\mathscr{I}_B$ in $\mathscr{O}_X$ such that the above map induces the surjection $\pi^*\pi_*\mathscr{O}_X(D) \twoheadrightarrow \mathscr{I}_B\mathscr{O}_X(D)$. We say that D is *relatively free* (*π-free*) if B is empty. We say that D is *relatively mobile* (*π-mobile*) if B is of codimension at least two. The definitions coincide with those on a normal complete variety. Unless $B = X$, there exists a maximal effective divisor F such that $\mathscr{I}_B \subset \mathscr{O}_X(-F)$. The divisors $D - F$ and F are called the *relative mobile part* (*π-mobile part*) and the *relative fixed part* (*π-fixed part*) of D respectively.

Definition 1.2.11 A Cartier divisor D on X/S is said to be *relatively semi-ample* (*π-semi-ample*) if aD is relatively free for some positive integer a. An $\mathbf{R}$-Cartier $\mathbf{R}$-divisor D on X is said to be *relatively semi-ample* (*π-semi-ample*) if it is expressed as a finite sum $D = \sum_i a_iA_i$ of relatively semi-ample Cartier divisors A_i with $a_i \in \mathbf{R}_{\geq 0}$. The definition is consistent by the next lemma.

Lemma 1.2.12 *Let $\pi\colon X \to S$ be a proper morphism from a normal variety to a variety. Let D and $A_1, \dots, A_n$ be Cartier divisors on X such that $D = \sum_i a_iA_i$ with $a_i \in \mathbf{R}_{\geq 0}$. If all A_i are relatively free, then aD is relatively free for some positive integer a.*

Proof Let $Z^1(X)_{\mathbf{Q}}$ denote the rational vector space of $\mathbf{Q}$-divisors on X. Let V be the vector subspace of $Z^1(X)_{\mathbf{Q}}$ spanned by $A_1, \dots, A_n$. Then D belongs to $(V \otimes_{\mathbf{Q}} \mathbf{R}) \cap Z^1(X)_{\mathbf{Q}} = V$ and hence we may assume that $a_i \in \mathbf{Q}$ and further $a_i \in \mathbf{Z}$ by multiplying D. The assertion in this case follows from the existence of the natural map $\bigoplus_i (\pi^*\pi_*\mathscr{O}_X(A_i))^{\oplus a_i} \to \pi^*\pi_*\mathscr{O}_X(D)$. □

We provide an alternative characterisation of semi-ampleness.

Lemma 1.2.13 *Let $\pi\colon X \to S$ be a proper morphism from a normal variety to a variety. An $\mathbf{R}$-divisor D on X is relatively semi-ample if and only if there exists a projective morphism $\pi_Y : Y \to S$ from a normal variety through which π factors as $\pi = \pi_Y \circ \varphi$ for $\varphi\colon X \to Y$ such that $D \sim_{\mathbf{R}} \varphi^*A$ by a relatively ample $\mathbf{R}$-divisor A on Y/S.*

Proof The if part is obvious. We shall prove the only-if part for a relatively semi-ample $\mathbf{R}$-divisor D. Write D as a finite sum $D = \sum_i a_iB_i$ of relatively free divisors B_i with $a_i \in \mathbf{R}_{>0}$. The morphism $\varphi_i \colon X \to Y_i = \operatorname{Proj}_S S\pi_*\mathscr{O}_X(B_i)$ provides a relation $B_i \sim \varphi_i^* A_i$ by a relatively ample divisor A_i on Y_i/S.

Let $\varphi \colon X \to Y$ be the Stein factorisation of $X \to \operatorname{Proj}_S S\pi_*\mathscr{O}_X(\sum_i B_i)$. A relative curve C in X/S is contracted to a point by φ if and only if $(\sum_i B_i \cdot C) = 0$. This is equivalent to $(B_i \cdot C) = 0$ for all i since B_i are relatively nef. By Lemma 1.1.1, every φ_i factors through φ as $\varphi_i = \psi_i \circ \varphi$ for $\psi_i \colon Y \to Y_i$ and $D \sim_{\mathbf{R}} \varphi^* \sum_i a_i \psi_i^* A_i$. Then C is contracted by φ if and only if $(\varphi^*(\sum_i a_i \psi_i^* A_i) \cdot C) = 0$. This shows the relative ampleness of $\sum_i a_i \psi_i^* A_i$ on Y/S. □

Definition 1.2.14 A Cartier divisor D on X/S is said to be *relatively big* (*π-big*) if there exists a positive integer a such that the rational map $X \dashrightarrow \operatorname{Proj}_S S\pi_*\mathscr{O}_X(aD)$ is birational to the image.

Assuming that π is projective, *Kodaira's lemma* characterises the bigness numerically.

Theorem 1.2.15 (Kodaira's lemma) *Let $\pi \colon X \to S$ be a projective morphism from a normal variety to a quasi-projective variety. A Cartier divisor D on X is relatively big if and only if there exist a relatively ample $\mathbf{Q}$-divisor A and an effective $\mathbf{Q}$-divisor B such that $D = A + B$.*

Proof The if part is obvious. We shall prove the only-if part for a relatively big divisor D. By Stein factorisation, we may assume that π has connected fibres. Multiplying D, we may assume that $X \dashrightarrow \operatorname{Proj}_S S\pi_*\mathscr{O}_X(D)$ is birational to the image Y. We write $\pi_Y \colon Y \to S$. Take an open subset U of X such that $\varphi \colon U \to Y$ is a morphism and such that the complement $X \setminus U$ is of codimension at least two. Then $\mathscr{O}_X(D)|_U \simeq \varphi^*\mathscr{O}_Y(1)$ and

$$\pi_{Y*}\mathscr{O}_Y(l) \subset \pi_{Y*}\varphi_*\varphi^*\mathscr{O}_Y(l) \simeq \pi_*(\mathscr{O}_X(lD)|_U) = \pi_*\mathscr{O}_X(lD)$$

for any $l \in \mathbf{Z}$. Hence there exists a positive rational constant c such that the rank of the $\mathscr{O}_S$-module $\pi_*\mathscr{O}_X(lD)$ is greater than cl^n for sufficiently large l, where $n = \dim X - \dim S$ is the dimension of the general fibre of π_Y.

Take a general very ample effective divisor H on X. Since the rank of $\pi_*\mathscr{O}_H(lD|_H)$ is estimated as $O(l^{n-1})$, the exact sequence

$$0 \to \pi_*\mathscr{O}_X(lD - H) \to \pi_*\mathscr{O}_X(lD) \to \pi_*\mathscr{O}_H(lD|_H)$$

yields the non-vanishing $\pi_*\mathscr{O}_X(lD - H) \neq 0$ for large l. Hence $H^0(\mathscr{O}_X(lD - H + \pi^*G)) = H^0(\pi_*\mathscr{O}_X(lD - H) \otimes \mathscr{O}_S(G)) \neq 0$ by a sufficiently very ample divisor G on S. Thus one can write $lD - H + \pi^*G = B_1 + (f)_X$ with an effective divisor B_1 and a principal divisor $(f)_X$ on X. Then $D = A + B$ with $A = l^{-1}(H - \pi^*G + (f)_X)$ and $B = l^{-1}B_1$. □

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

Remark 1.2.16 Without the quasi-projectivity of the base variety, Kodaira's lemma gives the decomposition $D = A + B$ into a relatively ample $\mathbf{Q}$-divisor A and a $\mathbf{Q}$-divisor B such that $\pi_*\mathscr{O}_X(lB) \neq 0$ for some positive integer l. In some literature, a Cartier divisor D on X/S with $\pi_*\mathscr{O}_X(D) \neq 0$ is said to be *relatively effective*. Provided that S is quasi-projective, this means that D is relatively linearly equivalent to an effective divisor. We do not use this terminology for the reason that a relatively effective divisor over $S = \operatorname{Spec} \mathbf{C}$ is not necessarily effective but effective up to linear equivalence.

By definition, a Cartier divisor D on X is relatively big if and only if so is the restriction $D|_{\pi^{-1}(U)}$ over some open subset U of S containing the generic point of $\pi(X)$. Thus Kodaira's lemma with Kleiman's criterion implies that bigness on a projective morphism is a numerical condition. This provides grounds for considering the cone of big divisors.

Definition 1.2.17 Assume that $\pi\colon X \to S$ is projective. The *big cone* $B(X/S)$ is the convex cone in $N^1(X/S)$ spanned by the classes of relatively big Cartier divisors on X. This is an open cone containing the ample cone $A(X/S)$. An $\mathbf{R}$-Cartier $\mathbf{R}$-divisor D on X is said to be *relatively big* (π*-big*) if the class of D belongs to the big cone $B(X/S)$. Namely, D is expressed as a finite sum $D = \sum_i b_i B_i$ of relatively big Cartier divisors B_i with $b_i \in \mathbf{R}_{>0}$.

One can formulate Kodaira's lemma for $\mathbf{R}$-divisors.

Corollary 1.2.18 *Let $\pi\colon X \to S$ be a projective morphism from a normal variety to a quasi-projective variety. An $\mathbf{R}$-Cartier $\mathbf{R}$-divisor D on X is relatively big if and only if there exist a relatively ample $\mathbf{Q}$-divisor A and an effective $\mathbf{R}$-divisor B such that $D = A + B$.*

Finally we introduce the notion of numerical limit of effective $\mathbf{R}$-divisors.

Definition 1.2.19 The *pseudo-effective cone* $P(X/S)$ is the closure of the convex cone in $N^1(X/S)$ spanned by the classes of Cartier divisors D on X with $\pi_*\mathscr{O}_X(D) \neq 0$. An $\mathbf{R}$-Cartier $\mathbf{R}$-divisor D on X is said to be *relatively pseudo-effective* (π*-pseudo-effective*) if the class of D belongs to the pseudo-effective cone $P(X/S)$.

By Kodaira's lemma, if π is projective, then the pseudo-effective cone $P(X/S)$ coincides with the closure of the big cone $B(X/S)$. If S is quasi-projective, then a relatively pseudo-effective $\mathbf{R}$-divisor on X is realised in the space $N^1(X/S)$ as a limit of a sequence of effective $\mathbf{R}$-divisors.

Example 1.2.20 Let C be a smooth projective curve of genus g. The *Jacobian* $J(C)$ of C represents the subgroup $\operatorname{Pic}^0 C$ of $\operatorname{Pic} C$ which consists of invertible sheaves of degree zero. Refer to [318] for example. It is an abelian variety of dimension g. Thus as far as $g \geq 1$, C has an invertible sheaf $\mathscr{L}$ of degree zero which is not a torsion in $\operatorname{Pic}^0 C$. The divisor $D \equiv 0$ such that $\mathscr{L} \simeq \mathscr{O}_C(D)$ is pseudo-effective but not $\mathbf{Q}$-linearly equivalent to zero.

1.3 The Program

The classification theory in birational geometry seeks to find a good representative of each birational class of varieties and to analyse it. We shall explain how the surface theory has matured into the minimal model program in higher dimensions. The program will be generalised logarithmically and relatively in the next section. We shall tacitly work over $\mathbf{C}$ as mentioned in the first section. We define a contraction in the following manner.

Definition 1.3.1 A *contraction* $\pi\colon X \to Y$ is a projective morphism of normal varieties with connected fibres, namely $\mathscr{O}_Y = \pi_*\mathscr{O}_X$. It is said to be *of fibre type* if $\dim Y < \dim X$. Thus a contraction is either birational or of fibre type. We say that the contraction π is *extremal* if $\rho(X/Y) = 1$.

We recall the surface theory in brief. Standard books are [28], [30] and [35]. The treatise [403] is also excellent. Let S be a smooth projective surface. Though one can construct a new surface from S by blowing it up at an arbitrary point, this operation is not essential in the birational study of S. Its exceptional curve is superfluous. We want to obtain a simple birational model of S by contracting superfluous curves. *Custelnuovo's criterion* for contraction enables us to do so.

Definition 1.3.2 A curve C in a smooth surface is called a (-1)-*curve* if it is isomorphic to $\mathbf{P}^1$ with self-intersection number $(C^2) = -1$.

Theorem 1.3.3 (Castelnuovo's contraction theorem) *Let C be a (-1)-curve in a smooth projective surface S. Then C is the exceptional curve of the blow-up $S \to T$ of a smooth projective surface T at a point.*

The contraction of C decreases the Picard number by one as $\rho(T) = \rho(S) - 1$. Hence one eventually attains a surface without (-1)-curves by contracting them successively. This surface is characterised as follows.

Theorem 1.3.4 *If a smooth projective surface S has no (-1)-curves, then either*

(i) *the canonical divisor K_S is nef or*
(ii) *there exists a contraction $S \to B$ of fibre type which is a $\mathbf{P}^1$-bundle over a smooth curve or isomorphic to $\mathbf{P}^2 \to \operatorname{Spec} \mathbf{C}$.*

Notice that a surface S_1 with the property (i) is never birational to a surface S_2 with (ii). If there were a birational map $S_1 \dashrightarrow S_2$, then a general relative curve C_2 in the fibration $S_2 \to B$ would be mapped regularly to a curve C_1 in S_1 with $(K_{S_1} \cdot C_1) \le (K_{S_2} \cdot C_2)$. This contradicts the intersection numbers $(K_{S_1} \cdot C_1) \ge 0$ and $(K_{S_2} \cdot C_2) < 0$.

In the case (ii), if B is a curve, then S/B is described as $\mathbf{P}(\mathscr{E})/B$ by a locally free sheaf $\mathscr{E}$ of rank two on B. When $B \simeq \mathbf{P}^1$, it is completely classified as below. The reader may refer to [377, 1 theorem 2.1.1] for the proof.

Theorem 1.3.5 (Grothendieck [159]) *Every locally free sheaf of rank r on $\mathbf{P}^1$ is isomorphic to a direct sum $\bigoplus_{i=1}^{r} \mathscr{O}_{\mathbf{P}^1}(a_i)$ with $a_i \in \mathbf{Z}$.*

Hence every $\mathbf{P}^1$-bundle over $\mathbf{P}^1$ is isomorphic to a Hirzebruch surface.

Definition 1.3.6 The *Hirzebruch surface* Σ_n for $n \in \mathbf{N}$ is the $\mathbf{P}^1$-bundle $\Sigma_n = \mathbf{P}(\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(-n))$ over $\mathbf{P}^1$.

The surface S in the case (i) is classified with showing the semi-ampleness of K_S, that is, lK_S is free for some positive integer l. Then the complete linear system $|lK_S|$ defines a morphism $S \to \mathbf{P}H^0(\mathscr{O}_S(lK_S))$. When l is sufficiently large and divisible, the induced surjection to the image in $\mathbf{P}H^0(\mathscr{O}_S(lK_S))$ is independent of l. One can refer to [30, chapter VI] for the classification below.

Theorem 1.3.7 *Let S be a smooth projective surface such that K_S is nef. Then K_S is semi-ample and S is one of the following in terms of $q = h^1(\mathscr{O}_S)$, $\chi = \chi(\mathscr{O}_S)$ and the dimension κ of the image of $S \to \mathbf{P}H^0(\mathscr{O}_S(lK_S))$ for sufficiently large and divisible l.*

(i) *$\kappa = 0$. S is a K3 surface defined by $K_S \sim 0$ and $q = 0$. $\chi = 2$.*
(ii) *$\kappa = 0$. S is an Enriques surface defined by $K_S \nsim 0$, $2K_S \sim 0$ and $q = 0$. $\chi = 1$.*
(iii) *$\kappa = 0$. S is an abelian surface. $K_S \sim 0$, $q = 2$ and $\chi = 0$.*
(iv) *$\kappa = 0$. S is a hyperelliptic surface defined as the quotient $(E \times F)/G$ of the product of elliptic curves E and F by a finite subgroup G of the translations of E which acts on F so that $F/G \simeq \mathbf{P}^1$. The least positive integer r such that $rK_S \sim 0$ is 2, 3, 4 or 6. $q = 1$ and $\chi = 0$.*
(v) *$\kappa = 1$. $\chi \ge 0$.*
(vi) *$\kappa = 2$. $\chi \ge 1$.*

Both the contraction $S \to T$ of a (-1)-curve and the fibration $S \to B$ in Theorem 1.3.4(ii) are extremal contractions in Definition 1.3.1 with respect to which the *anti-canonical divisor* $-K_S$ is relatively ample. Hence given a

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

smooth projective surface S, the program for finding a model in Theorem 1.3.4 by Theorem 1.3.3 is described as the following algorithm.

1 If K_S is nef, then output S, which belongs to the case (i) in Theorem 1.3.4.
2 If K_S is not nef, then there exists an extremal contraction $\pi\colon S \to T$ to a smooth projective variety such that $-K_S$ is relatively ample.
3 If π is of fibre type, then output S, which belongs to the case (ii) in Theorem 1.3.4.
4 If π is birational, then replace S by T and go back to 1.

The minimal model program is a higher dimensional extension of this program. However, a naive extension is confronted with several obstacles as will be seen. Let X be a smooth projective variety and suppose the existence of an extremal contraction $X \to Y$ such that $-K_X$ is relatively ample. The first obstacle is that Y may be singular.

Example 1.3.8 Let $\mathbf{A}^3 = \operatorname{Spec} \mathbf{C}[x_1, x_2, x_3]$. Consider the action of $\mathbf{Z}_2$ on $\mathbf{A}^3$ given by the involution which sends (x_1, x_2, x_3) to $(-x_1, -x_2, -x_3)$. The quotient $\mu\colon \mathbf{A}^3 \to Y = \mathbf{A}^3/\mathbf{Z}_2 = \operatorname{Spec} R$ is defined by the invariant ring $R = \mathbf{C}[x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1]$. The germ $o \in Y$ at the image o of the origin of $\mathbf{A}^3$ is an isolated singularity known as the cyclic quotient singularity of type $\frac{1}{2}(1, 1, 1)$ in Definition 2.2.10. The canonical sheaf ω_Y is the invariant part of $\mu_*\omega_{\mathbf{A}^3}$, generated by $x_1\theta, x_2\theta, x_3\theta$ for $\theta = dx_1 \wedge dx_2 \wedge dx_3$. This description shows that K_Y is not Cartier at o but $2K_Y$ is Cartier.

Let $\pi\colon X \to Y$ be the blow-up of Y at o, or to be precise, along the maximal ideal in $\mathscr{O}_Y$ defining o. Then X is smooth and the exceptional locus E in X is isomorphic to $\mathbf{P}^2$ with $\mathscr{O}_X(-E) \otimes \mathscr{O}_E \simeq \mathscr{O}_{\mathbf{P}^2}(2)$. It is an extremal contraction with $K_X = \pi^*K_Y + (1/2)E$ and $-K_X$ is π-ample.

The next example by Ueno reveals that we cannot avoid singularities. It exhibits a threefold which has no smooth birational models as in Theorem 1.3.4. We need the *negativity lemma*, which will be used at several places.

Theorem 1.3.9 (Negativity lemma) *Let $\pi\colon X \to Y$ be a proper birational morphism of normal varieties. Let E be a π-exceptional $\mathbf{R}$-divisor expressed as $E = M + F$ with a π-nef $\mathbf{R}$-divisor M and an effective $\mathbf{R}$-divisor F such that no π-exceptional prime divisor appears in F. Then $E \le 0$.*

Proof It suffices to work about the generic point η_i of the image $\pi(E_i)$ of each prime component E_i of E. Replacing π by the base change to the intersection of general hyperplane sections of Y about η_i, we may assume that $\pi(E_i)$ is a point in Y. By Chow's lemma, we may assume that π is a contraction. Then cutting

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

it with general hyperplane sections of X, we may assume that $\pi\colon X \to Y$ is a birational morphism of surfaces. By resolution, we may also assume that X is smooth.

After this reduction, the exceptional $\mathbf{R}$-divisor $E = M + F$ is π-nef on the smooth surface X. Since π is projective, there exists an effective divisor A on X supported on the exceptional locus of π such that $-A$ is π-ample. If $E \not\leq 0$, then there would exist a positive real number r and an exceptional curve C such that $E - rA \leq 0$ and such that C has coefficient zero in $E - rA$. Then $((E - rA) \cdot C) \leq 0$, which contradicts the π-ampleness of $E - rA$. □

Example 1.3.10 (Ueno [458, section 16]) Let A be an abelian threefold. It is described as the quotient $\mathbf{C}^3/\Gamma$ of the complex threefold $\mathbf{C}^3$ by a lattice $\Gamma \simeq \mathbf{Z}^6$ spanning $\mathbf{C}^3$ as a real vector space. The involution of A which sends x to $-x$ has $2^6 = 64$ fixed points. The associated quotient $X = A/\mathbf{Z}_2$ has at each fixed point a cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$ appeared in Example 1.3.8. The sheaf $\mathscr{O}_X(2K_X)$ is globally generated by $(dx_1 \wedge dx_2 \wedge dx_3)^{\otimes 2}$ for the coordinates x_1, x_2, x_3 of $\mathbf{C}^3$. Hence $2K_X \sim 0$. For the same reason as explained after Theorem 1.3.4, X is never birational to a threefold X' equipped with a contraction of fibre type with respect to which $-K_{X'}$ is relatively ample. Let Y be an arbitrary smooth projective variety birational to X. Using the negativity lemma, we shall prove that K_Y is never nef.

The blow-up B of X at all the 64 singular points is smooth and $2K_B \sim E$ for the sum $E = \sum_{i=1}^{64} E_i$ of the exceptional divisors $E_i \simeq \mathbf{P}^2$. Take a common log resolution W of (B, E) and Y with $p_1\colon W \to X$ and $p_2\colon W \to Y$. Let T_i denote the sum of p_i-exceptional prime divisors on W and let T denote the common part of T_1 and T_2. We write $2K_W = p_1^*(2K_X) + F_1 + G_1$ by effective divisors F_1 and G_1 with support T and $T_1 - T$ respectively. Similarly $K_W = p_2^*K_Y + F_2 + G_2$ by effective divisors F_2 and G_2 with support T and $T_2 - T$. Then

$$F_1 - 2F_2 + G_1 = -p_1^*(2K_X) + p_2^*(2K_Y) + 2G_2.$$

If K_Y were nef, then by the negativity lemma for p_1, the p_1-exceptional divisor $F_1 - 2F_2 + G_1$ would be negative in the sense that $F_1 - 2F_2 + G_1 \leq 0$. Hence $G_1 = 0$, by which $T = T_1$. It follows that the rational map $X \dashrightarrow Y$ produces no new divisors on Y. In particular, K_Y is the strict transform of K_X and hence $2K_Y \sim 0$. As a consequence, one obtains the relation $F_1 \sim 2F_2 + 2G_2$ of p_2-exceptional divisors. This is the equality $F_1 = 2F_2 + 2G_2$ by the negativity lemma for p_2. However, the strict transform of E_i appears in F_1 with coefficient one, which is absurd.

As a solution to these obstacles, the minimal model program admits mild singularities as in the above examples. We define *terminal* singularities by making the class of these singularities as small as possible.

Definition 1.3.11 Let $\pi: Y \to X$ be a proper birational morphism of normal varieties. Provided that X is $\mathbf{Q}$-Gorenstein, there exists a unique exceptional $\mathbf{Q}$-divisor $K_{Y/X}$ such that $K_Y = \pi^*K_X + K_{Y/X}$. We call $K_{Y/X}$ the *relative* canonical divisor.

Definition 1.3.12 A normal variety X is said to be *terminal*, or to have *terminal* singularities, if X is $\mathbf{Q}$-Gorenstein and for any resolution $X' \to X$, every exceptional prime divisor appears in $K_{X'/X}$ with positive coefficient.

The definition uses resolution of singularities. It is equivalent to the existence of some resolution $X' \to X$ satisfying the required property. See also Definition 1.4.3. Terminal threefold singularities will be classified in the next chapter.

Theorem 1.3.13 *A surface is terminal if and only if it is smooth.*

Proof Every surface S has the *minimal* resolution $S' \to S$, namely a unique resolution such that $K_{S'/S}$ is relatively nef. Since $K_{S'/S} \leq 0$ by the negativity lemma, if S is terminal, then S' has no exceptional curves, that is, S is smooth. The converse is obvious. □

Let X be a terminal projective variety. Unless K_X is nef, the *cone theorem* with the *contraction theorem* produces an extremal contraction $X \to Y$ such that $-K_X$ is relatively ample. Mori [332] first established the cone theorem for smooth varieties and constructed a contraction from a smooth threefold. Whereas he used the deformation theory of curves in positive characteristic, Kawamata [232] and others developed the extension to singular varieties by a cohomological method. Below we extract the part needed for the program. The precise statements will be provided in Theorems 1.4.7 and 1.4.9.

Theorem 1.3.14 *Let X be a terminal projective variety. If K_X is not nef, then there exists an extremal contraction $\pi: X \to Y$ to a normal projective variety such that $-K_X$ is relatively ample. It always satisfies* $\operatorname{Pic} X/\pi^* \operatorname{Pic} Y \simeq \mathbf{Z}$.

The most serious obstacle is that $\pi: X \to Y$ may be isomorphic in codimension one. We say that such π is *small* as defined below. In this case, the canonical divisor K_Y is never $\mathbf{Q}$-Cartier and it does not make sense to ask if K_Y is nef. If K_Y were $\mathbf{Q}$-Cartier, then the pull-back $K_X = \pi^*K_Y$ would contradict the π-ampleness of $-K_X$. Hence it is necessary to reconstruct a reasonable variety X^+ from Y, which is called the *flip* of X/Y. In the book, by a flip X is assumed to be terminal, and a generalised notion will be referred to as a log flip as defined in the next section.

Definition 1.3.15 Let $f\colon X \dashrightarrow Y$ be a birational map of normal varieties factorised as $f = q \circ p^{-1}$ with contractions $p\colon W \to X$ and $q\colon W \to Y$. We call f a *birational contraction map* if all p-exceptional prime divisors are q-exceptional. In other words, the strict transform defines a surjection $f_*\colon Z^1(X) \twoheadrightarrow Z^1(Y)$ of the groups of Weil divisors. We say that f is *small* if it is isomorphic in codimension one, that is, f_* is an isomorphism $Z^1(X) \simeq Z^1(Y)$.

Definition 1.3.16 A *flipping contraction* $\pi\colon X \to Y$ is a small contraction from a terminal variety such that $-K_X$ is π-ample. The *flip* of π is a small contraction $\pi^+\colon X^+ \to Y$ such that K_{X^+} is π^+-ample. The transformation $X \dashrightarrow X^+$ is also called the *flip* by abuse of language. We say that a flipping contraction $\pi\colon X \to Y$ and the flip of π are *elementary* if X is $\mathbf{Q}$-factorial and π is extremal.

Notation 1.3.17 For a Weil divisor D on a normal variety X, we write the graded $\mathscr{O}_X$-algebra $\mathscr{R}(X,D) = \bigoplus_{i\in\mathbf{N}} \mathscr{O}_X(iD)$.

The flip is described as $X^+ = \operatorname{Proj}_Y \mathscr{R}(Y, lK_Y)$ with a positive integer l such that lK_{X^+} is π^+-very ample. Note that $\mathscr{R}(Y, lK_Y) = \pi^+_*\mathscr{R}(X^+, lK_{X^+})$ as π^+ is small. Hence the flip is unique if it exists. Lemma 1.5.20 further provides the description $X^+ = \operatorname{Proj}_Y \mathscr{R}(Y, K_Y)$. One can consider the flip $X \dashrightarrow X^+$ to be the operation of replacing curves with negative intersection number with K_X by curves with positive intersection number with K_{X^+}. We shall explain the first example of a flip by Francia as a quotient of the Atiyah flop.

Example 1.3.18 (Atiyah [24]) Consider the germ $o \in Y = (x_1x_2 + x_3x_4 = 0) \subset \mathbf{A}^4$ discussed in Example 1.2.3. The canonical divisor K_Y is Cartier by the adjunction $K_Y = (K_{\mathbf{A}^4} + Y)|_Y$. Let $\pi\colon X \to Y$ be the blow-up along the ideal $(x_2, x_4)\mathscr{O}_Y$ in $\mathscr{O}_Y$. Then X is smooth and has exceptional locus $C = \pi^{-1}(o) \simeq \mathbf{P}^1$. The blow-up $\varphi\colon B \to Y$ at o factors through π as $\varphi = \pi \circ \mu$ for the contraction $\mu\colon B \to X$ of the exceptional locus $F \simeq \mathbf{P}^1 \times \mathbf{P}^1$ of φ to $C \simeq \mathbf{P}^1$.

Corresponding to the other contraction of F to $\mathbf{P}^1$, the morphism φ also factors through the blow-up $\pi^+\colon X^+ \to Y$ along the ideal $(x_1, x_4)\mathscr{O}_Y$ in $\mathscr{O}_Y$. It has exceptional locus $C^+ = (\pi^+)^{-1}(o) \simeq \mathbf{P}^1$. The transformation $X \to Y \leftarrow X^+$ is small with $K_X = \pi^*K_Y$ and $K_{X^+} = (\pi^+)^*K_Y$. This is known as the *Atiyah flop*, which is a classical example of a flop in Definition 5.1.6.

Example 1.3.19 (Francia [125]) We constructed the Atiyah flop $X \to Y \leftarrow X^+$ for $o \in Y = (x_1x_2 + x_3x_4 = 0) \subset \mathbf{A}^4$. Act $\mathbf{Z}_2$ on Y by the involution which sends (x_1, x_2, x_3, x_4) to $(-x_1, x_2, x_3, -x_4)$. The fixed locus in Y is the divisor

$D = (x_1 = x_4 = 0)$. This action extends to X and X^+ and one can consider the quotients $\pi' \colon X' = X/\mathbf{Z}_2 \to Y' = Y/\mathbf{Z}_2$ and $\pi'^+ \colon X'^+ = X^+/\mathbf{Z}_2 \to Y'$. Then X' has a cyclic quotient singularity of type $\frac{1}{2}(1, 1, 1)$ appeared in Example 1.3.8, whilst X'^+ remains smooth.

Let $C' = C/\mathbf{Z}_2 \simeq \mathbf{P}^1$ and $C'^+ = C^+/\mathbf{Z}_2 \simeq \mathbf{P}^1$ be the exceptional curves in X' and X'^+ respectively. We shall compute the intersection number $(K_{X'} \cdot C')$. One has $p_*C = 2C'$ as a cycle by $p \colon X \to X'$. On the other hand, the ramification formula in Theorem 2.2.20 gives $p^*K_{X'} = K_X - D_X$ with the strict transform D_X of D. Hence $(K_{X'} \cdot C') = ((K_X - D_X) \cdot C/2) = -1/2$ by the projection formula. In like manner, one has $(K_{X'^+} \cdot C'^+) = 1$ using $p^+ \colon X^+ \to X'^+$ with $p^+_*C^+ = C'^+$. Thus π'^+ is the flip of π', which is known as the *Francia flip*.

The flip $X \dashrightarrow X^+$ retains the property of being terminal. If it is elementary, then it also keeps the Picard number unchanged.

Lemma 1.3.20 *Let $\pi^+ \colon X^+ \to Y$ be the flip of a flipping contraction $\pi \colon X \to Y$. Then X^+ as well as X is terminal. If X and Y are projective and the flip $X \dashrightarrow X^+$ is elementary, then X^+ is $\mathbf{Q}$-factorial and projective and $\rho(X) = \rho(X^+)$.*

Proof Take a common resolution W of X and Y with $\mu \colon W \to X$ and $\mu^+ \colon W \to X^+$. Then $K_{W/X} - K_{W/X^+} = (\mu^+)^*K_{X^+} - \mu^*K_X$ is μ-exceptional and μ-nef. Hence $K_{W/X} \le K_{W/X^+}$ by the negativity lemma, showing that X^+ is terminal as is X.

Suppose that X and Y are projective and the flip is elementary. Then X^+ is projective. By Theorem 1.3.14, $(\operatorname{Pic} X/\pi^* \operatorname{Pic} Y) \otimes \mathbf{Q} \simeq \mathbf{Q}$ and it is generated by the π-ample divisor $-K_X$. Hence for any divisor D on X, there exists a rational number c such that $D_Y + cK_Y = \pi_*(D + cK_X)$ is $\mathbf{Q}$-Cartier where $D_Y = \pi_*D$. Then the strict transform D^+ in X^+ of D is expressed as $D^+ = (\pi^+)^*(D_Y + cK_Y) - cK_{X^+}$, which is $\mathbf{Q}$-Cartier. Thus X^+ is $\mathbf{Q}$-factorial and K_{X^+} generates $(\operatorname{Pic} X^+/(\pi^+)^* \operatorname{Pic} Y) \otimes \mathbf{Q}$. In particular, $\rho(X^+) = \rho(Y) + 1 = \rho(X)$. □

We are now in the position of stating the minimal model program. Given a $\mathbf{Q}$-factorial terminal projective variety X, it finds a birational contraction map $X \dashrightarrow Y$ such that Y is a minimal model or admits a structure of a Mori fibre space.

Definition 1.3.21 Let X be a $\mathbf{Q}$-factorial terminal projective variety. It is called a *minimal model* if K_X is nef. A *Mori fibre space* $X \to S$ is an extremal contraction of fibre type to a normal projective variety such that $-K_X$ is relatively ample. The base S will be proved to be $\mathbf{Q}$-factorial in Lemma 1.4.13.

If a $\mathbf{Q}$-factorial terminal projective variety X is not a minimal model, then Theorem 1.3.14 provides an extremal contraction $X \to Y$ to a normal projective variety such that $-K_X$ is relatively ample. By Lemma 1.3.23, it is a Mori fibre space, a flipping contraction or a divisorial contraction below.

Definition 1.3.22 A *divisorial contraction* is a birational contraction $\pi\colon X \to Y$ between terminal varieties such that $-K_X$ is π-ample and such that the exceptional locus E is a prime divisor on X. One can write $K_X = \pi^*K_Y + dE$ with a positive rational number d since Y is terminal. In particular, $-E$ is π-ample. We say that a divisorial contraction $\pi\colon X \to Y$ is *elementary* if X is $\mathbf{Q}$-factorial and π is extremal.

Lemma 1.3.23 *Let $\pi\colon X \to Y$ be an extremal contraction from a $\mathbf{Q}$-factorial terminal projective variety to a normal projective variety such that $-K_X$ is π-ample. If π is birational but not small, then π is a divisorial contraction, Y is $\mathbf{Q}$-factorial and $\rho(X) = \rho(Y) + 1$.*

Proof We shall prove that the exceptional locus is a prime divisor. The remaining assertions are derived from this in the same way as for Lemma 1.3.20.

Take a hyperplane section H_Y of Y such that π^*H_Y contains an exceptional prime divisor. Write $\pi^*H_Y = H + E$ with the strict transform H of H_Y and an exceptional divisor E. By $\rho(X/Y) = 1$, the divisor $-E \equiv_Y H$ is π-ample and the support of E contains a prime divisor F such that $-F$ is π-ample. If a curve C not in F were contracted to a point by π, then the intersection number $(F \cdot C) \geq 0$ would contradict the relative ampleness of $-F$. Thus F must coincide with the exceptional locus. □

Definition 1.3.24 The *minimal model program*, or the *MMP* for short, in the category $\mathscr{C}$ of $\mathbf{Q}$-factorial terminal projective varieties is the algorithm for $X \in \mathscr{C}$ which outputs $Y \in \mathscr{C}$ with a birational contraction map $X \dashrightarrow Y$ in the following manner.

1 If K_X is nef, then output $X \in \mathscr{C}$ as a minimal model.
2 If K_X is not nef, then there exists an extremal contraction $\pi\colon X \to Y$ as in Theorem 1.3.14.
3 If π is a Mori fibre space, then output $X \in \mathscr{C}$.
4 If π is a divisorial contraction, then $Y \in \mathscr{C}$ and $\rho(Y) = \rho(X) - 1$ by Lemma 1.3.23. Replace X by Y and go back to 1.
5 If π is a flipping contraction, then construct the flip $\pi^+\colon X^+ \to Y$ of π, for which $X^+ \in \mathscr{C}$ and $\rho(X^+) = \rho(X)$ by Lemma 1.3.20. Replace X by X^+ and go back to 1.

In order for the MMP to function, the existence and termination of flips are necessary. The termination means the non-existence of an infinite loop by the step 5 of the algorithm as stated in Conjecture 1.3.26. Assuming them, the MMP ends with a minimal model or a Mori fibre space by induction of the Picard number. The *existence of flips* was first proved by Mori [335] for threefold flips and fully settled by Birkar, Cascini, Hacon and McKernan [48] as in Theorem 1.4.11. However, the full form of Conjecture 1.3.26 is still open in dimension greater than four in spite of the termination of flips with scaling in the setting of Corollary 1.5.13. The termination in dimension three will be demonstrated in Theorem 1.6.3. The result in dimension four is in [249, theorem 5.1.15].

Theorem 1.3.25 *The flip in Definition* 1.3.16 *exists.*

Conjecture 1.3.26 (Termination of flips) *Let X be a $\mathbf{Q}$-factorial terminal projective variety. Then there exists no infinite sequence $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots$ of elementary flips $X_i \to Y_i \leftarrow X_{i+1}$ with X_i and Y_i projective.*

As is the case with the program for surfaces, which of a minimal model or a Mori fibre space the MMP outputs is determined by the input variety.

Proposition 1.3.27 *Let X be a $\mathbf{Q}$-factorial terminal projective variety. If K_X is pseudo-effective, then the output by the MMP from X is always a minimal model. If K_X is not pseudo-effective, then the output is always a Mori fibre space.*

Proof Let $X = X_0 \dashrightarrow \cdots \dashrightarrow X_n = Y$ be an output of the MMP from X, where each $f_i\colon X_i \dashrightarrow X_{i+1}$ is a divisorial contraction or a flip. It suffices to prove the equivalence of the pseudo-effectivity of K_X and that of K_Y.

Fix an ample divisor A on X. Its strict transform A_Y in Y is big. For a small positive rational number ε, the strict transform $-(K_{X_i} + \varepsilon A_i)$ in X_i of $-(K_X + \varepsilon A)$ is f_i-ample when f_i is a divisorial contraction. Then for a sufficiently large and divisible integer l, the multiple $l(K_X + \varepsilon A)$ is integral and $H^0(\mathscr{O}_{X_i}(l(K_{X_i} + \varepsilon A_i))) \simeq H^0(\mathscr{O}_{X_{i+1}}(l(K_{X_{i+1}} + \varepsilon A_{i+1})))$. It follows that

$$H^0(\mathscr{O}_X(l(K_X + \varepsilon A))) \simeq H^0(\mathscr{O}_Y(l(K_Y + \varepsilon A_Y))).$$

Thus the limit K_X of $K_X + \varepsilon A$ with ε approaching zero is pseudo-effective if and only if so is the limit K_Y of $K_Y + \varepsilon A_Y$. □

The problem subsequent to the MMP is to study Mori fibre spaces and minimal models. The Sarkisov program, which will be introduced in Chapter 6, is a standard tool for analysing birational maps of Mori fibre spaces. In the study of minimal models, the rational map defined by a multiple of the canonical divisor plays an important role as in Theorem 1.3.7. We expect the following *abundance* conjecture. It is only known up to dimension three as will be demonstrated in Chapter 10. The designation of abundance is by the assertion in Theorem 1.7.12 that the canonical divisor is semi-ample if and only if it is nef and *abundant* in the sense of Definition 1.7.10.

Conjecture 1.3.28 (Abundance) *If X is a minimal model, then K_X is semi-ample.*

1.4 Logarithmic and Relative Extensions

We formulated the MMP for projective varieties. However, it has turned out that the program becomes much more powerful by logarithmic and relative extensions. This perspective proves its worth even in the study of the original MMP. For example, the flip X^+/Y of a flipping contraction X/Y is considered to be the canonical model of X/Y. The purpose of this section is to outline the generalisations of the MMP. The books [277] and [307] elucidate the abstract side of the program. The treatise [249] is standard until now.

Let X be an algebraic scheme. Let Z be a closed subvariety of X and let $\mathfrak{p}$ denote the ideal sheaf in $\mathscr{O}_X$ defining Z. The *order* $\operatorname{ord}_Z \mathscr{I}$ along Z of a coherent ideal sheaf $\mathscr{I}$ in $\mathscr{O}_X$ is the maximal $l \in \mathbf{N} \cup \{\infty\}$ such that $\mathscr{I}\mathscr{O}_{X,\eta} \subset \mathfrak{p}^l \mathscr{O}_{X,\eta}$ at the generic point η of Z, where we define $\mathfrak{p}^\infty$ to be zero. We write $\operatorname{ord}_Z f = \operatorname{ord}_Z f\mathscr{O}_X$ for $f \in \mathscr{O}_X$. When X is normal, the *order* $\operatorname{ord}_Z D$ of a $\mathbf{Q}$-Cartier divisor D on X is defined as $r^{-1} \operatorname{ord}_Z \mathscr{O}_X(-rD)$ by a positive integer r such that rD is Cartier, which is independent of the choice of r. Beware of the difference between $\operatorname{ord}_Z D$ and $\operatorname{ord}_Z \mathscr{O}_X(-D)$. The notion of $\operatorname{ord}_Z D$ is extended linearly to $\mathbf{R}$-Cartier $\mathbf{R}$-divisors.

Let X be a variety. A divisor *over* X means the equivalence class of a prime divisor E on a normal variety Y equipped with a birational morphism $Y \to X$, where divisors E on Y/X and E' on Y'/X are equivalent if E is the strict transform of E' and vice versa, that is, E and E' define the same valuation on the function field of X. A divisor E over X is said to be *exceptional* if it is not realised as a prime divisor on X. The *order* $\operatorname{ord}_E \mathscr{I}$ along E of a coherent ideal sheaf $\mathscr{I}$ in $\mathscr{O}_X$ is defined as $\operatorname{ord}_E \mathscr{I}\mathscr{O}_Y$, which is independent of the realisation $E \subset Y/X$. The orders $\operatorname{ord}_E f$ and $\operatorname{ord}_E D$ are defined in like manner for a function f in $\mathscr{O}_X$ and an $\mathbf{R}$-Cartier $\mathbf{R}$-divisor D on X.

Definition 1.4.1 The *centre* $c_X(E)$ of a divisor E over X is the closure of the image $\pi(E)$ of E by $\pi\colon Y \to X$ on which E is realised as a prime divisor.

The logarithmic setting treats pairs (X, Δ) using $K_X + \Delta$ instead of K_X. A *pair* (X, Δ) consists of a normal variety X and an effective $\mathbf{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbf{R}$-Cartier, in which Δ is referred to as the *boundary*.

Definition 1.4.2 Let (X, Δ) be a pair and let E be a divisor over X realised on $\pi\colon Y \to X$. One can write $K_Y = \pi^*(K_X + \Delta) + A$ uniquely with an $\mathbf{R}$-divisor A such that $\pi_* A = -\Delta$. The *log discrepancy* $a_E(X, \Delta)$ of E with respect to (X, Δ) is defined as $a_E(X, \Delta) = 1 + \operatorname{ord}_E A$. When Δ is zero, we write $a_E(X)$ for $a_E(X, 0)$ and occasionally prefer the *discrepancy* of E which is defined as $a_E(X) - 1 = \operatorname{ord}_E K_{Y/X}$.

Definition 1.4.3 We say that a pair (X, Δ) is *terminal*, *canonical*, *purely log terminal* (*plt* for short) respectively if $a_E(X, \Delta) > 1$, ≥ 1, > 0 respectively for all divisors E exceptional over X. We say that (X, Δ) is *Kawamata log terminal* (*klt*), *log canonical* (*lc*) respectively if $a_E(X, \Delta) > 0$, ≥ 0 respectively for all divisors E over X. We say that (X, Δ) is *divisorially log terminal* (*dlt*) if it is lc and there exists a log resolution Y of (X, Δ) such that $a_E(X, \Delta) > 0$ for every prime divisor E on Y that is exceptional over X. When Δ is zero, in which X is $\mathbf{Q}$-Gorenstein, we simply say that X is terminal, canonical and so forth. This coincides with Definition 1.3.12. The notions of klt, plt and dlt singularities for $\Delta = 0$ are the same and we just say that X is *log terminal* (*lt*).

The vanishing of higher cohomologies enables us to compute the dimension of global sections of a sheaf numerically. The *Kodaira vanishing* and its generalisations are indispensable in complex birational geometry. The classical vanishing by Kodaira was proved in the theory of harmonic analysis. It does not hold in positive characteristic.

Theorem 1.4.4 (Kodaira vanishing [257]) *Let X be a smooth projective complex variety and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $H^i(\omega_X \otimes \mathscr{L}) = 0$ for all $i \geq 1$.*

The following generalisation is one of the most fundamental tools in the minimal model theory. It is obtained by the covering trick associated with a Kummer extension. It includes the *Grauert–Riemenschneider vanishing* $R^i\pi_*\omega_X = 0$ for $i \geq 1$ for a generically finite proper morphism π from a smooth variety X [152].

Theorem 1.4.5 (Kawamata–Viehweg vanishing [230], [463]) *Let (X, Δ) be a klt pair and let $\pi\colon X \to Y$ be a proper morphism to a variety. Let D be a $\mathbf{Q}$-Cartier integral divisor on X such that $D - (K_X + \Delta)$ is π-nef and π-big. Then $R^i\pi_*\mathscr{O}_X(D) = 0$ for all $i \geq 1$.*

We shall formulate the minimal model program for an lc pair (X, Δ) with a projective morphism $\pi\colon X \to S$ to a fixed variety. The *contraction* and *cone theorems* provide extremal contractions in the program.

Definition 1.4.6 Let C be a convex cone in a finite dimensional real vector space V. An *extremal face* F of C means a convex subcone of C such that if $v, w \in C$ and $v + w \in F$, then $v, w \in F$. An extremal face R is called an *extremal ray* if R is a half-line $\mathbf{R}_{\ge 0} v \cap C$ with one generator $v \in R \setminus 0$. For $\lambda \in V^\vee = \operatorname{Hom}(V, \mathbf{R})$, we say that an extremal face F is *negative* with respect to λ, or *λ-negative*, if $\lambda(v) < 0$ for all $v \in F \setminus 0$.

We consider the closed cone $\overline{\mathrm{NE}}(X/S)$ of curves in $N_1(X/S)$ and regard $N^1(X/S)$ as the dual of $N_1(X/S)$ via the intersection pairing.

Theorem 1.4.7 (Contraction theorem) *Let (X, Δ) be an lc pair with a projective morphism $\pi \colon X \to S$ to a variety. Let F be a $(K_X + \Delta)$-negative extremal face of $\overline{\mathrm{NE}}(X/S)$. Then there exists a unique contraction $\varphi \colon X \to Y$ through which π factors as $\pi = \pi_Y \circ \varphi$ for a projective morphism $\pi_Y : Y \to S$ such that a relative curve C in X/S is contracted to a point in Y if and only if $[C] \in F$. Further the natural sequence*

$$0 \to \operatorname{Pic} Y \xrightarrow{\varphi^*} \operatorname{Pic} X \to N^1(X/Y)$$

is exact and in particular $\rho(X/S) = \rho(Y/S) + \rho(X/Y)$.

We say that the contraction $\varphi \colon X \to Y$ above is *associated with* the face F. The dimension of F equals the relative Picard number $\rho(X/Y)$. By Kleiman's criterion, $K_X + \Delta$ is relatively ample with respect to φ.

The contraction theorem is a corollary to the *base-point free theorem*. Kawamata [232] completed it for klt pairs using the vanishing theorem and Shokurov's *non-vanishing theorem* [423].

Theorem 1.4.8 (Base-point free theorem) *Let (X, Δ) be a klt pair with a proper morphism $\pi \colon X \to S$ to a variety. Let D be a π-nef Cartier divisor on X such that $aD - (K_X + \Delta)$ is π-nef and π-big for some positive integer a. Then there exists a positive integer l_0 such that lD is π-free for all $l \ge l_0$.*

Thanks to the cone theorem, every extremal face F treated by the contraction theorem is spanned by the classes of rational curves. The cone theorem originated in the work of Mori [331] on the Hartshorne conjecture on a characterisation of the projective space. Kawamata [232] supplemented by [259] established the theorem for klt pairs by a cohomological method. It was extended to lc pairs by Fujino [129].

Theorem 1.4.9 (Cone theorem) *Let (X, Δ) be an lc pair with a projective morphism $\pi \colon X \to S$ to a variety. Let H be a π-ample $\mathbf{R}$-divisor on X. Then*

$$\overline{\mathrm{NE}}(X/S) = \overline{\mathrm{NE}}(X/S)_{K_X+\Delta+H \ge 0} + \sum_i \mathbf{R}_{\ge 0}[C_i],$$

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

where $\overline{\mathrm{NE}}(X/S)_{K_X+\Delta+H\geq 0} = \{z \in \overline{\mathrm{NE}}(X/S) \mid ((K_X + \Delta + H) \cdot z) \geq 0\}$ *and* $\sum_i \mathbf{R}_{\geq 0}[C_i]$ *is a finite sum of extremal rays generated by the class of a rational relative curve* C_i *in* X/S*.*

It is remarkable that the degree of the generator C_i of an extremal ray is bounded. This is essentially due to Kawamata [237] and extended in [129].

Theorem 1.4.10 *Let* (X, Δ) *be an lc pair with a projective morphism* $X \to S$ *to a variety. Then every* $(K_X + \Delta)$*-negative extremal ray of* $\overline{\mathrm{NE}}(X/S)$ *is generated by the class of a rational relative curve* C *in* X/S *such that* $0 < (-(K_X + \Delta) \cdot C) \leq 2 \dim X$.

Fix the base variety S and let $\mathscr{C}^l$ denote the category of **Q**-factorial lc pairs (X, Δ) projective over S, meaning that (X, Δ) is lc with X being **Q**-factorial and equipped with a projective morphism to S.

Let $(X/S, \Delta) \in \mathscr{C}^l$. We call $(X/S, \Delta)$ a *log minimal model* over S if $K_X + \Delta$ is relatively nef. Let $\pi\colon X \to Y/S$ be an extremal contraction associated with a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$. The contraction π is a *log Mori fibre space* if it is of fibre type, a *log divisorial contraction* if it is birational but not small and a *log flipping contraction* if it is small. If π is birational but not small, then the exceptional locus is a prime divisor similarly to Lemma 1.3.23. When X is terminal and Δ is zero, a log minimal model X/S and a log Mori fibre space $X \to Y/S$ are usually called a *relative minimal model* and a *relative Mori fibre space*. These generalise Definition 1.3.21 by relaxing the abstract projectivity of X and Y.

Let (X, Δ) be an lc pair. As in Definitions 1.3.16 and 1.3.22, we use the terminology of a log divisorial or flipping contraction $\pi\colon X \to Y$ without assuming that it is elementary. We say that π is *elementary* if X is **Q**-factorial and π is extremal. A *log flipping contraction* $\pi\colon X \to Y$ with respect to (X, Δ) is a small contraction such that $-(K_X + \Delta)$ is π-ample. The *log flip* of π is a small contraction $\pi^+\colon X^+ \to Y$ such that $K_{X^+} + \Delta^+$ is π^+-ample for the strict transform Δ^+ of Δ. The transformation $X \dashrightarrow X^+$ is also called the *log flip*. After Shokurov's establishment of threefold log flips [424] and his attempt at higher dimensional generalisation [426], Birkar, Cascini, Hacon and McKernan [48] proved the existence of log flips for klt pairs. This was extended to lc pairs substantially by Birkar [45] and Hacon–Xu [175].

Theorem 1.4.11 (Existence of log flips) *The log flip exists for an lc pair.*

Log divisorial contractions and log flips improve singularities as shown by the next lemma, which is proved essentially in the same manner as Lemmata 1.3.20 and 1.3.23 are proved.

Lemma 1.4.12 *Let $(X/S, \Delta) \in \mathscr{C}^l$ and let $f \colon X \dashrightarrow Y/S$ be a log divisorial contraction or a log flip associated with a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$. Then $(Y, f_*\Delta) \in \mathscr{C}^l$ and the inequality $a_E(X, \Delta) \le a_E(Y, f_*\Delta)$ of log discrepancies holds for any divisor E over X. Further $a_E(X, \Delta) < a_E(Y, f_*\Delta)$ if f is not isomorphic at the generic point of the centre $c_X(E)$ in X. In particular if (X, Δ) is klt, plt, dlt or lc, then so is $(Y, f_*\Delta)$. If (X, Δ) is terminal or canonical, and in addition $\Delta = f_*^{-1} f_*\Delta$, then so is $(Y, f_*\Delta)$.*

We remark that the base of a log Mori fibre space is $\mathbf{Q}$-factorial.

Lemma 1.4.13 *Let $\pi \colon X \to Y$ be a log Mori fibre space. Then Y is $\mathbf{Q}$-factorial. If $\mathrm{Cl}\, X/\mathrm{Pic}\, X$ is r-torsion for $r \in \mathbf{Z}_{>0}$, then so is $\mathrm{Cl}\, Y/\mathrm{Pic}\, Y$.*

Proof Let U be the smooth locus in Y and let $U_X = \pi^{-1}(U)$. For an arbitrary divisor D on Y, there exists a divisor D_X on X such that the restriction $D_X|_{U_X}$ equals the pull-back of $D|_U$. Suppose that rD_X is Cartier. Note that $rD_X \equiv_Y 0$ by $\rho(X/Y) = 1$. By the exact sequence in Theorem 1.4.7 for $S = Y$, the invertible sheaf $\mathscr{O}_X(rD_X)$ is isomorphic to the pull-back of some invertible sheaf $\mathscr{L}$ on Y. By the projection formula, $\mathscr{L}|_U \simeq (\pi|_{U_X})_*\mathscr{O}_X(rD_X)|_{U_X} = \mathscr{O}_Y(rD)|_U$. This implies that $\mathscr{L} \simeq \mathscr{O}_Y(rD)$ since $Y \setminus U$ is of codimension at least two. Thus rD is Cartier. □

The $(K_X + \Delta)$-*minimal model program* over S, or the $(K_X + \Delta)/S$-*MMP*, for $(X/S, \Delta) \in \mathscr{C}^l$ is the algorithm which outputs $(Y/S, \Gamma) \in \mathscr{C}^l$ with a birational contraction map $X \dashrightarrow Y/S$ in the following manner.

1 If $K_X + \Delta$ is relatively nef, then output $(X, \Delta) \in \mathscr{C}^l$ as a log minimal model.
2 If $K_X + \Delta$ is not relatively nef, then there exists a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$ and it defines an extremal contraction $\pi \colon X \to Y/S$ by the cone and contraction theorems.
3 If π is a log Mori fibre space, then output $(X, \Delta) \in \mathscr{C}^l$.
4 If π is a log divisorial contraction, then $(Y, \pi_*\Delta) \in \mathscr{C}^l$ and $\rho(Y/S) = \rho(X/S) - 1$. Replace (X, Δ) by $(Y, \pi_*\Delta)$ and go back to 1.
5 If π is a log flipping contraction, then construct the log flip $f \colon X \dashrightarrow X^+$ of π, for which $(X^+, f_*\Delta) \in \mathscr{C}^l$ and $\rho(X^+/S) = \rho(X/S)$. Replace (X, Δ) by $(X^+, f_*\Delta)$ and go back to 1.

Remark 1.4.14 Nakayama [368] formulated the minimal model program for klt pairs in the analytic category, including the Kawamata–Viehweg vanishing, the base-point free theorem and the cone and contraction theorems.

The full form of termination of log flips is known up to dimension three [425]. The log abundance conjecture will be discussed in the last section.

Conjecture 1.4.15 (Termination of log flips) *Let $(X/S, \Delta)$ be a $\mathbf{Q}$-factorial lc pair projective over a variety. Then there exists no infinite sequence $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots$ of elementary log flips $X_i \to Y_i \leftarrow X_{i+1}$ associated with an $(K_{X_i} + \Delta_i)$-negative extremal ray of $\overline{\mathrm{NE}}(X_i/S)$ for the strict transform Δ_i of Δ.*

The following is the same as Proposition 1.3.27.

Proposition 1.4.16 *Let $(X/S, \Delta)$ be a $\mathbf{Q}$-factorial lc pair projective over a variety. If $K_X + \Delta$ is relatively pseudo-effective, then the output by the $(K_X + \Delta)/S$-MMP is always a log minimal model. If $K_X + \Delta$ is not relatively pseudo-effective, then the output is always a log Mori fibre space.*

We shall supplement the notions of plt, dlt and lc singularities. Whereas lc pairs form the maximal class in which the MMP works, the variety of an lc pair is not even Cohen–Macaulay in general.

Example 1.4.17 Let S be an abelian surface and let $\mathscr{L}$ be an ample invertible sheaf on S. Take the line bundle $B = \operatorname{Spec}_S \bigoplus_{i \in \mathbf{N}} \mathscr{L}^{\otimes i} \to S$. The natural projection $\pi \colon B \to X$ to the *affine cone* $X = \operatorname{Spec} \bigoplus_{i \in \mathbf{N}} H^0(\mathscr{L}^{\otimes i})$ of S is the contraction which contracts a section $E \simeq S$ of B/S to the vertex o. One has $K_B + E \sim 0$ since $(K_B + E)|_E = K_E \sim 0$ by adjunction. Hence $K_X = \pi_*(K_B + E) \sim 0$ and $K_B + E = \pi^* K_X$. In particular, X is lc. The spectral sequence gives an isomorphism $R^1\pi_*\mathscr{O}_B \simeq H^1(\mathscr{O}_B) = H^1(\mathscr{O}_S) \simeq \mathbf{C}^2$.

We shall prove that X is not Cohen–Macaulay. Embed X into a projective variety $\bar{X}$ which is smooth outside o and extend π to the contraction $\bar{\pi} \colon \bar{B} \to \bar{X}$ isomorphic outside o. If X were Cohen–Macaulay, then by the duality and Serre vanishing, $H^2(\mathscr{O}_{\bar{X}}(-\bar{A})) = H^1(\omega_{\bar{X}} \otimes \mathscr{O}_{\bar{X}}(\bar{A}))^\vee = 0$ for a sufficiently ample Cartier divisor $\bar{A}$ on $\bar{X}$. On the other hand, $H^1(\mathscr{O}_{\bar{B}}(-\bar{\pi}^*\bar{A})) = H^2(\mathscr{O}_{\bar{B}}(K_{\bar{B}} + \bar{\pi}^*\bar{A}))^\vee = 0$ by Kawamata–Viehweg vanishing. Hence the spectral sequence $H^p(R^q\bar{\pi}_*\mathscr{O}_{\bar{B}}(-\bar{\pi}^*\bar{A})) \Rightarrow H^{p+q}(\mathscr{O}_{\bar{B}}(-\bar{\pi}^*\bar{A}))$ would give $R^1\pi_*\mathscr{O}_B \simeq R^1\bar{\pi}_*\mathscr{O}_{\bar{B}}(-\bar{\pi}^*\bar{A}) = 0$, which is a contradiction.

Dlt pairs form a class suited for inductive arguments on dimension. The variety of a dlt pair has *rational singularities* and it is Cohen–Macaulay as below. On the other hand, the definition of a dlt singularity is subtle. It is not an analytically local property as evidenced by Example 1.4.21.

Definition 1.4.18 Let X be a normal variety in characteristic zero. We say that X has *rational singularities* if $R^i\mu_*\mathscr{O}_{X'} = 0$ for all $i \geq 1$ for every resolution $\mu \colon X' \to X$. By Theorem 1.4.19, it is equivalent to the existence of some resolution μ having the required property.

Theorem 1.4.19 ([251, p.50 proposition]) *A resolution $\mu\colon X' \to X$ of a normal variety X satisfies $R^i\mu_*\mathscr{O}_{X'} = 0$ for all $i \geq 1$ if and only if X is Cohen–Macaulay with $\omega_X = \mu_*\omega_{X'}$.*

Theorem 1.4.20 ([249, theorem 1.3.6]) *If (X, Δ) is a dlt pair, then X has rational singularities and in particular it is Cohen–Macaulay.*

Example 1.4.21 The pair $(\mathbf{A}^2, D)$ with the sum D of the two axes given by x_1x_2 is dlt but $(\mathbf{A}^2, C)$ with the nodal curve C given by $x_1x_2 + x_1^3 + x_2^3$ is only lc. They are analytically isomorphic at origin. See also Example 2.1.2.

Szabó's result characterises a dlt singularity in an alternative way.

Theorem 1.4.22 (Szabó [437]) *A pair (X, Δ) is dlt if and only if there exists a smooth dense open subset U of X such that $\Delta|_U$ is reduced and snc and such that $a_E(X, \Delta) > 0$ for any divisor E over X with $c_X(E) \subset X \setminus U$. When X is quasi-projective, this implies the existence of an effective* $\mathbf{R}$*-divisor Δ' such that $(X, (1-\varepsilon)\Delta + \varepsilon\Delta')$ is klt for any sufficiently small positive real number ε.*

By the *connectedness lemma* of Shokurov and Kollár, a dlt pair (X, Δ) is plt if and only if all the connected components of $\lfloor \Delta \rfloor$ are irreducible.

Theorem 1.4.23 (Connectedness lemma) *Let (X, Δ) be an algebraic or analytic pair and let $\pi\colon X \to Y$ be a proper morphism to a variety with connected fibres such that $-(K_X + \Delta)$ is π-nef and π-big. Let $\mu\colon X' \to X$ be a log resolution of (X, Δ) and write $K_{X'} + N = \mu^*(K_X + \Delta) + P$ with effective* $\mathbf{R}$*-divisors P and N without common components such that $\mu_*P = 0$ and $\mu_*N = \Delta$. Then the natural map $\mathscr{O}_Y \to (\pi \circ \mu)_*\mathscr{O}_{\lfloor N \rfloor}$ is surjective.*

Proof This is an application of Kawamata–Viehweg vanishing to the klt pair (X', B) for $B = (N - P) - \lfloor N - P \rfloor$ with $\pi' = \pi \circ \mu\colon X' \to Y$. The vanishing also holds in the analytic category as noted in Remark 1.4.14. Since $\lceil P \rceil - \lfloor N \rfloor - (K_{X'} + B) = -\mu^*(K_X + \Delta)$ is π'-nef and π'-big, it follows from Theorem 1.4.5 that $R^1\pi'_*\mathscr{O}_{X'}(\lceil P \rceil - \lfloor N \rfloor) = 0$. Thus one has the surjection

$$\pi'_*\mathscr{O}_{X'}(\lceil P \rceil) \twoheadrightarrow \pi'_*\mathscr{O}_{\lfloor N \rfloor}(\lceil P \rceil|_{\lfloor N \rfloor}).$$

The left-hand side equals $\mathscr{O}_Y$ since $\mu_*\mathscr{O}_{X'}(\lceil P \rceil) = \mathscr{O}_X$. Hence the above map factors through the natural map $\mathscr{O}_Y \to \pi'_*\mathscr{O}_{\lfloor N \rfloor}$, which must be surjective. $\square$

Finally we discuss formulae of adjunction type. Let $(X, S + B)$ be a pair such that S is reduced and has no common components with B. Take a log resolution $\mu\colon X' \to X$ of $(X, S + B)$ and write $K_{X'} + S' + B' = \mu^*(K_X + S + B)$ with the strict transform S' of S and an $\mathbf{R}$-divisor B' such that $\mu_*B' = B$. Choose it so that $S' \to S$ factors through the normalisation $\nu\colon S^\nu \to S$ with a contraction $\mu_S\colon S' \to S^\nu$. We define the $\mathbf{R}$-divisor $B_{S^\nu} = \mu_{S*}(B'|_{S'})$ on S^ν. Then $K_{S^\nu}+B_{S^\nu}$ is an $\mathbf{R}$-Cartier $\mathbf{R}$-divisor such that $K_{S'}+B'|_{S'} = \mu_S^*(K_{S^\nu}+B_{S^\nu})$. If S is Cartier and normal and B is zero, then B_{S^ν} is zero by adjunction.

Definition 1.4.24 ([424, section 3]) The $\mathbf{R}$-divisor $B_{S^\nu} = \mu_{S*}(B'|_{S'})$ is called the *different* on S^ν of the pair $(X, S+B)$. The different B_{S^ν} is effective and independent of the choice of the log resolution μ. It satisfies the *adjunction* $\nu^*((K_X + S + B)|_S) = K_{S^\nu} + B_{S^\nu}$ via the pull-back $\operatorname{Pic} X \otimes \mathbf{R} \to \operatorname{Pic} S^\nu \otimes \mathbf{R}$.

Example 1.4.25 Consider the surface X in $\mathbf{A}^3$ given by $x_1^2 - x_2x_3$ and the x_3-axis D in Example 1.2.1. The blow-up $\mu\colon X' \to X$ at origin o is a log resolution of (X, D) with $K_{X'} + D' + (1/2)E = \mu^*(K_X + D)$ for the strict transform D' of D and the exceptional curve E. Hence $K_D + (1/2)o = (K_X + D)|_D$, that is, $(1/2)o$ is the different on D of the pair (X, D). One can regard X as the quotient $\mathbf{A}^2/\mathbf{Z}_2$ by the morphism $\mathbf{A}^2 \to X$ given by $(x_1, x_2, x_3) = (y_1y_2, y_1^2, y_2^2)$ for the coordinates y_1, y_2 of $\mathbf{A}^2$. In general for the cyclic quotient $X = \mathbf{A}^2/\mathbf{Z}_r(1, w)$ in Definition 2.2.10 with w coprime to r, each axis l of X satisfies the formula $K_l + (1 - r^{-1})o = (K_X + l)|_l$.

The *inversion of adjunction* compares the singularities on X with those on S^ν. The only-if part of each item in the next theorem is evident. The if part of the first item follows from the connectedness lemma.

Theorem 1.4.26 (Inversion of adjunction) *Let $(X, S + B)$ be an algebraic or analytic pair such that S is reduced and has no common components with B. Let S^ν be the normalisation of S and let B_{S^ν} denote the different on S^ν of $(X, S + B)$.*

(i) *The pair $(X, S+B)$ is plt about S if and only if (S^ν, B_{S^ν}) is klt. In this case, S is normal.*
(ii) ([225]) *The pair $(X, S + B)$ is lc about S if and only if (S^ν, B_{S^ν}) is lc.*

There is a generalisation of the adjunction. Let (X, Δ) be a pair. A *non-klt centre* of (X, Δ) means the centre $c_X(E)$ in X of a divisor E over X such that $a_E(X, \Delta) \le 0$. The union of all non-klt centres is called the *non-klt locus* of (X, Δ). It is the complement of the maximal open subset U of X where $(U, \Delta|_U)$ is klt. If (X, Δ) is lc, then a non-klt centre of (X, Δ) is often called an *lc centre*. An lc centre that is minimal with respect to inclusion is called a *minimal lc centre*. A minimal lc centre exists and it is normal [129, theorem 9.1].

Theorem 1.4.27 (Subadjunction formula [133, theorem 4.1], [244]) *Let (X, Δ) be an lc pair such that X is projective and let Z be a minimal lc centre of (X, Δ). Then Z admits a klt pair (Z, Δ_Z) which satisfies the adjunction $(K_X + \Delta)|_Z \sim_{\mathbf{R}} K_Z + \Delta_Z$.*

For fibrations, we state Ambro's adjunction formula. The extension to the case of **R**-divisors is in [133, theorem 3.1].

Theorem 1.4.28 (Ambro [13, theorem 0.2]) *Let (X, Δ) be a klt pair such that X is projective and let $\pi\colon X \to S$ be a contraction to a normal projective variety with $K_X + \Delta \sim_{\mathbf{R},S} 0$. Then S admits a klt pair (S, Δ_S) which satisfies the adjunction $K_X + \Delta \sim_{\mathbf{R}} \pi^*(K_S + \Delta_S)$ via $\pi^*\colon$* $\operatorname{Pic} S \otimes \mathbf{R} \to \operatorname{Pic} X \otimes \mathbf{R}$.

1.5 Existence of Flips

The existence of flips [48], [172] by Hacon and McKernan with Birkar and Cascini is a landmark in the minimal model theory. This section is an introduction to their work. The books [94] and [169] treat it.

Definition 1.5.1 Let $f\colon X \dashrightarrow Y$ be a birational contraction map defined in Definition 1.3.15. Let D be an **R**-Cartier **R**-divisor on X such that $D_Y = f_*D$ is also **R**-Cartier. We say that f is *non-positive* with respect to D (or *D-non-positive*) if $\operatorname{ord}_E D \ge \operatorname{ord}_E D_Y$ for all divisors E over X. We say that f is *negative* with respect to D (or *D-negative*) if it is D-non-positive and further $\operatorname{ord}_E D > \operatorname{ord}_E D_Y$ for every prime divisor E on X exceptional over Y.

We say that f is *crepant* with respect to D if $\operatorname{ord}_E D = \operatorname{ord}_E D_Y$ for all divisors E over X. This is equivalent to the equality $p^*D = q^*D_Y$ on a common resolution W with $p\colon W \to X$ and $q\colon W \to Y$. We simply say that f is *crepant* if it is crepant with respect to the canonical divisor K_X.

We define models of a pair over a fixed variety S.

Definition 1.5.2 Let X be a normal variety which is projective over a variety S. Let D be an **R**-Cartier **R**-divisor on X. Let $g\colon X \dashrightarrow Z/S$ be a rational map to a normal variety projective over S which is resolved as $g = q \circ p^{-1}$ by a resolution $p\colon W \to X$ and a contraction $q\colon W \to Z$. We call g the *ample model* of D over S if $p^*D \sim_{\mathbf{R},S} q^*A + E$ with a relatively ample **R**-divisor A on Z and an effective **R**-divisor E on W such that $E|_{U_W} \le B$ for any open subset U of S and any effective **R**-divisor $B \sim_{\mathbf{R},U} p^*D|_{U_W}$ on $U_W = W \times_S U$. The definition is independent of the choice of the resolution W.

Lemma 1.5.3 *The ample model is unique up to isomorphism if it exists.*

Proof Keep the notation in Definition 1.5.2. Suppose that another ample model $X \dashrightarrow Z'$ of D is realised with $q'\colon W \to Z'$ and $p^*D \sim_{\mathbf{R},S} (q')^*A' + E'$. If $E = E'$, then $q^*A \sim_{\mathbf{R},S} (q')^*A'$ and a relative curve in W/S is contracted by q if and only if it is contracted by q'. This means the isomorphism $q \simeq q'$

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

by Lemma 1.1.1. To see the equality $E = E'$, we may assume that S is quasi-projective. Since A' is relatively ample, there exists an effective **R**-divisor $B' \sim_{\mathbf{R},S} (q')^*A'$ which has no common components with E. It follows from the property of E that $E \le B' + E'$ and hence $E \le E'$. By symmetry, $E' \le E$ and thus $E = E'$. □

We have the following characterisation of the ample model when it is birational. In particular, the log flip $X \to Y \leftarrow X^+$ with respect to (X, Δ) is the ample model of $K_X + \Delta$ over Y.

Proposition 1.5.4 *Notation as in Definition* 1.5.2. *Suppose that g is birational. Then g is the ample model of D if and only if g is a D-non-positive birational contraction map such that g_*D is relatively ample.*

Proof The if part is easy. By assumption, one can write $p^*D = q^*(g_*D) + E$ with relatively ample g_*D and q-exceptional $E \ge 0$. Suppose that $B \sim_{\mathbf{R},S} p^*D$ is effective. Let $B' = B - q_*^{-1}q_*B$ be the q-exceptional part of B, for which $E - B' \sim_{\mathbf{R},Z} B - B' \ge 0$. Then $E \le B' \le B$ by the negativity lemma, showing that g is the ample model.

Conversely suppose that g is the ample model with $p^*D \sim_{\mathbf{R},S} q^*A + E$ as in Definition 1.5.2. Taking W suitably, we may assume that the q-exceptional locus Q equals the support of an effective **Q**-divisor F such that $q^*A - F$ is ample over S. Then $H = q^*A - F + \varepsilon E$ is still ample over S for small positive ε. Since $p^*D \sim_{\mathbf{R},S} H + F + (1-\varepsilon)E$, the property of the ample model shows that $E \le F + (1-\varepsilon)E$. Hence E is supported in Q.

Let P be a p-exceptional prime divisor. For a general curve C in P contracted by p, one has $(q^*A \cdot C) + (E \cdot C) = 0$ and $(q^*A \cdot C) \ge 0$. If $(E \cdot C) \ge 0$, then $(q^*A \cdot C) = 0$ and q contracts C. If $(E \cdot C) < 0$, then P appears in E. In both cases, P is q-exceptional. It follows that g is a birational contraction map and $g_*D \sim_{\mathbf{R},S} q_*(q^*A + E) = A$ is relatively ample. Then $p^*D = q^*(g_*D) + E$ and in particular g is D-non-positive. □

Definition 1.5.5 Let (X, Δ) be an lc pair projective over a variety S. Let $f \colon X \dashrightarrow Y/S$ be a birational contraction map over S to a normal variety Y projective over S. We call f (or Y) a *log minimal model* of $(X/S, \Delta)$ if f is $(K_X + \Delta)$-negative, $K_Y + f_*\Delta$ is relatively nef and Y is **Q**-factorial. If the identity $X \to X/S$ is a log minimal model, then $(X/S, \Delta)$ is a log minimal model in the sense in the preceding section. We call f (or Y) the *log canonical model* of $(X/S, \Delta)$ if f is the ample model of $K_X + \Delta$ over S. When X is terminal and Δ is zero, a log minimal model and the log canonical model of X/S are called a *minimal model* and the *canonical model* of X/S.

Log minimal models are isomorphic in codimension one and crepant with respect to the pairs.

Proposition 1.5.6 *Let $(X/S, \Delta)$ be an lc pair projective over a variety. Let $f: X \dashrightarrow X_1/S$ be a $(K_X + \Delta)$-negative birational contraction map over S to a normal variety projective over S, which makes an lc pair $(X_1/S, \Delta_1)$ with $\Delta_1 = f_*\Delta$. Then every log minimal model of $(X/S, \Delta)$ is a log minimal model of $(X_1/S, \Delta_1)$ and vice versa.*

Proof The vice-versa part is obvious. We shall prove the former part. Let $(X_2/S, \Delta_2)$ be a log minimal model of $(X/S, \Delta)$. It suffices to show that $g: X_1 \dashrightarrow X_2$ is a $(K_{X_1} + \Delta_1)$-negative birational contraction map.

The idea has appeared in Example 1.3.10. Take a common log resolution Y of (X, Δ), (X_1, Δ_1) and (X_2, Δ_2) with $\mu: Y \to X$ and $\mu_i: Y \to X_i$. Let T_i denote the sum of μ_i-exceptional prime divisors and let T denote the common part of T_1 and T_2. Write $\mu^*(K_X + \Delta) = \mu_i^*(K_{X_i} + \Delta_i) + E_i + F_i$ with $\mathbf{R}$-divisors E_i and F_i supported in T and $T_i - T$ respectively. Since $X \dashrightarrow X_i$ is $(K_X + \Delta)$-negative, E_i and F_i are effective and the support of F_i equals $T_i - T$.

By the negativity lemma, the μ_1-exceptional $\mathbf{R}$-divisor

$$E_1 - E_2 + F_1 = -\mu_1^*(K_{X_1} + \Delta_1) + \mu_2^*(K_{X_2} + \Delta_2) + F_2$$

is negative, that is, $E_1 \le E_2$ and $F_1 \le 0$. In particular $F_1 = 0$ and thus $T_1 = T \le T_2$, which means that g is a birational contraction map. The $(K_{X_1}+\Delta_1)$-negativity of g follows from the expression $\mu_1^*(K_{X_1}+\Delta_1) = \mu_2^*(K_{X_2}+\Delta_2)+(E_2-E_1)+F_2$ where $E_2 - E_1 \ge 0$. □

Corollary 1.5.7 *Let $(X/S, \Delta)$ be an lc pair projective over a variety. Let $(X_i/S, \Delta_i)$ for $i = 1, 2$ be log minimal models of $(X/S, \Delta)$. Then $X_1 \dashrightarrow X_2$ is small and crepant with respect to $K_{X_1} + \Delta_1$.*

Proof By Proposition 1.5.6, $(X_2/S, \Delta_2)$ is a log minimal model of $(X_1/S, \Delta_1)$ and vice versa. □

Let $(X/S, \Delta)$ be a $\mathbf{Q}$-factorial lc pair projective over a variety. Assume that $K_X + \Delta$ is relatively pseudo-effective. Then every output $(Y/S, \Gamma)$ of the $(K_X + \Delta)/S$-MMP is a log minimal model of $(X/S, \Delta)$. If the log abundance holds for $(Y/S, \Gamma)$, that is, $K_Y + \Gamma$ is relatively semi-ample, then by Lemma 1.2.13, Y admits a contraction $\varphi: Y \to Z/S$ for which $K_Y + \Gamma \sim_{\mathbf{R}} \varphi^* A$ with a relatively ample $\mathbf{R}$-divisor A on Z. The rational map $X \dashrightarrow Z$ is the ample model of $K_X + \Delta$ over S.

Birkar, Cascini, Hacon and McKernan proved the finiteness of models simultaneously with the existence of log flips for klt pairs. In Definition 1.5.5, we

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

call f a *weak log canonical model* of $(X/S, \Delta)$ if f is $(K_X + \Delta)$-non-positive and $K_Y + f_*\Delta$ is relatively nef. Let $V_{\mathbf{Q}}$ be a finite dimensional rational vector space and let $V = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$. A *rational polytope* in V is the convex hull of a finite number of points in $V_{\mathbf{Q}}$.

Theorem 1.5.8 ([48, corollary 1.1.5, theorem E]) *Let X be a normal variety projective over a quasi-projective variety S. Let $V = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$ be the extension of a finite dimensional vector subspace $V_{\mathbf{Q}}$ of the rational vector space $Z^1(X) \otimes \mathbf{Q}$ of $\mathbf{Q}$-divisors on X. Let C be a rational polytope in V such that for all $\Delta \in C$, (X, Δ) is klt and $A_\Delta \le \Delta$ for some relatively ample $\mathbf{Q}$-divisor A_Δ. Define the set $\mathscr{E}$ of $\Delta \in C$ such that $K_X + \Delta$ is relatively pseudo-effective.*

(i) *There exist finitely many rational maps $g_i \colon X \dashrightarrow Y_i/S$ such that*

$$\mathscr{E} = \bigsqcup_i \mathscr{A}_i, \qquad \mathscr{A}_i = \{\Delta \in \mathscr{E} \mid g_i \text{ the ample model of } K_X + \Delta \text{ over } S\}.$$

Every $\mathscr{A}_i$ has the closure $\bar{\mathscr{A}_i}$ which is a finite union of rational polytopes. Moreover if $\bar{\mathscr{A}_i} \cap \mathscr{A}_j \neq \emptyset$, then there exists a contraction $g_{ji} \colon Y_i \to Y_j$ such that $g_j = g_{ji} \circ g_i$.

(ii) *There exist finitely many birational maps $f_i \colon X \dashrightarrow W_i/S$ such that*

$$\mathscr{E} = \bigcup_i \mathscr{B}_i, \qquad \mathscr{B}_i = \{\Delta \in \mathscr{E} \mid f_i \text{ a weak lc model of } (X/S, \Delta)\}$$

and such that every weak lc model of $(X/S, \Delta)$ with $\Delta \in \mathscr{E}$ is isomorphic to some f_i. Every $\mathscr{B}_i$ is a rational polytope. For any $\mathscr{B}_i$, there exist some $\mathscr{A}_j$ and a contraction $h_{ji} \colon W_i \to Y_j$ such that $\mathscr{B}_i \subset \bar{\mathscr{A}_j}$ and $g_j = h_{ji} \circ f_i$.

Remark 1.5.9 In the theorem with the quasi-projective condition relaxed,

$$\mathscr{P} = \{\Delta \in V \mid (X, \Delta) \text{ lc}, \ K_X + \Delta \text{ relatively nef}\}$$

is also a rational polytope and called *Shokurov's polytope* [425]. The proof in [44, proposition 3.2], [131, theorem 4.7.2] uses the boundedness of length of an extremal ray in Theorem 1.4.10.

By the finiteness of models, one can run the MMP *with scaling* practically.

Definition 1.5.10 Let (X, Δ) be a $\mathbf{Q}$-factorial klt pair projective over a variety S. Fix a relatively big $\mathbf{R}$-divisor A such that $(X, \Delta + A)$ is klt and such that $K_X + \Delta + A$ is relatively nef. The $(K_X + \Delta)$-*minimal model program* over S with *scaling* of A is a $(K_X + \Delta)/S$-MMP $(X, \Delta) = (X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots$ running in the following manner, where $A_0 = A$ and $t_{-1} = 1$.

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

1 If $K_{X_i} + \Delta_i$ is relatively nef, then output $(X_i/S, \Delta_i)$.
2 If $K_{X_i} + \Delta_i$ is not relatively nef, then define the *nef threshold* $t_i \in \mathbf{R}_{>0}$ as the least real number such that $K_{X_i} + \Delta_i + t_i A_i$ is relatively nef. It satisfies $t_i \le t_{i-1}$. By Lemma 1.5.11, there exists a $(K_{X_i} + \Delta_i)$-negative extremal ray $\mathbf{R}_{\ge 0}[C_i]$ of $\overline{\mathrm{NE}}(X_i/S)$ such that $((K_{X_i} + \Delta_i + t_i A_i) \cdot C_i) = 0$. Take the extremal contraction $\pi_i \colon X_i \to Y_i/S$ associated with this ray.
3 If π_i is a log Mori fibre space, then output $(X_i/S, \Delta_i)$.
4 If π_i is a log divisorial contraction, then take $X_{i+1} = Y_i$. If π_i is a log flipping contraction, then construct the log flip $X_{i+1} \to Y_i$ of π_i by Theorem 1.4.11. In each case, set $f_i \colon X_i \dashrightarrow X_{i+1}$, $\Delta_{i+1} = f_{i*}\Delta_i$ and $A_{i+1} = f_{i*}A_i$. Then f_i is crepant with respect to $K_{X_i} + \Delta_i + t_i A_i$ and hence $K_{X_{i+1}} + \Delta_{i+1} + t_i A_{i+1}$ is relatively nef. Replace (X_i, Δ_i) by (X_{i+1}, Δ_{i+1}) and go back to 1.

Lemma 1.5.11 *Let $(X/S, \Delta)$ be a klt pair projective over a variety. Let A be a relatively big $\mathbf{R}$-divisor such that $(X, \Delta + A)$ is klt and such that $K_X + \Delta + A$ is relatively nef but $K_X + \Delta + tA$ is not relatively nef for any $t < 1$. Then there exists a $(K_X + \Delta)$-negative extremal ray $\mathbf{R}_{\ge 0}[C]$ of $\overline{\mathrm{NE}}(X/S)$ such that $((K_X + \Delta + A) \cdot C) = 0$.*

Proof We shall verify the finiteness of the number of $(K_X + \Delta + A/2)$-negative extremal rays of $\overline{\mathrm{NE}}(X/S)$, which implies the lemma. We may assume that S is quasi-projective. By Kodaira's lemma, one can write $A = H + E$ with a relatively ample $\mathbf{Q}$-divisor H and an effective $\mathbf{R}$-divisor E. Take $D = \Delta + (1-\varepsilon)A/2 + \varepsilon E/2$ with small positive ε. Then (X, D) is klt and $K_X + \Delta + A/2 = K_X + D + \varepsilon H/2$. Thus the finiteness follows from the cone theorem. □

Remark 1.5.12 The lemma actually holds for an lc pair $(X/S, \Delta)$ and an $\mathbf{R}$-divisor A such that $(X, \Delta + A)$ is lc. This is an elementary consequence [43, lemma 3.1] of Theorem 1.4.10 and Remark 1.5.9. Hence the MMP with scaling may be formulated for lc pairs without relative bigness of A.

The advantage of the MMP in Definition 1.5.10 is that each step X_i is a weak lc model of $(X/S, \Delta + t_i A)$. This yields the following termination. As a special case, the MMP with scaling functions when $X \to S$ is generically finite because all divisors are relatively big.

Corollary 1.5.13 *Let $(X/S, \Delta)$ be a $\mathbf{Q}$-factorial klt pair projective over a variety. If Δ is relatively big or $K_X + \Delta$ is not relatively pseudo-effective, then the $(K_X + \Delta)$-MMP over S with scaling terminates.*

Proof Keep the notation in Definition 1.5.10. When Δ is relatively big, let $\varepsilon = 0$. When $K_X + \Delta$ is not relatively pseudo-effective, choose a rational number $0 < \varepsilon \le 1$ such that $K_X + \Delta + \varepsilon A$ is not relatively pseudo-effective. Take the

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

closed interval $I = [\varepsilon, 1]$. Then $\Delta + tA$ is relatively big for all $t \in I$, and locally on S, each step X_i of the $(K_X + \Delta)/S$-MMP with scaling of A is a weak lc model of $(X/S, \Delta + t_i A)$ with $t_i \in I$. By Theorem 1.5.8(ii), the number of weak lc models of $(X/S, \Delta + tA)$ with $t \in I$ is finite and thus so is the number of models X_i. Hence the MMP must terminate. Note that $X_i \dashrightarrow X_j$ with $i < j$ is never an isomorphism by Lemma 1.4.12. □

Corollary 1.5.14 *Let $(X/S, \Delta)$ be a* **Q***-factorial klt pair projective over a variety. If $K_X + \Delta$ is relatively big, then the $(K_X + \Delta)$-MMP over S with scaling terminates with a log minimal model $(Y/S, \Gamma)$ such that $K_Y + \Gamma$ is relatively semi-ample.*

Proof We may assume that S is quasi-projective. Then $(X, \Delta + B)$ is klt for a general effective **R**-divisor $B \sim_{\mathbf{R},S} \varepsilon(K_X + \Delta)$ with small positive ε. Every $(K_X + \Delta)/S$-MMP with scaling of A is a $(K_X + \Delta + B)/S$-MMP with scaling of $(1+\varepsilon)A$. By Corollary 1.5.13, it terminates with a log minimal model $(Y/S, \Gamma)$ with relatively big $K_Y + \Gamma$, which is relatively semi-ample by the base-point free theorem. □

Combining this with the work of Fujino and Mori [134], one obtains the finite generation of the log canonical ring.

Definition 1.5.15 The *canonical ring* of a normal variety X is the graded ring $\bigoplus_{i \in \mathbf{N}} H^0(\mathscr{O}_X(iK_X))$. The *log canonical ring* of a pair (X, Δ) is the graded ring $\bigoplus_{i \in \mathbf{N}} H^0(\mathscr{O}_X(\lfloor i(K_X + \Delta) \rfloor))$.

Theorem 1.5.16 *Let (X, Δ) be a klt pair such that X is complete and Δ is a* **Q***-divisor. Then the log canonical ring of (X, Δ) is finitely generated.*

The next proposition with $I = \emptyset$ shows the existence of a **Q**-factorialisation of X which admits a klt pair (X, Δ). A **Q***-factorialisation* of a normal variety means a small contraction from a **Q**-factorial normal variety to it.

Proposition 1.5.17 ([48, corollary 1.4.3]) *Let (X, Δ) be a klt pair and let I be an arbitrary subset of divisors E exceptional over X with $a_E(X, \Delta) \le 1$. Then there exists a birational contraction $\pi\colon Y \to X$ from a* **Q***-factorial normal variety such that I coincides with the set of π-exceptional prime divisors on Y.*

Proof Take a log resolution $\mu\colon X' \to X$ of (X, Δ) on which every $E \in I$ is realised as a divisor. Write $K_{X'} + N = \mu^*(K_X + \Delta) + P$ with effective **R**-divisors P and N without common components such that $\mu_* P = 0$ and $\mu_* N = \Delta$. Take the sum $S = \sum_i S_i$ of all the μ-exceptional prime divisors S_i not in I and make a klt pair $(X', N + \varepsilon S)$ with small positive ε.

We run the $(K_{X'} + N + \varepsilon S)/X$-MMP with scaling. It ends with a log minimal model (Y, N_Y) over X for the strict transform N_Y of $N + \varepsilon S$. Since $K_{X'} + N +$

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

$\varepsilon S \equiv_X P + \varepsilon S$, the negativity lemma shows that $f\colon X' \dashrightarrow Y$ contracts all prime divisors that appear in $P + \varepsilon S$. On the other hand, the $(P + \varepsilon S)$-negative map f does not contract any divisors in I. Hence f exactly contracts μ-exceptional prime divisors not in I and the result $Y \to X$ is a desired contraction. $\square$

We quote a clever application which transforms an lc pair to a dlt pair.

Theorem 1.5.18 (Hacon [270, theorem 3.1]) *Let (X, Δ) be an lc pair such that X is quasi-projective. Then there exists a birational contraction $\pi\colon Y \to X$ from a $\mathbf{Q}$-factorial normal variety such that (Y, Δ_Y) is dlt for the $\mathbf{R}$-divisor Δ_Y defined by $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ with $\pi_*\Delta_Y = \Delta$.*

In the remainder of the section, we shall explain how the existence of flips is derived from the lower dimensional MMP. We follow the exposition [248]. The argument requires familiarity with the abstract theory of the MMP. The reader may treat this part only for reference.

Shokurov reduced the existence to that of pre-limiting flips. A *pre-limiting flip* (*pl flip*) is the log flip of a log flipping contraction from a $\mathbf{Q}$-factorial dlt pair (X, Δ) such that $\lfloor \Delta \rfloor$ contains a prime divisor S with $-S$ relatively ample. The following assertion is sufficient after perturbation of Δ.

Theorem 1.5.19 *Let (X, Δ) be a $\mathbf{Q}$-factorial plt pair such that Δ is a $\mathbf{Q}$-divisor and $S = \lfloor \Delta \rfloor$ is a prime divisor. Let $\pi\colon X \to Z$ be an elementary log flipping contraction with respect to (X, Δ) such that $-S$ is π-ample. Then the log flip of π exists.*

Fix a positive integer l such that $l(K_X + \Delta)$ is integral and Cartier. The log flip of π is, if it exists, described as $\operatorname{Proj}_Z \mathscr{R}(Z, l(K_Z + \pi_*\Delta)) \to Z$ by the graded $\mathscr{O}_Z$-algebra $\mathscr{R}(Z, l(K_Z + \pi_*\Delta))$ in Notation 1.3.17. As will be seen in Lemma 5.1.2, the existence of the log flip is equivalent to the finite generation of $\mathscr{R}(Z, l(K_Z + \pi_*\Delta))$. Since $\rho(X/Z) = 1$ and $-S$ is π-ample, it is also equivalent to the finite generation of $\mathscr{R}(Z, S_Z)$ for $S_Z = \pi_*S$. The lemma below is used implicitly.

Lemma 1.5.20 *Let X be a variety and let $\mathscr{R} = \bigoplus_{i \in \mathbf{N}} \mathscr{R}_i$ be a graded $\mathscr{O}_X$-algebra which is an integral domain. Fix a positive integer l. Then $\mathscr{R}$ is a finitely generated $\mathscr{O}_X$-algebra if and only if so is the truncation $\mathscr{S} = \bigoplus_{i \in \mathbf{N}} \mathscr{R}_{il}$.*

Proof The only-if part is obvious. To see the converse, suppose that $\mathscr{S}$ is finitely generated, by which $\mathscr{S}$ is a noetherian domain. It suffices to prove that $\mathscr{M}_j = \bigoplus_{i \in \mathbf{N}} \mathscr{R}_{il+j}$ is a finite $\mathscr{S}$-module for each $0 \le j < l$. We may assume that $\mathscr{M}_j \ne 0$ and locally take a non-zero member m_j of $\mathscr{M}_j$. The multiplication by m_j^{l-1} defines an injection $\mathscr{M}_j \hookrightarrow \mathscr{S}$ of $\mathscr{S}$-modules. Since $\mathscr{S}$ is noetherian, the finiteness of $\mathscr{M}_j$ follows. $\square$

By Theorem 1.4.26(i), S is normal. By Proposition 2.2.23, the quotient $\mathscr{Q}_i = \mathscr{O}_X(iS)/\mathscr{O}_X((i-1)S)$ is a divisorial sheaf on S. Let $\mathscr{S}_i$ denote the image of the induced map $\mathscr{O}_Z(iS_Z) = \pi_*\mathscr{O}_X(iS) \to \pi_*\mathscr{Q}_i$. It fits into the exact sequence

$$0 \to \mathscr{O}_Z((i-1)S_Z) \to \mathscr{O}_Z(iS_Z) \to \mathscr{S}_i \to 0.$$

Then $\mathscr{R}(Z, S_Z)$ is finitely generated if and only if so is the graded $\mathscr{O}_{S_Z}$-algebra $\bigoplus_{i\in\mathbf{N}} \mathscr{S}_i$. This is also equivalent to the finite generation of $\bigoplus_{i\in\mathbf{N}} \mathscr{R}_i$ for the image $\mathscr{R}_i$ of the restriction map

$$\pi_*\mathscr{O}_X(il(K_X+\Delta)) \to \pi_*\mathscr{O}_S(il(K_X+\Delta)|_S).$$

We shall demonstrate this finite generation applying the *extension theorem* due to Hacon–McKernan [170] and Takayama [438]. What will be used is the following variant.

Theorem 1.5.21 (Extension theorem [171, theorem 5.4.21]) *Let $\pi\colon X \to Z$ be a projective morphism from a smooth variety to a variety. Let (X, Δ) be a plt pair such that Δ is a $\mathbf{Q}$-divisor with snc support and $S = \lfloor\Delta\rfloor$ is a prime divisor. Take a positive integer l such that $L = l(K_X+\Delta)$ is integral. Suppose that $\Delta = S + A + C$ with a π-ample $\mathbf{Q}$-divisor A and an effective $\mathbf{Q}$-divisor C in which S does not appear. Suppose that the relative base locus of aL contains no lc centres of $(X, \lceil\Delta\rceil)$ for some positive integer a. Then the natural map $\pi_*\mathscr{O}_X(L) \to \pi_*\mathscr{O}_S(L|_S)$ is surjective.*

We fix a log resolution $\mu\colon Y \to X$ of (X, Δ) with $\pi_Y = \pi\circ\mu\colon Y \to Z$. Let T denote the strict transform in Y of S and let $\pi_T = \pi_Y|_T\colon T \to S_Z$, which is birational. We write $K_Y + \Delta_Y = \mu^*(K_X+\Delta) + P$ with effective $\mathbf{Q}$-divisors P and Δ_Y without common components such that $\mu_*P = 0$ and $\mu_*\Delta_Y = \Delta$. We may choose μ so that the support of $\Delta_Y - T$ is a disjoint union of prime divisors. Indeed for the expression $\Delta_Y = T + \sum_i e_iE_i$ with prime divisors E_i and $e_i \in \mathbf{Q}_{>0}$, let a be the maximum of $e_i + e_j$ such that $E_i \cap E_j \neq \emptyset$ and let b be the number of pairs (E_i, E_j) such that $E_i \cap E_j \neq \emptyset$ and $e_i + e_j = a$. Then the blow-up of Y along $E_i \cap E_j$ decreases (a, b) with respect to lexicographic order. We write $K_T + \Delta_T = (K_Y + \Delta_Y)|_T$ by $\Delta_T = (\Delta_Y - T)|_T$. Then (T, Δ_T) is terminal.

We build a tower $\cdots \to Y_2 \to Y_1 \to Y_0 = Y$ of log resolutions inductively.

Lemma 1.5.22 *There exists a sequence $\{(\mu_i, D_i)\}_{i\geq 1}$ of pairs of a log resolution $\mu_i\colon Y_i \to X$ of (X, Δ) factoring through Y_{i-1} as $Y_i \to Y_{i-1} \to X$ and a $\mathbf{Q}$-divisor D_i on Y_i which satisfies the following. Let $\pi_i = \pi\circ\mu_i\colon Y_i \to Z$ and $T_i = \mu_{i*}^{-1}S$ and write $K_{Y_i} + \Delta_i = \mu_i^*(K_X+\Delta) + P_i$ with $\mu_{i*}P_i = 0$ and $\mu_{i*}\Delta_i = \Delta$ in the same manner as above. Then for all i and j,*

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

- *the induced morphism $T_i \to T$ is an isomorphism,*
- *$T_i \le D_i \le \Delta_i$ and ilD_i is integral,*
- *$il(K_{Y_i} + D_i)$ and $il(K_{Y_i} + \Delta_i)$ have the same π_i-mobile part M_i,*
- *$\pi_{i*}\mathscr{O}_{Y_i}(ail(K_{Y_i}+D_i)) \to \pi_{T*}\mathscr{O}_T(ail(K_T+C_i))$ is surjective for any positive integer a, where $C_i = (D_i - T_i)|_T$ via $T_i \simeq T$ and*
- *$iC_i + jC_j \le (i+j)C_{i+j}$.*

Proof For a log resolution $\mu_i : Y_i \to X$ which factors through Y_{i-1} and induces an isomorphism $T_i \simeq T$, we take the decomposition $il(K_{Y_i} + \Delta_i) = M_i + F_i$ into the π_i-mobile part M_i and the π_i-fixed part F_i. Note that F_i is π_i-exceptional. Let Σ_i denote the support of $\Delta_i - T_i$. We take Y_{i-1} as Y_i initially and blow up Y_i along $G \cap T_i$ for a prime divisor $G \subset \Sigma_i$ successively until the relative base locus of M_i contains no components of $\Sigma_i \cap T_i$. This blow-up keeps T_i isomorphic to T. We further blow up Y_i along $G \cap G'$ for prime divisors $G, G' \subset \Sigma_i$ until Σ_i is a disjoint union of prime divisors.

Let E_i be the maximal divisor on Y_i such that $E_i \le il(\Delta_i - T_i)$ and $E_i \le F_i$. In other words, E_i is the componentwise minimum of $il(\Delta_i - T_i)$ and F_i. We take $D_i = \Delta_i - E_i/il$. Then $T_i \le D_i \le \Delta_i$ and

$$il(K_{Y_i} + D_i) = M_i + (F_i - E_i)$$

with the effective divisor $F_i - E_i$ which has no common components with D_i. In particular, $il(K_{Y_i} + D_i)$ still has the π_i-mobile part M_i. One can apply Theorem 1.5.21 to the plt pair $(Y_i/Z, D_i)$, which asserts the surjection $\pi_{i*}\mathscr{O}_{Y_i}(ail(K_{Y_i} + D_i)) \twoheadrightarrow \pi_{T*}\mathscr{O}_T(ail(K_T + C_i))$ for any $a \in \mathbf{Z}_{>0}$ with $C_i = (D_i - T_i)|_T$ via $T_i \simeq T$.

The inequality $K_{Y_j} + \Delta_j \ge \mu_{ij}^*(K_{Y_i} + \Delta_i)$ by $\mu_{ij} : Y_j \to Y_i$ for $i \le j$ is an equality about T_j, and $\Delta_j - T_j$ equals $\mu_{ij}^*(\Delta_i - T_i)$ about T_j. Hence about T_{i+j}, one has $F_{i+j} \le \mu_{i,i+j}^* F_i + \mu_{j,i+j}^* F_j$ and $i\mu_{i,i+j}^*(D_i - T_i) + j\mu_{j,i+j}^*(D_j - T_j) \le (i+j)(D_{i+j} - T_{i+j})$. This yields $iC_i + jC_j \le (i+j)C_{i+j}$. □

By the convexity $iC_i + jC_j \le (i+j)C_{i+j}$ and the boundedness $C_i \le \Delta_T$, the sequence $\{C_i\}_i$ converges to some effective $\mathbf{R}$-divisor $C \le \Delta_T$. The pair (T, C) is terminal as so is (T, Δ_T).

Take a log minimal model $f_i : T \dashrightarrow \bar{T}_i/S_Z$ of the terminal pair $(T, C_i)/S_Z$ by the lower dimensional MMP. We apply the finiteness of log minimal models in Theorem 1.5.8(ii) in the neighbourhood at C in the real vector space generated by prime divisors appearing in Δ_T. It shows that there exist only finitely many models $\bar{T}_i$ up to isomorphism. Then replacing μ by a higher log resolution, we can assume that every f_i is a contraction $T \to \bar{T}_i/S_Z$.

By the base-point free theorem, the relatively nef and big divisor $il(K_{\bar{T}_i} + f_{i*}C_i)$ is relatively semi-ample. Hence there exists a positive integer a_i such

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

that $a_i il(K_{\bar{T}_i} + f_{i*}C_i)$ is relatively free. The pull-back $Q_i = a_i il f_i^*(K_{\bar{T}_i} + f_{i*}C_i)$ coincides with the π_T-mobile part of $a_i il(K_T + C_i)$ and it is relatively free. By the surjection in Lemma 1.5.22, the π_i-mobile part R_i of $a_i il(K_{Y_i} + D_i)$ becomes relatively free with $Q_i = R_i|_T$ after blowing up Y_i away from T_i if necessary.

We keep the notation M_i in Lemma 1.5.22 and define $L_i = M_i|_T$. Write $K_T = (\mu|_T)^*((K_X + \Delta)|_S) + F$ with $F = P|_T - \Delta_T$. Then $\lceil F \rceil \geq 0$.

Lemma 1.5.23 *The sequence $\{L_i/i\}_i$ converges to an* **R***-divisor A on T. The limit A is π_T-semi-ample and the π_T-mobile part of $\lceil iA + F \rceil$ is at most L_i.*

Proof Keep the notation $\mu_{ij} \colon Y_j \to Y_i$. Observe that $\mu_{i,i+j}^* M_i + \mu_{j,i+j}^* M_j \leq M_{i+j}$ and hence $L_i + L_j \leq L_{i+j} \leq (i+j)l(K_T + \Delta_T)$. Thus the limit A of $\{L_i/i\}_i$ exists with $A \leq l(K_T + \Delta_T)$. From $a_i M_i \leq R_i$ and $\mu_{i,a_i i}^* R_i \leq M_{a_i i}$, one has $a_i L_i \leq Q_i \leq L_{a_i i}$, that is, $L_i/i \leq Q_i/a_i i \leq L_{a_i i}/a_i i$. Hence A is also the limit of the sequence of $Q_i/a_i i = l f_i^*(K_{\bar{T}_i} + f_{i*}C_i)$. Choosing one of the finitely many models $\bar{T}_i$ up to isomorphism, one can describe A as $l f_i^*(K_{\bar{T}_i} + f_{i*}C)$. This is the pull-back of the relatively nef and big **R**-divisor $l(K_{\bar{T}_i} + f_{i*}C)$, which is relatively semi-ample by the base-point free theorem.

For a divisor B on Y_j or T, we write $\mathrm{Mb}(B)$ for the relative mobile part of B over Z or S_Z. For all $i, j \geq 1$, since

$$\left(\frac{iR_j}{a_j j} + P_j - \Delta_j\right) - K_{Y_j} = \frac{iR_j}{a_j j} - \mu_j^*(K_X + \Delta)$$

is relatively nef and big, $R^1\pi_{j*}\mathscr{O}_{Y_j}(\lceil iR_j/a_j j + P_j - \Delta_j \rceil) = 0$ by Kawamata–Viheweg vanishing. This provides the surjection $\pi_{j*}\mathscr{O}_{Y_j}(\lceil iR_j/a_j j + \Gamma_j \rceil) \twoheadrightarrow \pi_{T*}\mathscr{O}_T(\lceil iQ_j/a_j j + F \rceil)$ for $\Gamma_j = P_j + T_j - \Delta_j$. Hence

$$\mathrm{Mb}\left(\left\lceil \frac{iQ_j}{a_j j} + F \right\rceil\right) \leq \mathrm{Mb}\left(\left\lceil \frac{iR_j}{a_j j} + \Gamma_j \right\rceil\right)\bigg|_T.$$

From $R_j/a_j j \leq l(K_{Y_j} + \Delta_j) = l\mu_j^*(K_X + \Delta) + lP_j$ and $\lceil \Gamma_j \rceil = \lceil P_j \rceil$, one has

$$\begin{aligned}\mathrm{Mb}\left(\left\lceil \frac{iR_j}{a_j j} + \Gamma_j \right\rceil\right)\bigg|_T &\leq \mathrm{Mb}(il\mu_j^*(K_X + \Delta) + ilP_j + \lceil P_j \rceil)|_T \\ &= \mathrm{Mb}(il\mu_j^*(K_X + \Delta))|_T = \mathrm{Mb}(il\mu_i^*(K_X + \Delta))|_T = L_i.\end{aligned}$$

Join these two inequalities into $\mathrm{Mb}(\lceil iQ_j/a_j j + F \rceil) \leq L_i$. This tends to the inequality $\mathrm{Mb}(\lceil iA + F \rceil) \leq L_i$ when j goes to infinity. □

Proof of Theorem 1.5.19 We have observed that the theorem is equivalent to the finite generation of $\mathscr{R} = \bigoplus_{i \in \mathbf{N}} \mathscr{R}_i$ for the image $\mathscr{R}_i$ of $\pi_*\mathscr{O}_X(il(K_X+\Delta)) \to \pi_*\mathscr{O}_S(il(K_X + \Delta)|_S)$. By Lemma 1.5.22, $\mathscr{R}_i = \pi_{T*}\mathscr{O}_T(L_i)$. Note that $L_i \leq iA$

for the π_T-semi-ample $\mathbf{R}$-divisor A in Lemma 1.5.23 since $\{L_{2^p i}/2^p i\}_{p\in\mathbf{N}}$ is non-decreasing and converges to A.

If A is a $\mathbf{Q}$-divisor, then there exists a positive integer n such that nA is integral, Cartier and π_T-free. Then $inA \le L_{in}$ by Lemma 1.5.23 and hence $L_{in} = inA$. Now $\mathscr{R}_{in} = \pi_{T*}\mathscr{O}_T(inA)$ and nA is π_T-free. Thus $\bigoplus_{i\in\mathbf{N}}\mathscr{R}_{in}$ is finitely generated and so is $\mathscr{R}$ by Lemma 1.5.20.

If A were not a $\mathbf{Q}$-divisor, then by the Diophantine approximation in [48, lemma 3.7.6], for any $\varepsilon > 0$, there would exist $i \ge 1$ and a π_T-free Cartier divisor G such that $G \not\le iA$ and such that every prime divisor has coefficient at least $-\varepsilon$ in $iA - G$. The latter property implies that $\lceil F + iA - G\rceil$ is effective if ε is sufficiently small. Consequently $G \le G + \lceil F + iA - G\rceil = \lceil iA + F\rceil$ and hence $G \le L_i \le iA$ by Lemma 1.5.23, which contradicts $G \not\le iA$. □

1.6 Termination of Flips

In spite of the establishment of the termination of the MMP with scaling in the setting of Corollary 1.5.13, we still do not know whether an arbitrary MMP terminates or not. The full termination is only known up to dimension three.

Theorem 1.6.1 ([240], [424, theorem 4.1]) *The termination of log flips in Conjecture* 1.4.15 *holds in dimension three.*

The proof relies on the decreasing property of an invariant named the *difficulty*, which grew out of the work of Shokurov [423, definition 2.15]. We shall demonstrate the termination for terminal threefold pairs. Before this, we see that a terminal threefold singularity is isolated.

Lemma 1.6.2 *If* (X, Δ) *is a terminal threefold pair, then* X *has isolated singularities.*

Proof We may assume that X is affine. Take a log resolution $\mu\colon X' \to X$ of (X, Δ) and write $K_{X'} + \Delta' = \mu^*(K_X + \Delta) + P$ with the strict transform Δ' of Δ and an exceptional $\mathbf{R}$-divisor P. Every exceptional prime divisor has positive coefficient in P. Let C be an arbitrary curve in X. Take the general hyperplane section S of X, which intersects C properly at a general point x in C. It suffices to prove that S is smooth at x, from which we deduce that X is smooth at x.

By the general choice of S, μ is also a log resolution of $(X, \Delta + S)$ and $K_{S'} + \Delta'|_{S'} \equiv_S P|_{S'}$ with $S' = \mu_*^{-1}S = \mu^*S$. Unless $P|_{S'}$ is zero, $K_{S'}$ is not μ-nef by the negativity lemma and one can find and contract a relative (-1)-curve in S'/S. One eventually attains a smooth surface T/S in which the strict transform of $P|_{S'}$ becomes zero. Then $\mu_T : T \to S$ is finite. It is an isomorphism because S is Cohen–Macaulay by Theorem 1.4.20, or from the following direct argument. The natural map $\mathscr{O}_X(-S) \otimes R^1\mu_*\mathscr{O}_{X'} = R^1\mu_*\mathscr{O}_{X'}(-S') \to R^1\mu_*\mathscr{O}_{X'}$ is surjective by $R^1\mu_*\mathscr{O}_{S'} \simeq R^1\mu_{T*}\mathscr{O}_T = 0$. Hence $R^1\mu_*\mathscr{O}_{X'}$ is zero about S and $\mathscr{O}_X \to \mu_*\mathscr{O}_{S'} = \mu_{T*}\mathscr{O}_T$ is surjective. Then $T \simeq S$ and S is smooth. □

We do not assume relative setting in the following theorem. We do not need $\mathbf{Q}$-factoriality if the boundary is zero.

Theorem 1.6.3 *Let (X, Δ) be a $\mathbf{Q}$-factorial canonical threefold pair such that $\lfloor \Delta \rfloor = 0$. Then there exists no infinite sequence $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots$ of elementary log flips with respect to (X_i, Δ_i) for the strict transform Δ_i of Δ.*

Proof Write $\Delta = \sum_{j=1}^n d_j D_j$ with prime divisors D_j and $d_j \in \mathbf{R}_{>0}$. We shall prove the theorem by induction on the number n of components of Δ. We write the log flip as $X_i \to Y_i \leftarrow X_{i+1}$.

Let d be the maximum of d_j, where we define $d = 0$ if $n = 0$, that is, $\Delta = 0$. Discussing on a log resolution of (X, Δ), we have the finiteness of the number of divisors E over X with log discrepancy $a_E(X, \Delta) < 2 - d$. Define the finite set $S = \{s \in d + \sum_j \mathbf{N} d_j \mid s < 1\}$, where $S = \{0\}$ if $n = 0$. For $s \in S$, let $N(i, s)$ denote the number of divisors E over X_i such that $a_E(X_i, \Delta_i) < 2 - s$. We define the *difficulty*

$$D(i) = \sum_{s \in S} N(i, s).$$

The sequence $\{D(i)\}_i$ is non-increasing since the inequality $a_E(X_i, \Delta_i) \le a_E(X_{i+1}, \Delta_{i+1})$ holds in our situation similarly to Lemma 1.4.12.

Assume that $n = 0$, in which $D(i) = N(i, 0)$. Take a curve C^+ in X_{i+1} contracted to a point in Y_i. In the same manner as for Lemma 1.4.12, every divisor E over X_{i+1} with centre $c_{X_{i+1}}(E) = C^+$ has $1 \le a_E(X_i) < a_E(X_{i+1})$. Hence X_{i+1} is terminal at the generic point η of C^+ and thus smooth at η by Lemma 1.6.2. By blowing up X_{i+1} along C^+ about η, one obtains the exceptional divisor F with $a_F(X_{i+1}) = 2$. Since $a_F(X_i) < a_F(X_{i+1}) = 2$, F is counted in $N(i, 0)$ but not in $N(i+1, 0)$. Thus the strict inequality $D(i+1) < D(i)$ holds and the sequence must terminate.

When $n \ge 1$, we assume without loss of generality that $d_n = d$. Let D_{ji} be the strict transform in X_i of D_j. If there exists a curve C^+ in $D_{n,i+1}$ contracted to a point in Y_i, then the same argument as above shows that X_{i+1} is smooth at the generic point η of C^+. The divisor F obtained at η by the blow-up of X_{i+1} along C^+ has $a_F(X_i, \Delta_i) < a_F(X_{i+1}, \Delta_{i+1}) = 2 - s$ for $s = \sum_j d_j \operatorname{ord}_F D_{j,i+1} \in S$. It is counted in $N(i, s)$ but not in $N(i+1, s)$. Hence for the normalisation T_i of D_{ni}, the induced map $T_i \dashrightarrow T_{i+1}$ is a morphism for all i after truncation of the

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

sequence. Take a compactification $\bar{T}_0$ of T_0 which is smooth about $\bar{T}_0 \setminus T_0$, and compactify $T_i \to T_{i+1}$ to $\bar{T}_i \to \bar{T}_{i+1}$ naturally.

If D_{ni} contains a curve C contracted to a point in Y_i, then $\bar{T}_i \to \bar{T}_{i+1}$ is not isomorphic above C and hence $\rho(\bar{T}_{i+1}) < \rho(\bar{T}_i)$ as $(-K_{X_i} \cdot C) \neq 0$. Truncating the sequence again, we attain the situation where $T_i \simeq T_{i+1}$ for all i. Then D_{ni} is nef over Y_i and $X_i \dashrightarrow X_{i+1}$ is a log flip with respect to $(X_i, \sum_{j=1}^{n-1} d_j D_{ji})$. By the assumption of induction on n, the sequence terminates. □

Shokurov reduced the termination in an arbitrary dimension to the two properties in Conjecture 1.6.5 of an invariant of singularity called the minimal log discrepancy. We include the proof for the sake of completion.

Definition 1.6.4 Let (X, Δ) be a pair. Let η be a scheme-theoretic point in X with closure $Z = \overline{\{\eta\}}$ in X. The *minimal log discrepancy* $\mathrm{mld}_\eta(X, \Delta)$ of the pair (X, Δ) at η is defined as

$$\mathrm{mld}_\eta(X, \Delta) = \inf\{a_E(X, \Delta) \mid E \text{ divisor over } X,\ c_X(E) = Z\}.$$

Taking a log resolution of $(X, \Delta, \mathfrak{p})$ for the ideal sheaf $\mathfrak{p}$ in $\mathscr{O}_X$ defining Z, one sees that $\mathrm{mld}_\eta(X, \Delta)$ is either a non-negative real number or minus infinity and that it is actually the minimum unless $\mathrm{mld}_\eta(X, \Delta) = -\infty$. It satisfies the relation $\mathrm{mld}_z(X, \Delta) = \mathrm{mld}_\eta(X, \Delta) + \dim Z$ for a general closed point z in Z. It is sometimes convenient to use the *minimal log discrepancy* of (X, Δ) *in* a closed subset W of X which is defined as

$$\mathrm{mld}_W(X, \Delta) = \inf\{a_E(X, \Delta) \mid E \text{ divisor over } X,\ c_X(E) \subset W\}.$$

We say that a subset I of the real numbers satisfies the *ascending chain condition*, or the *ACC* for short, if there exists no strictly increasing infinite sequence in I. We say that I satisfies the *descending chain condition*, or the *DCC*, if there exists no strictly decreasing infinite sequence in I.

Conjecture 1.6.5 (i) (Lower semi-continuity) *For a pair (X, Δ), the function $X^m \to \mathbf{R}_{\geq 0} \cup \{-\infty\}$ from the set X^m of closed points in X which sends x to $\mathrm{mld}_x(X, \Delta)$ is lower semi-continuous.*

(ii) (ACC) *Fix a positive integer n and a subset I of the positive real numbers which satisfies the DCC. Then the set*

$$\{\mathrm{mld}_x(X, \Delta) \mid (X, \Delta) \text{ pair},\ \dim X = n,\ \Delta \in I\}$$

satisfies the ACC, where $\Delta \in I$ means that all coefficients in Δ belong to I.

The lower semi-continuity was verified in dimension three by Ambro [12] as an easy consequence of the MMP. On the other hand, the ACC was only proved

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

in dimension two by Alexeev [7]. The proof requires deep numerical analysis of surface singularities.

We assume the projectivity of the variety in the following theorem.

Theorem 1.6.6 (Shokurov [427]) *Let* (X, Δ) *be an lc pair such that* X *is projective. Assume Conjecture* 1.6.5 *in the dimension of* X*. Then there exists no infinite sequence* $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots$ *of log flips* $X_i \dashrightarrow X_{i+1}$ *associated with an* $(K_{X_i} + \Delta_i)$*-negative extremal face of* $\overline{\mathrm{NE}}(X_i)$ *for the strict transform* Δ_i *of* Δ.

Proof We write the log flip $f_i \colon X_i \dashrightarrow X_{i+1}$ as $X_i \to Y_i \leftarrow X_{i+1}$ where X_i and Y_i are projective. Let Z_i and Z_i^+ denote the exceptional loci of $X_i \to Y_i$ and $X_{i+1} \to Y_i$ respectively.

Step 1 We use the ACC in this step. Let $a_i = \mathrm{mld}_{Z_i}(X_i, \Delta_i)$. We claim the existence of a real number a such that $a \le a_i$ for all i after truncation and such that $a = a_i$ for infinitely many i.

Consider the non-decreasing sequence $\alpha_0 \le \alpha_1 \le \cdots$ of $\alpha_i = \inf\{a_j \mid i \le j\}$. It suffices to show that for any i, there exists $j \ge i$ such that $\alpha_i = a_j$. Indeed, then $\alpha_i = \mathrm{mld}_\eta(X_j, \Delta_j) = \mathrm{mld}_z(X_j, \Delta_j) - \dim \overline{\{\eta\}}$ with some scheme-theoretic point η in Z_j and a general point z in $\overline{\{\eta\}}$. The ACC provides a real number a such that $a = \alpha_i$ for all sufficiently large i, which satisfies the property in the claim.

Let $\alpha_{il} = \min\{a_j \mid i \le j \le l\}$, which equals $a_j = a_E(X_j, \Delta_j)$ for some $i \le j \le l$ and a divisor E over X_j with $c_{X_j}(E) \subset Z_j$. Similarly to Lemma 1.4.12,

$$a_E(X_i, \Delta_i) \le a_E(X_{i+1}, \Delta_{i+1}) \le \cdots \le a_E(X_j, \Delta_j).$$

If $c_{X_p}(E)$ were contained in Z_p for some $i \le p < j$, then the strict inequality $a_E(X_p, \Delta_p) < a_E(X_{p+1}, \Delta_{p+1})$ would hold as in the same lemma and hence $a_p \le a_E(X_p, \Delta_p) < a_j$, which contradicts $a_j = \alpha_{il} \le a_p$. Thus $c_{X_p}(E) \not\subset Z_p$ for all $i \le p < j$. It follows that $X_i \dashrightarrow X_j$ is isomorphic at the generic point η_E of $c_{X_i}(E)$ and $\alpha_{il} = \mathrm{mld}_{\eta_E}(X_i, \Delta_i)$. This number belongs to a finite set given by the fixed pair (X_i, Δ_i). Hence the non-increasing sequence $\{\alpha_{il}\}_{l \ge i}$ stabilises at the infimum α_i.

Step 2 Take the truncation as in Step 1. For infinitely many i, Z_i has a scheme-theoretic point η_i with $\mathrm{mld}_{\eta_i}(X_i, \Delta_i) = \mathrm{mld}_{Z_i}(X_i, \Delta_i) = a$. Let d be the maximum number such that for infinitely many i, there exists $\eta_i \in Z_i$ with $\mathrm{mld}_{\eta_i}(X_i, \Delta_i) = a$ and $d = \dim \overline{\{\eta_i\}}$. We take the Zariski closure W_i in X_i of the subset of scheme-theoretic points

$$\{\eta \in X_i \mid \mathrm{mld}_\eta(X_i, \Delta_i) = a,\ d = \dim \overline{\{\eta\}}\}.$$

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

Then $W_{i+1} = \overline{W_{i+1} \setminus Z_i^+}$ because every divisor E over X_{i+1} with $c_{X_{i+1}}(E) \subset Z_i^+$ has $a \le a_i \le a_E(X_i, \Delta_i) < a_E(X_{i+1}, \Delta_{i+1})$. Thus f_i induces a birational map $W_i' \dashrightarrow W_{i+1}$ from the closure W_i' of $f_i^{-1}(W_{i+1} \setminus Z_i^+) \subset W_i$. By noetherian induction, every f_i induces a birational map $W_i \dashrightarrow W_{i+1}$ after truncation.

Step 3 We use the lower semi-continuity in this step. By the definition of W_i, closed points z in W_i with $\mathrm{mld}_z(X_i, \Delta_i) = a + d$ form a Zariski dense subset of W_i. By the lower semi-continuity, every closed point z in W_i has $\mathrm{mld}_z(X_i, \Delta_i) \le a + d$.

We shall prove that $W_i \cap Z_i$ is of dimension at most d and $W_{i+1} \cap Z_i^+$ is of dimension less than d. Let C be an irreducible component of $W_i \cap Z_i$ with generic point η. Then for a general point z in C,

$$a \le \mathrm{mld}_\eta(X_i, \Delta_i) = \mathrm{mld}_z(X_i, \Delta_i) - \dim C \le a + d - \dim C,$$

by which $\dim C \le d$. In like manner for an irreducible component C^+ of $W_{i+1} \cap Z_i^+$ with generic point η^+,

$$a \le \mathrm{mld}_{Z_i}(X_i, \Delta_i) < \mathrm{mld}_{\eta^+}(X_{i+1}, \Delta_{i+1}) \le a + d - \dim C^+$$

where the middle inequality follows as in Lemma 1.4.12. Thus $\dim C^+ < d$.

Therefore $W_i \dashrightarrow W_{i+1}$ is isomorphic outside loci in W_i and W_{i+1} of dimension at most d and it generates no new d-cycles on W_{i+1}. Further by the definition of d, it really contracts a d-cycle for infinitely many i. One can find an irreducible component of W_0 such that for the normalisation V_i of its strict transform in X_i, the induced map $V_i \dashrightarrow V_{i+1}$ has the same properties as $W_i \dashrightarrow W_{i+1}$ has, namely the properties mentioned above. Write $V_i \to U_i \leftarrow V_{i+1}$ with the normalisation U_i of the same image in Y_i of V_i and V_{i+1}.

Step 4 This step follows [127]. We consider the *Borel–Moore homology* $H_{2d}^{\mathrm{BM}}(V_i)$ of V_i as an analytic space [141, example 19.1.1]. Let $\Lambda_{2d}(V_i)$ be the subgroup of $H_{2d}^{\mathrm{BM}}(V_i)$ generated by algebraic d-cycles on V_i. We also define $\Lambda_{2d}(U_i) \subset H_{2d}^{\mathrm{BM}}(U_i)$ in the same manner.

Take the induced map

$$\Lambda_{2d}(V_i) \twoheadrightarrow \Lambda_{2d}(U_i) \simeq \Lambda_{2d}(V_{i+1}),$$

where the isomorphism $\Lambda_{2d}(V_{i+1}) \simeq \Lambda_{2d}(U_i)$ follows from that of $V_{i+1} \to U_i$ outside a locus in V_{i+1} of dimension less than d. By the projectivity of V_i, whenever $V_i \to U_i$ contracts a d-cycle, which occurs for infinitely many i, this cycle generates a non-trivial sublattice $\mathbf{Z}$ in the kernel of $\Lambda_{2d}(V_i) \twoheadrightarrow \Lambda_{2d}(U_i)$. Consequently the rank of $\Lambda_{2d}(V_i)$ would drop infinitely many times. This is a contradiction. □

The minimal log discrepancy can be described in terms of the arc space. We quickly explain this when the variety is smooth.

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

For any natural number i, the functor on the category of algebraic schemes over an algebraically closed field k which sends X to $X \times \operatorname{Spec} k[t]/(t^{i+1})$ has the right adjoint functor J_i. A closed point in J_iX corresponds to a morphism $\operatorname{Spec} k[t]/(t^{i+1}) \to X$. There exists a natural morphism $J_{i+1}X \to J_iX$. Note that $J_0X = X$. We define the *arc space* of X as the inverse limit $J_\infty X = \varprojlim_i J_iX$. It is a noetherian scheme but not of finite type over $\operatorname{Spec} k$ in general. A closed point in $J_\infty X$ corresponds to a morphism $\gamma\colon \operatorname{Spec} k[[t]] \to X$ from the spectrum of the ring of formal power series. For a function f in $\mathcal{O}_X$, the order function $J_\infty X \to \mathbf{N} \cup \{\infty\}$ is defined by sending γ to the order of $\gamma^* f$ in the discrete valuation ring $k[[t]]$.

Let X be a smooth variety of dimension n. Then $J_{i+1}X \to J_iX$ is a vector bundle of rank n. A subset S of $J_\infty X$ is called a *cylinder* if it is the inverse image $\pi_i^{-1}(S_i)$ by $\pi_i\colon J_\infty X \to J_iX$ of a constructible subset S_i of J_iX for some i, for which S_i equals $\pi_i(S)$. The *dimension* of S is defined as

$$\dim S = \dim \pi_i(S) - (i+1)n \in \mathbf{Z}$$

for any such i. An effective $\mathbf{R}$-divisor Δ on X defines the order function $\operatorname{ord}_\Delta\colon J_\infty X \to \mathbf{R}_{\ge 0} \cup \{\infty\}$, and the inverse image $(\operatorname{ord}_\Delta)^{-1}(r)$ is a cylinder in $J_\infty X$ for all $r \in \mathbf{R}_{\ge 0}$.

Theorem 1.6.7 (Ein–Mustaţă–Yasuda [117]) *Let (X, Δ) be a klt pair such that X is smooth and let W be a closed subset of X. Then for a real number a,* $\operatorname{mld}_W(X, \Delta) < a$ *if and only if there exists a non-negative real number r such that the cylinder* $(\operatorname{ord}_\Delta)^{-1}(r) \cap \pi^{-1}(W)$ *in $J_\infty X$ is of dimension greater than $-r - a$, where π denotes the projection $J_\infty X \to X$.*

This was used to prove the lower semi-continuity of minimal log discrepancies and the *precise inversion of adjunction*, a more precise version of Theorem 1.4.26, on local complete intersection (lci) varieties.

Conjecture 1.6.8 (Precise inversion of adjunction) *Let (X, Δ) be a pair such that $S = \lfloor \Delta \rfloor$ is reduced and normal. Let Δ_S denote the different on S of (X, Δ). Then* $\operatorname{mld}_x(X, \Delta) = \operatorname{mld}_x(S, \Delta_S)$ *for any $x \in S$.*

Theorem 1.6.9 *Conjecture* 1.6.8 *holds in the following cases.*

(i) ([48]) *(X, B) is klt for some B, S is* $\mathbf{Q}$*-Cartier and* $\operatorname{mld}_x(X, \Delta) \le 1$.
(ii) ([116], [117]) *X and S are lci.*

Whilst the ACC for minimal log discrepancies remains open, that for log canonical thresholds has been established completely.

Definition 1.6.10 Let (X, Δ) be an lc pair and let A be a non-zero effective $\mathbf{R}$-Cartier $\mathbf{R}$-divisor on X. The *log canonical threshold* of A on (X, Δ) is the greatest real number t such that $(X, \Delta + tA)$ is lc.

Theorem 1.6.11 (Hacon–McKernan–Xu [174]) *Fix a positive integer n and a subset I of the positive real numbers which satisfies the DCC. Then the set of log canonical thresholds of A on (X, Δ) where X is of dimension n and the coefficients in Δ and A belong to I satisfies the ACC.*

The study of $\mathrm{mld}_x(X, \Delta)$ of an lc pair pertains to the analysis of a divisor E over X which *computes* $\mathrm{mld}_x(X, \Delta)$ in the sense that $a_E(X, \Delta) = \mathrm{mld}_x(X, \Delta)$ with $c_X(E) = x$. The perspective of the above theorem tempts us to want A such that $a_E(X, \Delta + A) = \mathrm{mld}_x(X, \Delta + A) = 0$, but it is not always possible.

Example 1.6.12 ([229, example 2]) Consider $o \in X = \mathbf{A}^2$ with coordinates x_1, x_2. Let C be the cuspidal curve in X defined by $x_1^2 + x_2^3$ and take the pair (X, Δ) with $\Delta = (2/3)C$. Let X_1 be the blow-up of X at o and let E_1 be the exceptional curve. For $i = 2$ and 3, let X_i be the blow-up of X_{i-1} at the point at which the strict transform of C meets E_{i-1} and let E_i be the exceptional curve in X_i. Then X_3 is a log resolution of (X, Δ) and $a_{E_1}(X, \Delta) = 2/3$, $a_{E_2}(X, \Delta) = 1, a_{E_3}(X, \Delta) = 1$. Thus E_1 is a unique divisor over X that computes $\mathrm{mld}_o(X, \Delta) = 2/3$. If an effective $\mathbf{R}$-divisor A satisfies $a_{E_1}(X, \Delta + A) = 0$, then $\mathrm{ord}_{E_1} A = a_{E_1}(X, \Delta) = 2/3$. Since $\mathrm{ord}_{E_3} A \geq \mathrm{ord}_{E_1} A \cdot \mathrm{ord}_{E_3} E_1 = (2/3) \cdot 2 = 4/3$, one has $\mathrm{mld}_o(X, \Delta + A) \leq a_{E_3}(X, \Delta + A) \leq 1 - 4/3 < 0$.

1.7 Abundance

Recall that a *log minimal model* is a $\mathbf{Q}$-factorial lc pair $(X/S, \Delta)$ projective over a variety such that $K_X + \Delta$ is relatively nef. The *log abundance* is the following generalisation of Conjecture 1.3.28. It is known up to dimension three. We remark that the boundary Δ may be assumed to be a $\mathbf{Q}$-divisor by the rationality of Shokurov's polytope in Remark 1.5.9.

Conjecture 1.7.1 (Log abundance) *If $(X/S, \Delta)$ is a log minimal model, then $K_X + \Delta$ is relatively semi-ample.*

Theorem 1.7.2 (Keel–Matsuki–McKernan [250]) *The log abundance in Conjecture* 1.7.1 *holds in dimension three.*

We shall explain the reduction of the abundance to the equality of Kodaira dimension κ and numerical Kodaira dimension ν.

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

Definition 1.7.3 Let X be a normal complete variety and let D be a **Q**-Cartier **Q**-divisor on X. The *Iitaka dimension* $\kappa(X, D)$ of D is defined as follows. We define $\kappa(X, D) = -\infty$ if $H^0(\mathscr{O}_X(\lfloor lD \rfloor)) = 0$ for all positive integers l. Otherwise we take the rational map $\varphi_l \colon X \dashrightarrow \mathbf{P}H^0(\mathscr{O}_X(\lfloor lD \rfloor))$ for each $l \in \mathbf{Z}_{>0}$ such that $H^0(\mathscr{O}_X(\lfloor lD \rfloor)) \neq 0$. There exists a contraction $\varphi' \colon X' \to Y'$ with a birational contraction $\mu \colon X' \to X$ such that for any sufficiently large and divisible l, the rational map $X \dashrightarrow \varphi_l(X)$ to the image of φ_l is birational to φ'. Namely, $\varphi_l \circ \mu = f_l \circ \varphi'$ for a birational map $f_l \colon Y' \dashrightarrow \varphi_l(X)$. We define $\kappa(X, D)$ to be the dimension of Y'. Such φ' is called the *Iitaka fibration* associated with D. Note that $\kappa(F', \mu^*D|_{F'}) = 0$ for the fibre F' of φ' at a *very* general point in Y'.

If X is terminal, then we write $\kappa(X)$ for $\kappa(X, K_X)$ and call $\kappa(X)$ the *Kodaira dimension* of X. The Kodaira dimension is a birational invariant because every proper birational morphism $\mu \colon X' \to X$ of terminal varieties satisfies $\mu_*\mathscr{O}_{X'}(lK_{X'}) = \mathscr{O}_X(lK_X)$ for all $l \geq 0$. The variety X is said to be of *general type* if the Kodaira dimension $\kappa(X)$ equals the dimension of X.

Standard references for Iitaka fibrations are [196, chapter 10], [334] and [458]. The Iitaka dimension $\kappa(X, D)$ is minus infinity or a natural number up to the dimension of X. It attains the dimension of X if and only if D is big. One has $\kappa(X, D) = \kappa(X, qD)$ for any positive rational number q.

Theorem 1.7.4 (Easy addition) *Let $X \to Y$ be a contraction between normal projective varieties and let F be the general fibre. Let D be a Cartier divisor on X. Then $\kappa(X, D) \leq \kappa(F, D|_F) + \dim Y$.*

Proof We write $\kappa = \kappa(X, D)$. The inequality is trivial if $\kappa = -\infty$. If $\kappa \geq 0$, then $\kappa(F, D|_F) \geq 0$. In particular, the inequality is also evident when $\kappa = 0$. We shall assume that $\kappa \geq 1$. Take a positive integer l such that $\varphi_l \colon X \dashrightarrow \varphi_l(X) \subset P = \mathbf{P}H^0(\mathscr{O}_X(lD))$ is birational to the Iitaka fibration associated with D.

Let Z be the image of the rational map $X \dashrightarrow P \times Y$ defined by φ_l and $X \to Y$. The first projection $Z \to \varphi_l(X)$ is surjective, by which the dimension κ of $\varphi_l(X)$ satisfies

$$\kappa \leq \dim Z = \dim Z_y + \dim Y$$

for the general fibre $Z_y = Z \times_Y y$ at the image y in Y of F. Let V be the image of the restriction map $H^0(\mathscr{O}_X(lD)) \to H^0(\mathscr{O}_F(lD|_F))$. Then $Z_y \subset \mathbf{P}V \subset P$, and for the rational map $\varphi_{Fl} \colon F \dashrightarrow P_F = \mathbf{P}H^0(\mathscr{O}_F(lD|_F))$ given by $lD|_F$, the projection $P_F \dashrightarrow \mathbf{P}V$ induces a dominant map $\varphi_{Fl}(F) \dashrightarrow Z_y$. Hence

$$\dim Z_y \leq \dim \varphi_{Fl}(F) \leq \kappa(F, D|_F).$$

These two inequalities are combined into the one in the theorem. □

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

Iitaka originally defined $\kappa(X,D)$ as the order of growth of $h^0(\mathscr{O}_X(\lfloor lD \rfloor))$ as below. The proof is found in [196, theorem 10.2] and [288, corollary 2.1.38].

Theorem 1.7.5 *Let D be a Cartier divisor on a normal complete variety X and let $\kappa = \kappa(X,D)$. Then there exist a positive integer l_0 and positive real numbers a and b such that $al^\kappa \le h^0(\mathscr{O}_X(ll_0D)) \le bl^\kappa$ for all $l \in \mathbf{Z}_{>0}$.*

Definition 1.7.6 Let X be a normal complete variety and let D be a nef $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on X. The *numerical Iitaka dimension* $\nu(X,D)$ of D is the maximal number ν such that $D^\nu \not\equiv 0$, that is, $(D^\nu \cdot Z) \neq 0$ for some ν-cycle Z. We define $\nu(X,D) = 0$ if $D \equiv 0$. If X is terminal and K_X is nef, then we write $\nu(X)$ for $\nu(X,K_X)$ and call $\nu(X)$ the *numerical Kodaira dimension* of X.

The numerical Iitaka dimension $\nu(X,D)$ is a natural number up to the dimension of X. The intersection number (D^n) of the nef $\mathbf{Q}$-divisor D, where n is the dimension of X, is the *volume* of D.

Definition 1.7.7 Let X be a normal complete variety of dimension n. The *volume* of a Cartier divisor D on X is the non-negative real number

$$\operatorname{vol}(D) = \limsup_{l\to\infty} \frac{h^0(\mathscr{O}_X(lD))}{l^n/n!}.$$

The volume of a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor D is defined as $a^{-n}\operatorname{vol}(aD)$ by a positive integer a such that aD is integral and Cartier. By Theorem 1.7.5, D has positive volume if and only if D is big. The volume $\operatorname{vol}(D)$ is actually defined by the numerical equivalence class of D and it extends to a continuous function $\operatorname{vol}\colon N^1(X) \to \mathbf{R}_{\ge 0}$. Refer to [288, subsection 2.2.C] for the proof.

Example 1.7.8 Cutkosky and Srinivas [99, example 4] constructed an explicit example of a divisor with irrational volume. Let $S = C \times C$ be the product of an elliptic curve C. Let Δ be the diagonal on S and let $A = o \times C$ and $B = C \times o$ by a point o in C. Take the $\mathbf{P}^1$-bundle $\pi\colon X = \mathbf{P}(\mathscr{O}_S(-A-B-\Delta) \oplus \mathscr{O}_S) \to S$. The projection $\mathscr{O}_S(-A-B-\Delta) \oplus \mathscr{O}_S \twoheadrightarrow \mathscr{O}_S(-A-B-\Delta)$ gives a section S in X with $\mathscr{O}_X(S) \simeq \mathscr{O}_X(1)$. They computed the volume of the divisor $D = 2S + \pi^*(A + 2B + 3\Delta)$ as $\operatorname{vol}(D) = 36 + 4/\sqrt{3}$.

Lemma 1.7.9 *Let X be a normal complete variety of dimension n and let D be a nef $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on X. Then $\kappa(X,D) \le \nu(X,D)$. Moreover, $\kappa(X,D) = n$ if and only if $\nu(X,D) = n$, and* $\operatorname{vol}(D) = (D^n)$ *in this case.*

Proof We write $\kappa = \kappa(X,D)$ and $\nu = \nu(X,D)$. By Chow's lemma and Hironaka's resolution, we have a resolution $\mu\colon X' \to X$ such that X' is projective. Replacing X by X' and D by $l\mu^*D$ for suitable l, we may and shall

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press

assume that X is a smooth projective variety and D is a Cartier divisor such that $\varphi\colon X \dashrightarrow \varphi(X) \subset P = \mathbf{P}H^0(\mathscr{O}_X(D))$ is birational to the Iitaka fibration associated with D. Resolving φ, we may assume that it is a morphism $X \to P$.

One can write $D = \varphi^*H + E$ with a hyperplane H in P and an effective divisor E on X. Take an ample divisor A on X. Since D is nef, one has

$$0 < (\varphi^*H^\kappa \cdot A^{n-\kappa}) \le (\varphi^*H^{\kappa-1} \cdot DA^{n-\kappa}) \le \cdots \le (D^\kappa A^{n-\kappa}),$$

which implies that $D^\kappa \not\equiv 0$. Thus $\kappa \le \nu$.

Suppose that $\nu = n$. We take as A the general hyperplane section of X such that $A - K_X$ is ample. By Kleiman's criterion, $lD + A - K_X$ is ample for any $l \ge 0$ and hence $H^i(\mathscr{O}_X(lD + A)) = 0$ for all $i \ge 1$ by Kodaira vanishing. Thus $h^0(\mathscr{O}_X(lD + A)) = \chi(\mathscr{O}_X(lD + A))$, which is the sum of $\chi(\mathscr{O}_X(lD))$ and $\chi(\mathscr{O}_A((lD+A)|_A))$. The first summand $\chi(\mathscr{O}_X(lD))$ is expressed as $cl^n + O(l^{n-1})$ with $c = (D^n)/n! > 0$ by the asymptotic Riemann–Roch theorem. The second is $O(l^{n-1})$ by an application of Grothendieck's dévissage. Hence $h^0(\mathscr{O}_X(lD + A)) = cl^n + O(l^{n-1})$. With $h^0(\mathscr{O}_A((lD + A)|_A)) = O(l^{n-1})$, the exact sequence

$$0 \to H^0(\mathscr{O}_X(lD)) \to H^0(\mathscr{O}_X(lD + A)) \to H^0(\mathscr{O}_A((lD + A)|_A))$$

yields $h^0(\mathscr{O}_X(lD)) = cl^n + O(l^{n-1})$. Then $\kappa = n$ by Theorem 1.7.5, and $\mathrm{vol}(D) = n! \cdot c = (D^n)$. $\square$

The inequality $\kappa(X, D) \le \nu(X, D)$ is not an equality in general. For instance, the divisor $D \equiv 0$ on a curve C of positive genus in Example 1.2.20 has $\kappa(C, D) = -\infty$ but $\nu(C, D) = 0$. On the other hand if D is semi-ample, then $\kappa(X, D) = \nu(X, D)$ since a multiple of D is the pull-back of an ample divisor. Thus the abundance includes the following conjecture.

Definition 1.7.10 Let X be a normal complete variety. We say that a nef $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor D on X is *abundant* if $\kappa(X, D) = \nu(X, D)$.

Conjecture 1.7.11 *Let (X, Δ) be a log minimal model such that Δ is a $\mathbf{Q}$-divisor. Then $K_X + \Delta$ is abundant.*

Kawamata [234] derived the abundance from the corresponding statement of this conjecture. This was generalised logarithmically and relatively by Nakayama [367, theorem 5] and reproved in [130].

Theorem 1.7.12 (Kawamata) *Let $(X/S, \Delta)$ be a klt pair such that $X \to S$ has connected fibres and Δ is a $\mathbf{Q}$-divisor. If $K_X + \Delta$ is relatively nef and the restriction $(K_X+\Delta)|_F$ to the general fibre F of $X \to S$ is abundant, then $K_X+\Delta$ is relatively semi-ample.*

The base-point free theorem provides the log abundance for a klt log minimal model $(X/S,\Delta)$ such that $K_X+\Delta$ is relatively big. In view of Lemma 1.7.9, this amounts to the case when the numerical Iitaka dimension ν is maximal. We also have the log abundance at the opposite extreme when ν is zero as stated below, proved by Nakayama [369, V theorem 4.8] for klt pairs and extended to lc pairs in [65], [144], [247]. This is formulated without the minimality of the pair.

Theorem 1.7.13 *Let (X,Δ) be an lc projective pair such that Δ is a $\mathbf{Q}$-divisor. Suppose that $K_X+\Delta$ is pseudo-effective and for any ample Cartier divisor A on X, there exists a constant c such that $h^0(\mathscr{O}_X(\lfloor l(K_X+\Delta)\rfloor+A))\le c$ for all positive l. Then $\kappa(X,K_X+\Delta)=0$. In particular if (X,Δ) is a log minimal model with $\nu(X,K_X+\Delta)=0$, then $\kappa(X,K_X+\Delta)=0$ and $K_X+\Delta\sim_{\mathbf{Q}}0$.*

Definition 1.7.14 Let $(X/S,\Delta)$ be an lc pair projective over a variety such that $K_X+\Delta$ is relatively pseudo-effective. We say that a log minimal model $(Y/S,\Gamma)$ of $(X/S,\Delta)$ is *good* if $K_Y+\Gamma$ is relatively semi-ample. If $(X/S,\Delta)$ has a good log minimal model, then by Corollary 1.5.7, every log minimal model of $(X/S,\Delta)$ is good.

We shall explain Lai's inductive approach to the abundance in the case when the Iitaka dimension is non-negative. Let (X,Δ) be a klt projective pair such that Δ is a $\mathbf{Q}$-divisor. Suppose that $\kappa(X,K_X+\Delta)\ge 0$. Take a log resolution $\mu\colon X'\to X$ which admits the Iitaka fibration $\varphi'\colon X'\to Y'$ associated with $K_X+\Delta$. One has a klt pair (X',Δ') with a $\mathbf{Q}$-divisor Δ' such that $\mu_*\Delta'=\Delta$ and such that μ is $(K_{X'}+\Delta')$-negative. Namely, $K_{X'}+\Delta'=\mu^*(K_X+\Delta)+P$ with an effective $\mathbf{Q}$-divisor P the support of which equals the μ-exceptional locus. Then by Proposition 1.5.6, every log minimal model of (X,Δ) is a log minimal model of (X',Δ') and vice versa. By the following theorem, (X,Δ) has a good log minimal model if so does the restriction $(F,\Delta'|_F)$ to the very general fibre of φ', where $\Delta'|_F$ can be defined since F meets the singular locus of X' in codimension at least two in F.

Theorem 1.7.15 (Lai [282]) *Let (X,Δ) be a klt projective pair such that Δ is a $\mathbf{Q}$-divisor. Suppose that $\kappa(X,K_X+\Delta)\ge 0$ and X admits the Iitaka fibration $\varphi\colon X\to Y$ associated with $K_X+\Delta$. Let F be the very general fibre of φ. If the klt pair $(F,\Delta|_F)$ has a good log minimal model, then so does (X,Δ).*

Thus by induction on dimension, the abundance for a klt projective pair (X,Δ) with $K_X+\Delta$ nef is reduced to the two statements which are

- the *non-vanishing* $\kappa(X,K_X+\Delta)\ge 0$ and
- the implication from $\nu(X,K_X+\Delta)\ge 1$ to $\kappa(X,K_X+\Delta)\ge 1$.

The proof of Theorem 1.7.15 uses the next application [282, proposition 2.7] of the work of Birkar, Cascini, Hacon and McKernan.

Proposition 1.7.16 *Let $(X/S, \Delta)$ be a $\mathbf{Q}$-factorial klt pair with a contraction $X \to S$. Suppose that the restriction $(F, \Delta|_F)$ to the very general fibre F of $X \to S$ has a good log minimal model. Then after finitely many steps in the $(K_X + \Delta)/S$-MMP with scaling, one attains a $\mathbf{Q}$-factorial klt pair $(X'/S, \Delta')$ such that the restriction $(F', \Delta'|_{F'})$ to the very general fibre F' of $X' \to S$ has semi-ample $K_{F'} + \Delta'|_{F'}$.*

Proof of Theorem 1.7.15 The Iitaka dimension $\kappa = \kappa(X, K_X + \Delta)$ equals the dimension of Y. The assertion is trivial if $\kappa = 0$ in which $F = X$. If κ equals the dimension of X, then the theorem follows from Corollary 1.5.14. We shall assume that $1 \le \kappa < \dim X$. By $\mathbf{Q}$-factorialisation in Proposition 1.5.17, X may be assumed to be $\mathbf{Q}$-factorial.

Step 1 There exists a positive integer l such that $l(K_X + \Delta)$ is integral and such that $l(K_X + \Delta) = \varphi^* A + G$ with a hyperplane section A of Y and the fix part G of $l(K_X + \Delta)$. Applying Proposition 1.7.16 to φ and replacing X, we may assume that $K_F + \Delta|_F$ is semi-ample. Then $G|_F \sim l(K_F + \Delta|_F) \sim_{\mathbf{Q}} 0$ by $\kappa(F, K_F + \Delta|_F) = 0$. Hence $G|_F = 0$. In other words, G does not dominate Y. It suffices to derive the equality $G = 0$.

The reduction above is made by running the $(K_X + \Delta)/Y \equiv_Y l^{-1}G$-MMP with scaling of some big H. After further replacement of X, we may assume that this MMP no longer contracts any divisors on X. Then for any $\varepsilon \in \mathbf{Q}_{>0}$, a multiple of $G + \varepsilon H$ is mobile over Y. Indeed thanks to Corollary 1.5.13, the $(K_X + \Delta)/Y$-MMP with scaling of H produces a log minimal model $(X'/Y, \Delta' + l^{-1}\varepsilon H')$ of $(X/Y, \Delta + l^{-1}\varepsilon H)$. By the base-point free theorem, the big $\mathbf{Q}$-divisor $K_{X'} + \Delta' + l^{-1}\varepsilon H'$ is semi-ample over Y. This implies the relative mobility of a multiple of $K_X + \Delta + l^{-1}\varepsilon H \sim_{\mathbf{Q},Y} l^{-1}(G + \varepsilon H)$ because the rational map $X \dashrightarrow X'$ is small by assumption.

Step 2 Let E be a prime component of G. If the image $P = \varphi(E)$ were of codimenison at least two in Y, then as in the proof of Theorem 1.3.9, we would construct a birational contraction $\varphi|_S \colon S \to T$ from a normal surface by taking the base change to the intersection T of general hyperplane sections of Y and cutting it with general hyperplane sections of X. Then $G|_S$ is a non-zero exceptional divisor on S/T. By the negativity lemma, $G|_S$ is not $\varphi|_S$-nef and hence contains a curve C with $(G \cdot C) < 0$. Curves C realised in this manner cover some divisor E' on X with $\varphi(E') = P$. They have $((G + \varepsilon H) \cdot C) < 0$ for small positive ε, contradicting the relative mobility in Step 1.

Thus $P = \varphi(E)$ is a prime divisor on Y. Using Step 1 in like manner, one can also verify that $G|_{\varphi^{-1}(U)}$ is proportional to the pull-back $\varphi^*P|_U$ restricted to some smooth neighbourhood U in Y at the generic point of P.

Step 3 By Theorem 1.5.16, we may choose l so that jG is the fix part of $j\varphi^*A + jG$ for all $j \geq 0$. Then for any $0 \leq i \leq j$,

$$H^0(\mathscr{O}_X(j\varphi^*A)) = H^0(\mathscr{O}_X(j\varphi^*A + iG)) = H^0(\mathscr{O}_Y(jA) \otimes \varphi_*\mathscr{O}_X(iG)).$$

Thus for $0 \leq i < j$, the quotient $\mathscr{Q}_i = \varphi_*\mathscr{O}_X((i+1)G)/\varphi_*\mathscr{O}_X(iG)$ satisfies the inclusion

$$H^0(\mathscr{O}_Y(jA) \otimes \mathscr{Q}_i) \subset H^1(\mathscr{O}_Y(jA) \otimes \varphi_*\mathscr{O}_X(iG)).$$

Given $i \geq 0$, whenever j is sufficiently large, $\mathscr{O}_Y(jA) \otimes \mathscr{Q}_i$ is generated by global sections and $H^1(\mathscr{O}_Y(jA) \otimes \varphi_*\mathscr{O}_X(iG)) = 0$. Hence $\mathscr{Q}_i = 0$ from the above inclusion. It follows that $\mathscr{O}_Y = \varphi_*\mathscr{O}_X(iG)$ for all $i \geq 0$.

If $G \neq 0$ and has a component E, then by Step 2, $P = \varphi(E)$ is a divisor and there exists a smooth open subset $\iota\colon U \hookrightarrow Y$ such that the complement $Y \setminus U$ is of codimension at least two and such that $\varphi^*P|_U \leq iG|_{\varphi^{-1}(U)}$ for some $i \geq 1$. But then

$$\mathscr{O}_Y(P) = \iota_*\mathscr{O}_U(P|_U) \subset \iota_*(\varphi_*\mathscr{O}_X(iG))|_U = \iota_*\mathscr{O}_U = \mathscr{O}_Y,$$

which is absurd. Thus G must be zero. □

The existence of good minimal models includes the *Iitaka conjecture*. Below we chronicle several known results. The case (iv) contains (i) and now also contains (v) by Corollary 1.5.14.

Conjecture 1.7.17 (Iitaka conjecture $C_{n,m}$) *Let $X \to Y$ be a contraction between smooth projective varieties, where X and Y are of dimension n and m respectively, and let F be the very general fibre. Then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.*

Theorem 1.7.18 *Conjecture* 1.7.17 *holds in the following cases.*

(i) ($C_{n,n-1}$, Viehweg [460]) *F is a curve.*
(ii) ($C_{n,1}$, Kawamata [231]) *Y is a curve.*
(iii) (Viehweg [462], [464, corollary IV]) *Y is of general type.*
(iv) (Kawamata [233]) *F has a good minimal model.*
(v) (Kollár [261]) *F is of general type.*

https://doi.org/10.1017/9781108933988.002 Published online by Cambridge University Press