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PARALLEL LINES

BY
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(The sixth Jeffery-Williams Lecture)

CONTENTS

1. **Introduction.** About a hundred years ago, the author of *Through the Looking-Glass* wrote another book called *Euclid and his Modern Rivals.* Were these rivals Lobachevsky and Bolyai, Riemann and Schläfli? No, they were merely the authors of dull school textbooks that would soon be forgotten. The sad truth is that, in 1873, hardly anyone in England knew of the breakthrough that had occurred on the Continent some fifty years before: only Cayley in Cambridge, Clifford in London, and a few students. Even if Cayley or Clifford had visited Oxford and given a lecture there, it is doubtful that he would have succeeded in convincing the conservative Dodgson that Euclid's postulates could be modified to yield two new worlds, surpassing in strangeness the worlds of the two Alice books and yet just as logically consistent as Euclid.

At the very time when Dodgson in England was defending Euclid, Klein in Germany was bringing together the two "opposite" worlds by embedding them in a projective space and specializing a *polarity* or, as von Staudt called it, a

This paper is one of a series of survey papers written at the invitation of the Editors of the Canadian Mathematical Bulletin.

Polarsystem. Klein's reasons for calling one of these geometries hyperbolic and the other elliptic are somewhat obscure. I like to justify the names in terms of their Greek roots, using a simple figure suggested by the title of this talk.

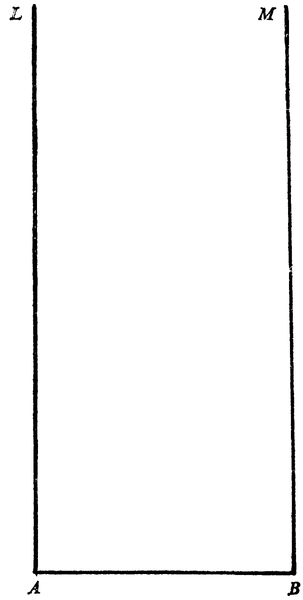

Figure 1

Figure 1 shows two rays in a plane, drawn from points A and B, perpendicular to the line AB on the same side of it. According to the familar notions of Euclidean geometry, these rays are parallel. Measuring the distance from a point L on the first ray to the nearest point M on the second, we find this distance equal to AB. But is this only a first approximation? Can we be sure that the distance will remain the same when L is very far away? Suppose the rays are extended millions of miles: the measurement of LM ceases to be practical. If in fact LM is less than AB, or greater than AB, which would be strange but not really inconceivable, then we are living in a non-Euclidean universe.

If $LM < AB$, so that the rays converge and ultimately intersect, the geometry is *elliptic* (from *elleipein*, to fall short). If $LM > AB$, so that the rays diverge, the geometry is *hyperbolic* (from *hyperballein*, to exceed, or to throw beyond. I have been tempted to speculate that the verb *ballein*, to throw, may be the origin of our word *ball*, or that the two words sprang from a common Aryan root).

2. **The hyperbolic plane.** In the latter case, Euclid's "fifth postulate" is denied. For, if the ray AL, making a right angle with AB, is replaced by AJ, making a very slightly smaller angle with AB (as in Fig. 2), then this new ray

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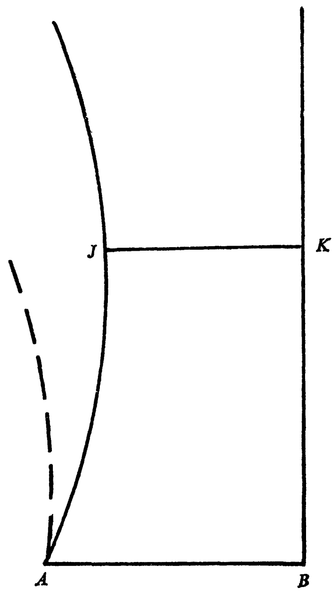

Figure 2

from A and the old one from B will converge at first, attain a minimal distance JK, and then diverge. In the terminology of Eduard Study, these lines AJ and BK are *ultraparallel*, and we have the germ of a proof that *ultraparallel lines have a common perpendicular which measures the shortest distance between them.* (Compare [**1**], p. 133, Fig. 7.2A.)

Somewhat similarly, we may consider what happens to a ray AM (Fig. 3) when M recedes from B so that the distance BM tends to infinity. In the terminology and notation of Lobachevsky, the limiting ray AN is *parallel* to BM, and the angle BAN, which is the limit of BAM, is the *angle of parallelism* $\Pi(p)$ corresponding to the distance $p = AB$. Thus the situation illustrated in Figure 2 will arise if the ray AJ satisfies

$$\Pi(p) < \angle BAJ < 90°.$$

Both Lobachevsky and Bolyai used a horosphere, that is, a sphere of infinite radius, in order to prove that

$$\Pi(p) = 2 \operatorname{arc} \tan e^{-p}$$

when p is measured in terms of a suitable unit of distance. Later on I hope to present an easy proof of this important formula.

3. **A historical digression.** But first I would like to quote a paragraph about Lobachevsky and Bolyai from the non-Euclidean geometry textbook by Stefan Kulczycki ([**4**], pp. 54–55):

It is truly amazing to what extent the trains of thought of the two scholars were related; they were in essence both based on the properties of the horosphere.... They were both aware of the

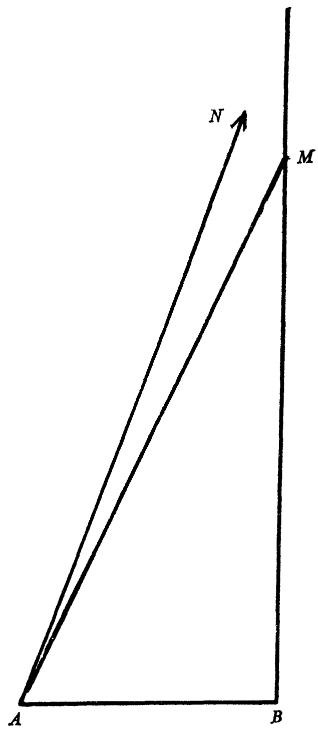

Figure 3

value and importance of their work.... Both met with complete indifference or even, in the case of Lobachevsky, with jeers from people who ... failed to comprehend what it was about. [They] were both bitterly disappointed but reacted ... in different ways. Bolyai, exasperated, closed his mouth and withdrew from scientific activity. Lobachevsky ... in publication after publication ... doggedly justified his non-Euclidean geometry from every point of view.... The extraordinary steadfastness of spirit shown by Lobachevsky during his twenty-five year struggle in utter isolation has very few equals in the history of science.

This account is, perhaps slightly exaggerated; for we know that Gauss sent Lobachevsky a letter of genuine praise and arranged a corresponding membership for him in the Göttingen Academy.

4. **Hyperbolic space.** Believing that astronomical space might be non-Euclidean, Lobachevsky looked for an experimental proof using parallax. Suppose (Fig. 4) M is a star while A and C are two opposite positions of the Earth in its orbit round the Sun, so chosen that the angles at A and C are equal. Since the parallax $90° - \angle BAM$ is greater than $90° - \Pi(AB)$, hyperbolic space would require a positive lower bound for all parallaxes. The fact that no such bound was observed merely means that, if astronomical space is hyperbolic, 93 million miles must be very small in comparison with the absolute unit of hyperbolic distance.

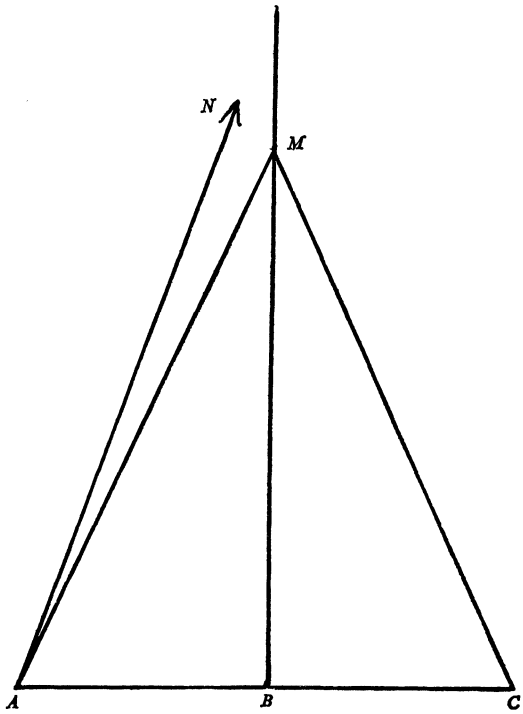

Figure 4

In hyperbolic space, two non-intersecting planes may be either parallel or ultraparallel. Figures 1, 2, 3 may be thought of as normal sections of arrangements of planes. Most of Euclid's Book XI remains valid when the space is hyperbolic. (See [**1**], pp. 180–184.) Through A, and likewise through B, there is a unique plane perpendicular to the line AB; but these two planes (Fig. 1) are ultraparallel, having AB for their shortest distance, and the whole arrangement is symmetrical for rotations about the line AB. The sections of these ultraparallel planes by the plane through AB perpendicular to ABM are two ultraparallel lines: say a through A, and b through B. While M recedes from B (Fig. 3), the plane aM rotates about a, and its limiting position aN is *parallel* to bM. When the plane, still rotating about a, has proceeded beyond this critical stage, it becomes, for a while, ultraparallel to the fixed plane through b.

To sum up, there are three types of plane-pair: *intersecting* (so as to contain a common line), *parallel* (so as to contain parallel lines in a perpendicular plane), and *ultraparallel* (having a common perpendicular line).

5. **The celestial sphere.** These ideas become clearer when we imagine ourselves to be surrounded by a celestial sphere, the kind of sphere on which people long ago believed the stars to be embedded: a sphere so large that we

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can move about and still remain at the centre. Parallel rays converge to a point on the celestial sphere, the kind of ideal point that Hilbert called an *end*. A complete line has thus two ends, infinitely far apart, where it cuts the celestial sphere. This is more intuitively acceptable than what we were taught in the usual introduction to projective geometry, where the two ends are supposed to coincide.

Since a plane intersects the celestial sphere in a circle, we may regard the ∞^3 planes in space as being in one-to-one correspondence with the ∞^3 circles on the sphere. (For this purpose we make no measurements on the sphere itself and thus make no distinction between the so-called great circles and small circles.) Two intersecting planes yield two circles, intersecting at the same pair of supplementary angles. When the smaller angle tends to zero, we obtain two parallel planes yielding two tangent circles. Finally, two ultraparallel planes yield two disjoint circles, that is, two circles having no common point.

Now that we have represented all the planes in the hyperbolic space by all the circles on the sphere, we can overlook the infinite size of the sphere and project it stereographically on an ordinary plane or, more precisely, an *inversive* plane. Such a plane is derived from the Euclidean plane by adding one "point at infinity" and regarding all the straight lines as circles that happen to pass through this "ideal" point.

6. **The inversive plane.** It thus appears that the ∞^3 planes in hyperbolic space correspond to the ∞^3 circles in the inversive plane, and we can develop these two geometries "along parallel lines", with enhanced understanding of both.

According to Klein's *Erlangen programme*, a geometry is determined by the group of transformations under which its essential properties remain invariant. For hyperbolic space this is the group of hyperbolic isometries, generated by reflections in all the planes. For the inversive plane it is the group of circle-preserving transformations, generated by inversions in all the circles. The correspondence that we have been discussing is justified by the fact that these two continuous groups are isomorphic.

The transformation called *inversion* is commonly ascribed to someone called Magnus in 1831. But J. J. Burckhardt recently told me that its true inventor was Jakob Steiner, a few years earlier. Inversion in a circle is usually defined in terms of Euclidean distances from the centre, but a more significant definition is the following. Two points P and P' are inverses of each other in a circle β if every circle through P and P' is orthogonal to β. Given β and P, we merely have to draw two circles orthogonal to β through P, and their remaining point of intersection is P'. This definition is "more significant" because it remains valid in a non-Euclidean plane, or on a sphere such as our celestial sphere. A circle β decomposes the sphere into two parts, such as hemispheres, which are interchanged by inversion in β. The corresponding plane decomposes the hyperbolic space into two half-spaces which are interchanged by reflection in the plane. The isomorphism is now clear.

7. **Pencils of circles and pencils of planes.** Orthogonal to any two circles in the inversive plane, there is a pencil of *coaxal* circles: intersecting or tangent or disjoint according as the first two circles are disjoint or tangent or intersecting. Analogously in hyperbolic space, perpendicular to any two planes there is a pencil of planes: intersecting or parallel or ultraparallel according as the first two planes are ultraparallel or parallel or intersecting. In particular, any three planes perpendicular to one line belong to a pencil of ultraparallel planes, and the corresponding three disjoint circles are coaxal.

If three disjoint circles are not coaxal, their three radical axes concur in a point, called their *radical centre*, from which tangents drawn to the three circles all have the same length (or possibly the radical axes are parallel, and the radical centre is at infinity). Hence there is a unique circle (or line) orthogonal to the three disjoint circles. This familiar theorem of Euclidean geometry yields the following interesting property of planes. *If three planes are ultraparallel in pairs, their three common perpendicular lines, if distinct, are coplanar:* there is a unique plane perpendicular to the three given planes.

8. **Liebmann and Poincaré.** This representation of hyperbolic space on the inversive plane was originally developed by Heinrich Liebmann in the first edition of his book ([**5**], p. 54). Strangely, he abandoned it in later editions. This may have been because, although a plane is represented by a circle and a line by a point pair, there is no very simple image for a single point.

To investigate the geometry of one hyperbolic plane, he represented this plane by a circle ω and regarded its lines as sections of planes perpendicular to the one plane. The circles representing these planes are, of course, orthogonal to ω. In this way the circles orthogonal to ω represent the lines of the hyperbolic plane. We have thus derived Poincaré's conformal model in a perfectly natural manner.

In particular, the angles between two intersecting lines, being equal to the dihedral angles between two intersecting planes, can be measured as the angles of intersection of the two corresponding circles. Similarly, the distance between two points A and B may be regarded as the distance between two ultraparallel planes, namely the planes through A and B perpendicular to the line l that joins these points.

9. **Hyperbolic distance expressed as inversive distance.** The distance AB thus appears as the so-called *inversive distance* $(\alpha\beta)$ between two disjoint circles α and β. To investigate this kind of distance, we observe that it must

have two essential properties. It must be invariant for the group generated by all inversions, and it must be properly additive, so that if B lies between A and C on the line l,

$$AC = AB + BC.$$

This means that, if three disjoint coaxal α, β, γ are named in order, so that β lies *between* α and γ in the sense that every circle intersecting α and γ intersects β too, then

$$(\alpha\gamma) = (\alpha\beta) + (\beta\gamma).$$

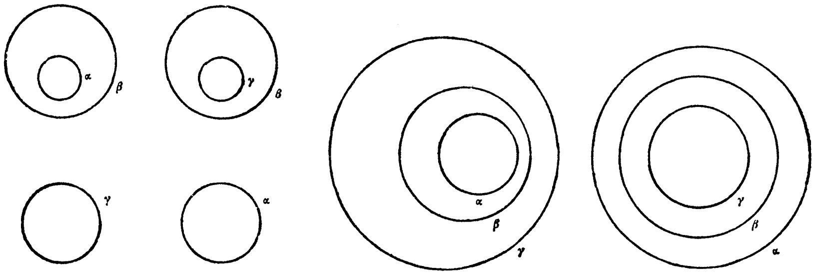

Figure 5

Three such circles may reasonably be called *nested*, through various inversions will make their Euclidean appearance change considerably (see Fig. 5). The coaxal pencil to which they belong has two *limiting points* representing the two *ends* of the line l. Inversion in a circle whose centre is one of these limiting points will yield *concentric* circles whose radii satisfy $a > b > c$ or $a < b < c$ according to which of the two limiting points is used. Different radii of inversion will yield similar figures, but the two ratios a/b and b/a are inversively invariant together, though they may be interchanged. In other words, $|\log a/b|$ is truly invariant. This remark suggests an appropriate definition for the *inversive distance* $p = (\alpha\beta)$ between any two disjoint circles α and β: it is *the absolute magnitude of the natural logarithm of the ratio of the radi of any two concentric circles into which the given circles can be inverted:*

$$p = \left|\log \frac{a}{b}\right|.$$

This kind of distance is properly additive, for, if $a > b > c$ or $a < b < c$,

$$\left|\log \frac{a}{b}\right| + \left|\log \frac{b}{c}\right| = \left|\log \frac{a}{c}\right|.$$

It is an inversive invariant even though it is most easily defined in terms of the Euclidean concept of radius.

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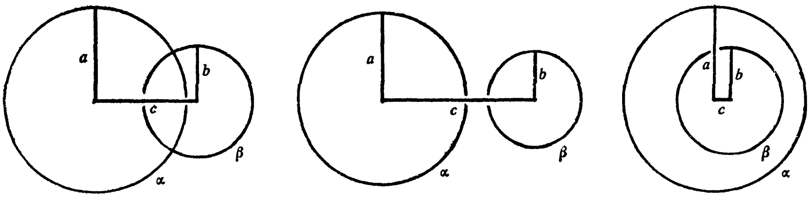

Figure 6

10. **The inversive distance between two Euclidean circles.** Figure 6 shows, in the Euclidean plane, two circles of radii a and b whose centres are distant c apart. (Please forgive my using c in a new sense.) If

$$a<b+c \quad \text{and} \quad b<c+a \quad \text{and} \quad c<a+b,$$

so that the circles intersect, either point of intersection forms, with the two centres, a triangle with sides a, b, c; hence the cosines of the two supplementary angles of intersection are

$$\pm\frac{a^2+b^2-c^2}{2ab}.$$

Since angles are preserved by inversion, it follows that the expression

$$\left|\frac{a^2+b^2-c^2}{2ab}\right|$$

is an inversive invariant. Since inversion is an algebraic transformation of Cartesian coordinates, this invariance is independent of the "triangle inequalities" and thus remains valid when the two circles are disjoint. In particular, the above expression is unchanged when the circles have been inverted into concentric circles. Suppose this inversion changes a, b, c into a', b', 0. Then

$$\left|\frac{a^2+b^2-c^2}{2ab}\right|=\frac{a'^2+b'^2}{2a'b'}=\frac{1}{2}\left(\frac{a'}{b'}+\frac{b'}{a'}\right).$$

If p is the inversive distance between the two circles (before or after inversion), so that a'/b' and b'/a' are e^p and e^{-p} or vice versa, we conclude that

$$\left|\frac{a^2+b^2-c^2}{2ab}\right|=\tfrac{1}{2}(e^p+e^{-p})=\cosh p.$$

This is a useful formula for p in terms of a, b, c.

11. **The inversive distance between a line and a circle.** It is interesting to see what happens when the first circle is replaced by a straight line. (Since a and c are now infinite, the formula cannot be applied directly.) Since the group generated by all inversions includes all similarities, the inversive distance

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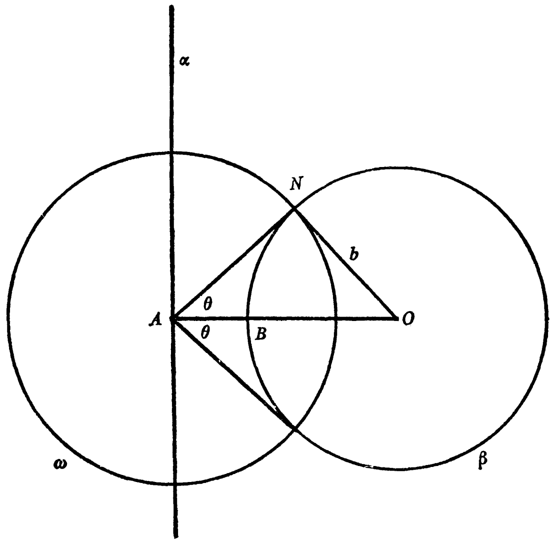

Figure 7

between a line and a circle must be invariant for any similarity: it must be a function of the angle subtended by the circle at the nearest point A on the line. Let this angle be 2θ, as in Figure 7, so that, if O and b are the centre and radius of the circle β,

$$OA = b \operatorname{cosec} \theta.$$

Inversion in β leaves β itself unchanged, but the line becomes a circle having diameter

$$OA' = \frac{b^2}{OA} = b \sin \theta,$$

that is, a circle α for which $a = c = \frac{1}{2}b \sin \theta$. (See Figure 8.) Hence the inversive distance p is given by

$$\cosh p = \left| \frac{a^2 + b^2 - a^2}{2ab} \right| = \frac{b}{2a} = \frac{b}{b \sin \theta} = \operatorname{cosec} \theta.$$

Let us record this formula

$$\cosh p = \operatorname{cosec} \theta$$

for future use.

12. **The angle of parallelism.** Returning to Poincaré's conformal model of the hyperbolic plane by circles orthogonal to a fixed circle ω, we see that the hyperbolic distance AB, which Poincaré expressed as the logarithm of a cross

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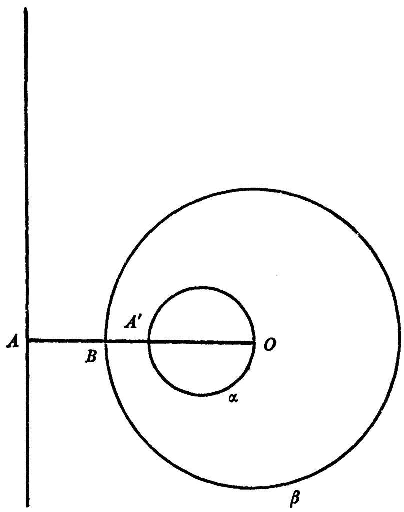

Figure 8

ratio, is now simply the inversive distance between two circles, one through A and one through B, orthogonal to both ω and the circle that represents the line AB. If one of the two circles reduces to a line, this line is, of course, a diameter of ω.

Two parallel lines of the hyperbolic plane, being sections of parallel planes in hyperbolic space, are represented by tangent circles. As they are both orthogonal to ω, their point of contact N (representing the common *end* of the lines) is on ω. Thus Figure 7 represents Figure 3 when we have drawn ω as the circle with centre A and radius AN. The parallel rays are now the radius AN of ω and the arc BN of β. The hyperbolic distance AB is the inversive distance p between the line α (through A) and the circle β (through B). The corresponding angle of parallelism $\Pi(p)$ is the angle $BAN = \theta$. By the formula that we recorded for future use,

$$\operatorname{cosec} \theta = \cosh p.$$

This implies

$$\cot \theta = \sinh p,$$

$$\operatorname{cosec} \theta - \cot \theta = \cosh p - \sinh p,$$

$$\tan \tfrac{1}{2}\theta = e^{-p},$$

$$\Pi(p) = 2 \arctan e^{-p}.$$

We have obtained the classical expression, which is what we set out to do.

13. **The non-triangle inequality for inversive distances.** Six years ago, in my Presidential Address ([**3**], p. 11), I sketched a proof that the inversive distances between pairs of three nested circles (with β between α and γ) satisfy the "non-triangle inequality"

$$(\alpha\gamma) \geq (\alpha\beta) + (\beta\gamma)$$

with equality only when the circles are coaxal. I would like to explain, in conclusion, how the analogous situation in hyperbolic space makes this almost obvious.

In saying that the given disjoint circles are *nested*, we mean that every circle intersecting both α and γ intersects β too. By considering three sections of a sphere, we see that the corresponding property of three planes (ultraparallel in pairs, with β between α and γ) is that every line joining a point of α to a point of γ must intersect β. Since these planes α, β, γ are ultraparallel in pairs, there is at least one plane perpendicular to all three. Their sections by such a plane are three mutually ultraparallel lines, to which we may assign the same labels α, β, γ. Of course it is still true that every line joining a point of α to a point of γ intersects β. This holds, in particular, for the common perpendicular of α and γ. If the three circles are coaxal (Fig. 5), this common perpendicular is perpendicular to β too, and on this line we measure

$$(\alpha\gamma) = (\alpha\beta) + (\beta\gamma).$$

More interestingly, if the three nested circles are not coaxal, the shortest distances between pairs of the lines are alternate sides of a crossed hexagon having a right angle at each vertex, as in Figure 9. Since $(\alpha\beta)$ is the shortest distance from α to β, and $(\beta\gamma)$ from β to γ, it is clear that the segment marked $(\alpha\gamma)$ is the sum of two parts, one greater than $(\alpha\beta)$ and the other greater than $(\beta\gamma)$. Hence

$$(\alpha\gamma) > (\alpha\beta) + (\beta\gamma).$$

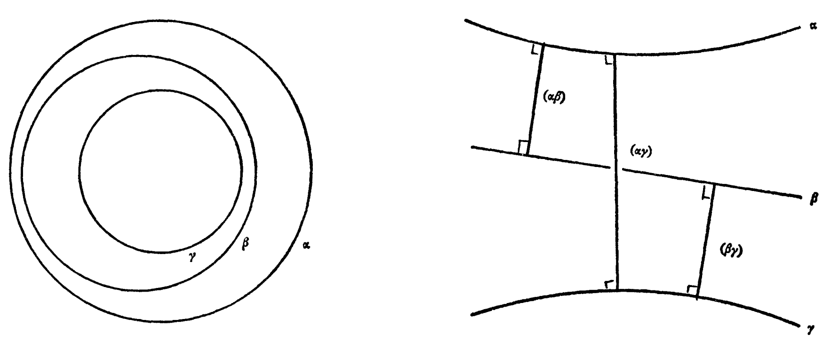

Figure 9

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4. Stefan Kulczycki, *Non-Euclidean Geometry* (translated by S. Knapowski), Pergamon, 1961.

5. Heinrich Liebmann, *Nichteuklidische Geometrie*, Leipzig, 1905.

* I take this opportunity to correct an error on page 240, just after Fig. 5: The words "regular tetrahedron" should be replaced by "*disphenoid*, whose pairs of opposite edges are congruent." The rest of the sentence should be deleted.

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