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THEORETICAL PEARLS
An unsolvable numeral system in lambda calculus

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Abstract

For numeral systems in untyped λ-calculus the definability of a successor, a predecessor and a test for zero implies the definability of all recursive functions on that system. Towards a disproof of the converse statement, H. P. Barendregt and the author constructed a numeral system consisting of unsolvable λ-terms, being adequate for unary functions. Then, independently, B. Intrigila found an analogous system for all computable functions.

1 Notation

We suppose the reader has some basic knowledge about untyped λ-calculus. The set of *lambda terms* is denoted by Λ, and Λ° is the set of *closed* terms. *Syntactical equality* on Λ is denoted by $\equiv$, and (beta) *convertibility* by $=_\beta$ or simply by $=$. The following *standard combinators* are used:

$$\begin{aligned}\mathbf{I} &\equiv \lambda x.x,\\ \mathbf{K} &\equiv \lambda xy.x,\\ \mathbf{B} &\equiv \lambda fgx.f(gx),\\ \mathbf{\Omega} &\equiv (\lambda x.xx)(\lambda x.xx),\\ \mathbf{true} &\equiv \lambda xy.x \quad (\equiv \mathbf{K}),\\ \mathbf{false} &\equiv \lambda xy.y.\end{aligned}$$

2 Numeral systems

Numeral systems are used to represent natural numbers and numeric functions in λ-calculus.

Definition 2.1
A *numeral system* is a sequence of λ-terms

$$\boldsymbol{d} = \boldsymbol{d}_0, \boldsymbol{d}_1, \ldots$$

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such that

(1) each d_n is a closed term;
(2) $\forall m, n \in \mathbb{N}\ [d_m =_\beta d_n \Rightarrow m = n]$.

The best known numerals are the Church numerals, which can be considered as function iterators.

Definition 2.2

(i) Let $F, M \in \Lambda$, and $n \in \mathbb{N}$. Then the *n-fold iteration of F on M* (notation $F^n(M)$) is defined inductively as follows:

$$F^0(M) \equiv M,$$
$$F^{n+1}(M) \equiv F(F^n(M)).$$

(ii) The system of *Church numerals* $\boldsymbol{c} = \boldsymbol{c}_0, \boldsymbol{c}_1, \ldots$ is defined by

$$\boldsymbol{c}_n \equiv \lambda fx . f^n(x).$$

Definition 2.3

Let $\boldsymbol{d}$ be a numeral system.

(i) Let $f: \mathbb{N}^k \to \mathbb{N}$ be a numeric function. Then f is *λ-definable* with respect to $\boldsymbol{d}$ if for some $F \in \Lambda^\circ$

$$\forall n_1, \ldots, n_k \in \mathbb{N}\ F\boldsymbol{d}_{n_1} \ldots \boldsymbol{d}_{n_k} =_\beta \boldsymbol{d}_{f(n_1, \ldots, n_k)}.$$

In that case f is said to be *λ-defined by F*.

(ii) $\boldsymbol{d}$ is *adequate* if all recursive functions are λ-definable with respect to $\boldsymbol{d}$.

Definition 2.4

Let $\boldsymbol{d}$ be a numeral system.

(i) $\boldsymbol{d}$ has a *successor* if there exists a term $S_d^+ \in \Lambda^\circ$ such that for all $n \in \mathbb{N}$

$$S_d^+ \boldsymbol{d}_n = \boldsymbol{d}_{n+1}.$$

(ii) $\boldsymbol{d}$ has a *predecessor* if for some $P_d^- \in \Lambda^\circ$ one has

$$P_d^- \boldsymbol{d}_{n+1} = \boldsymbol{d}_n.$$

(iii) $\boldsymbol{d}$ has a *test for zero* if for some $Zero_d \in \Lambda^\circ$

$$Zero_d \boldsymbol{d}_0 = \mathbf{true},$$
$$Zero_d \boldsymbol{d}_{n+1} = \mathbf{false}.$$

Proposition 2.5

The system of Church numerals **c** *has a successor, a predecessor, and a test for zero.*

Proof

Take

$$S_c^+ \equiv \lambda xfy . f(xfy),$$
$$P_c^- \equiv \lambda xfy . x(\lambda pq . q(pf))(\mathbf{K}y)\mathbf{I},$$
$$Zero_c \equiv \lambda x . x(\lambda y . \mathbf{false})\,\mathbf{true}. \quad \square$$

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Theorem 2.6
*Let **d** be a numeral system. If **d** has a successor, a predecessor, and a test for zero, then **d** is adequate.*

Proof
See Barendregt (1984, Section 6.4). □

Corollary 2.7
*The system of Church numerals **c** is adequate.*

One may wonder if the converse of Theorem 2.6 holds. In particular, if existence of a test for zero is necessary for a numeral system to be adequate.

Question 2.8 (H. P. Barendregt and E. Barendsen, 1989).
Is there an adequate numeral system without a test for zero?

Barendregt and the author partially solved the problem in 1989 by constructing a numeral system consisting of unsolvable terms, which was adequate with respect to unary functions. Recently, Intrigila (1990) described a numeral system without a test for zero which is adequate for all recursive functions. Below we present a proof combining Intrigila's construction and that by Barendregt and the author (which are very similar), after giving the necessary background theory on solvability. Therefore this paper is mainly self-contained.

In view of the paradigm that unsolvable terms internalize the notion 'undefined' or 'meaningless' (see Barendregt, 1984, pp. 40–43), the idea of representing natural numbers by unsolvables seems a little perverse.

3 Solvability

Definition 3.1

(i) Let $M \in \Lambda^{\circ}$. Then M is *solvable* if for some sequence $\vec{N}$

$$M\vec{N} = \mathbf{I}.$$

(ii) A term $M \in \Lambda$ is *solvable* if a closure $\lambda\vec{x}.M$ is solvable.
(iii) M is *unsolvable* if M is not solvable.

Example 3.2
(i) $\mathbf{K}$ is solvable: $\mathbf{KII} = \mathbf{I}$.
(ii) $x\mathbf{\Omega}$ is solvable: $(\lambda x.x\mathbf{\Omega})(\mathbf{KI}) = \mathbf{I}$.
(iii) $\mathbf{\Omega}$ is unsolvable.

Below an equivalent characterization of solvability, using the reduction behaviour of a term, will be given. See Barendregt (1984) for details.

Lemma 3.3
Each $M \in \Lambda$ is either of the form

$$M \equiv \lambda x_1 \ldots x_n . yP_1 \ldots P_m, \quad n \geqslant 0, \quad m \geqslant 0, \tag{1}$$

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or

$$M \equiv \lambda x_1 \ldots x_n . (\lambda y . P_0) P_1 \ldots P_m, \quad n \geqslant 0, \quad m \geqslant 1. \tag{2}$$

Proof

By a straightforward case distinction. □

Definition 3.4

(i) A term M is a *head normal form* (hnf) if M is of the form (1) in Lemma 3.3.

(ii) *M has a* hnf if $M =_\beta N$ with N a hnf.

(iii) If M is of the form (2) in Lemma 3.3 then $(\lambda y . P_0) P_1$ is called the *head redex* of M.

Theorem 3.5 (C. P. Wadsworth)

For all $M \in \Lambda$,

$$M \textit{ is solvable} \Leftrightarrow M \textit{ has a } \text{hnf}.$$

The connection with reduction strategies is established in the following.

Definition 3.6

(i) Suppose M has Δ as head redex. We write

$$M \to_{\text{h}} N$$

if N results from M by contracting Δ. This is called a *one step head reduction.*

(ii) $\twoheadrightarrow_{\text{h}}$ is the reflexive transitive closure of $\to_{\text{h}}$.

(iii) The *head reduction path* of M is the sequence $M_0, M_1, \ldots$ such that

$$M \equiv M_0 \to_{\text{h}} M_1 \to_{\text{h}} M_2 \to_{\text{h}} \ldots.$$

If M_i is a hnf for some i then the head reduction of M is said to *terminate* at M_i. Otherwise M has an *infinite* head reduction.

Theorem 3.7 (C. P. Wadsworth)

M has a hnf *iff the head reduction path of M terminates.*

The following is a 'topological' result, stressing that an unsolvable cannot really be used as a meaningful argument in a computation.

(Genericity) *Lemma 3.8*

Let M, $N \in \Lambda$ with M unsolvable and N having a normal form. Then for all $F \in \Lambda$

$$FM = N \quad \Rightarrow \quad \forall L \in \Lambda \quad FL = N.$$

Proof

See Barendregt (1984, proposition 14.3.24). □

4 An unsolvable numeral system

Definition 4.1

For F, $G \in \Lambda$, define

$$F \circ G \equiv \lambda x . F(Gx).$$

Note that $\mathbf{B}FG = F \circ G$.

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Definition 4.2
The combinator P is defined by

$$P \equiv \mathbf{\Theta}(\lambda f . f \circ f),$$

where $\mathbf{\Theta}$ is Turing's fixedpoint combinator, defined as follows.

$$A \equiv \lambda xy . y(xxy),$$
$$\mathbf{\Theta} \equiv AA.$$

Lemma 4.3
(i) $P \circ P = P$.
(ii) *Px is unsolvable.*

Proof
(i) By definition.
(ii) Note that

$$Px \twoheadrightarrow_{\mathrm{h}} (\lambda f . f \circ f)\, Px \rightarrow_{\mathrm{h}} (P \circ P)\, x \rightarrow_{\mathrm{h}} P(Px),$$

so Px has an infinite head reduction path. Hence Px is unsolvable by theorems 3.7 and 3.5. □

Definition 4.4
The numeral system $\boldsymbol{u} = \boldsymbol{u}_0, \boldsymbol{u}_1, \ldots$ is defined as follows. For each $n \in \mathbb{N}$

$$\boldsymbol{u}_n \equiv \lambda x . P(x\boldsymbol{c}_n).$$

Lemma 4.5
Each $\boldsymbol{u}_n$ is unsolvable.

Proof
By Lemma 4.3 (ii). □

In order to show (in a uniform way) that $\boldsymbol{u}$ is adequate we need a sequence of combinators.

Definition 4.6
For each $n \geqslant 1$, define

$$\mathbf{B}_n \equiv \lambda z x_1 \ldots x_n . z(x_1 \ldots x_n).$$

Note that $\mathbf{B}_2 \equiv \mathbf{B}$.

Theorem 4.7
$\boldsymbol{u}$ is an adequate numeral system.

Proof
Suppose $f: \mathbb{N}^k \rightarrow \mathbb{N}$ is recursive. Let F λ-define f with respect to $\boldsymbol{c}$. Define

$$F^* \equiv \lambda x_1 \ldots x_k z . \mathbf{B}_{2k+1} x_1 \mathbf{B}_{2k} x_2 \ldots \mathbf{B}_{k+2} x_k \mathbf{B}_{k+1} zF.$$

Claim. F^* λ-defines f with respect to $\boldsymbol{u}$.

Proof (example: $k = 3$). Suppose F represents $f: \mathbb{N}^3 \to \mathbb{N}$ w.r.t. $\boldsymbol{c}$. Define

$$F^* \equiv \lambda uvwz . \mathbf{B}_7 u \mathbf{B}_6 v \mathbf{B}_5 w \mathbf{B}_4 z F.$$

Let $p, q, r \in \mathbb{N}$. Then

$$\begin{aligned}
F^* \boldsymbol{u}_p \boldsymbol{u}_q \boldsymbol{u}_r &= \lambda z . \mathbf{B}_7 \boldsymbol{u}_p \mathbf{B}_6 \boldsymbol{u}_q \mathbf{B}_5 \boldsymbol{u}_r \mathbf{B}_4 z F \\
&= \lambda z . \boldsymbol{u}_p (\mathbf{B}_6 \boldsymbol{u}_q \mathbf{B}_5 \boldsymbol{u}_r \mathbf{B}_4 z F) \\
&= \lambda z . P(\mathbf{B}_6 \boldsymbol{u}_q \mathbf{B}_5 \boldsymbol{u}_r \mathbf{B}_4 z F \boldsymbol{c}_p) \\
&= \lambda z . P(\boldsymbol{u}_q (\mathbf{B}_5 \boldsymbol{u}_r \mathbf{B}_4 z F \boldsymbol{c}_p)) \\
&= \lambda z . P(P(\mathbf{B}_5 \boldsymbol{u}_r \mathbf{B}_4 z F \boldsymbol{c}_p \boldsymbol{c}_q)) \\
&= \lambda z . P(P(\boldsymbol{u}_r (\mathbf{B}_4 z F \boldsymbol{c}_p \boldsymbol{c}_q))) \\
&= \lambda z . P(P(P(\mathbf{B}_4 z F \boldsymbol{c}_p \boldsymbol{c}_q \boldsymbol{c}_r))) \\
&= \lambda z . P(P(P(z(F \boldsymbol{c}_p \boldsymbol{c}_q \boldsymbol{c}_r)))) \\
&= \lambda z . P(P(P(z \boldsymbol{c}_{f(p,q,r)}))) \quad \text{since } F \ \lambda\text{-defines } f, \\
&= \lambda z . P(z \boldsymbol{c}_{f(p,q,r)}) \quad \text{by Lemma 4.3 (i)}, \\
&\equiv \boldsymbol{u}_{f(p,q,r)}. \quad \Box
\end{aligned}$$

Now we can answer Question 2.8 affirmatively.

Proposition 4.8
$\boldsymbol{u}$ does not have a test for zero.

Proof.
By the genericity lemma one has for each $Z \in \Lambda^\circ$

$$Z\boldsymbol{u}_0 = \mathbf{true} \quad \Rightarrow \quad \forall n \in \mathbb{N} \quad Z\boldsymbol{u}_n = \mathbf{true}. \quad \Box$$

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