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Relations Among Heegner Cycles on Families of Abelian Surfaces

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Abstract. We compute relations of rational equivalence among special codimension 2 cycles on families of Abelian surfaces using elements of a higher Chow group. These relations are similar to those between Heegner points and special divisors obtained by Zagier, Van der Geer and others.

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1. Introduction

A conjecture of Beilinson and Bloch asserts that if $\mathbf{V}$ is a smooth, geometrically irreducible, projective variety defined over a number field F, then

$$rank\ CH^p_{hom}(\mathbf{V}_F) = ord_{s=p} L_F(H^{2p-1}(\mathbf{V}), s).$$

Here CH^p_{hom} denotes the group of codimension p cycles homologically equivalent to zero modulo rational equivalence on the variety. In particular, this says that the rank of the Chow group is finite. In the case when $p = 1$ the finiteness of the rank is the Mordell–Weil theorem. However very little is known about this conjecture in the case when $p > 1$. The purpose of this paper is to make some progress towards this conjecture in the case of codimension 2 cycles on certain types of varieties.

The varieties in question are compactifications of modular families of Abelian surfaces over modular or Shimura curves. Schoen [Sc1], in the case of modular curves and later Besser [Be], in the case of Shimura curves, constructed certain codimension 2 nullhomologous cycles supported in fibres over complex multiplication points on the modular curve. (In fact they showed that these cycles are in general not algebraically equivalent to 0 so they are actually in the so called Griffiths group). These 'CM cycles' are the analogues of Heegner points on modular curves and are similarly defined over number fields. Further, one can take certain traces of these cycles, analogous to Heegner divisors, to get cycles defined over fixed number fields F and hence they give elements of the groups in question. The purpose of this paper is to construct relations of rational equivalence between these cycles in the hope that one can construct enough relations to prove that the groups are finite dimensional.

Our strategy is to use the localization sequence for higher Chow groups [Bl1]. From that one sees that to construct relations between codimension 2 cycles it is enough to construct elements of the group $CH^2(\mathcal{A}_\eta, 1)$ where $\mathcal{A}_\eta$ is the generic Abelian surface of the family and then compute their boundary. It turns out that there are some very natural elements constructed by Collino [Co] and interestingly, their boundary can be expressed in terms of the CM cycles.

We then have the following:

THEOREM 1.1. *Let* $\mathbf{X}(\mathbf{D_0}, \mathbf{N})$ *be a Shimura curve parametrising Abelian surfaces with endomorphism ring an Eichler order (loc. cit. Section 2) of level N in a division algebra of discriminant* D_0. *Let* $\mathbf{W}(\mathbf{D_0}, \mathbf{N})$ *denote (the non-singular compactification of) the universal Abelian surface over* $\mathbf{X}(\mathbf{D_0}, \mathbf{N})$. *Let* p, a, b *be the invariants which determine Hashimoto's model (ibid.) and let* P *and* Q *denote two* 2*-torsion points in the generic fibre.*

Then there are relations in $CH^2(\mathbf{W}(\mathbf{D_0}, \mathbf{N})) \otimes \mathbb{Q}$ *of the form*

$$\sum_{[\mathcal{L}]} \varepsilon_{P,Q}([\mathcal{L}]) \sum_{r,s} \sum_{d^2|\Delta} d\mathcal{Z}_{\Delta/d^2,[\mathcal{L}],s_1/d,-s_1/d} \equiv 0$$

where $\mathcal{L}$ *runs through all even theta characteristics,* $r, s \in \mathbb{Z}$ *such that*

$$\frac{n^2p - r^2}{4} \in \mathbb{Z}_{\geqslant 0}, \qquad \frac{4bDn^2 - s^2}{4} \in \mathbb{Z}_{\geqslant 0},$$

n *is coprime with* $2k$, r, s, n *mutually coprime,* $s_1 = p(s/2) - aDr)$ *and*

$$\Delta = \frac{ps^2 - 4aDrs + 4bDr^2 - 4Dn^2}{4}$$

with d *running through all* $d^2|\Delta$ *such that* Δ/d^2 *is still a discriminant.*

Here $\mathcal{Z}_\Delta$ is a *Heegner Cycle of discriminant* Δ (loc. cit Section 5), a codimension 2 cycle which is homologous to zero in $CH^2(\mathbf{W}(\mathbf{D_0}, \mathbf{N})) \otimes \mathbb{Q}$ and $\varepsilon_{P,Q}([\mathcal{L}])$ is a sign function which depends on the level and the points P and Q (loc. cit. Section 4).

Relations between Heegner *points* on modular curves have been studied before by many other authors. In Zagier [Za] and Van der Geer [Ge], they construct relations between these points which are very similar to ours, the main difference being that our relations involve the level 2 structure while theirs do not.

As is known [Za], [G-Z], Heegner points are very closely related to coefficients of modular forms of weight $\frac{3}{2}$ and relations of rational equivalence between Heegner points imply relations between coefficients of these forms. Similarly, Heegner cycles are supposed to be related to coefficients of modular forms of weight $\frac{5}{2}$. Some evidence of this can be found in [Sc2], [Zh] and [Ne]. So relations of rational equivalence between these cycles should give rise to relations between coefficients of such modular forms.

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More interestingly, recently Borcherds [Bo] has used his new constructions of automophic forms to construct relations between Heegner points on modular and Shimura curves and more generally, relations between special divisors on modular varieties, and *prove* that these special subvarieties are related to coefficients of modular forms. Starting from certain meromorphic modular forms he constructs other automorphic forms whose divisors are on such special subvarieties. It would be very interesting to see if his methods can be generalized to construct Collino type elements of the higher Chow groups and use them to prove that the Heegner cycles are related to coefficients of modular forms. At the moment it is not clear to me how this can be done, but a weaker statement, namely computation of the regulator of Collino's element may be a little more tractable.

The outline of the paper is as follows: In Section 2 we describe the varieties and cycles in question. In Section 3 we introduce the higher Chow groups and the localization sequence that we use. In Section 4 we describe Collino's construction of the elements of the higher Chow groups. In Section 5 we compute the boundary of the elements and relate them to the CM cycles. Finally, in section 6 we put them all together to get our main result and give some examples.

2. Preliminaries

2.1. EICHLER ORDERS

The varieties we consider are universal families of Abelian surfaces whose endomorphism rings contain an order in a quaternionic division algebra. To describe precisely we use the theorems of Hashimoto [Ha] and Roberts [Ro]. Another standard reference is [Vi].

Let $\mathbf{B}$ be an indefinite quaternion algebra over $\mathbb{Q}$ with discriminant D_0. An *Eichler Order of level* $D = D_0N$ is an order $\mathcal{O}$ in $\mathbf{B}$ such that

$$\mathcal{O} \otimes \mathbb{Z}_\ell \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod \ell \quad \text{for all } \ell | N$$

Let $\mathcal{O}$ be an Eichler order of level $D = D_0N$ where N is a positive integer prime to D_0. Let $\mathcal{S}$ be the set of primes where the division algebra is ramified (i.e. those primes ℓ such that $\mathbf{B} \otimes \mathbb{Q}_\ell \neq M_2(\mathbb{Q}_\ell)$). Choose an auxiliary prime p such that the Hilbert symbol $(-D, p)_\ell = -1$ if and only if $\ell \in \mathcal{S}$. Such a p is guaranteed by Dirichlet's theorem on primes in arithmetic progressions. Let a and b be such that $a^2D + 1 = bp$. Then we have the following theorem described in Hashimoto, [Ha], though parts of it existed in some form in earlier papers.

THEOREM 2.1. *Let* $\mathbf{B}$ *be a quaternionic division algebra of discriminant* D_0. *Then:*

(1) $\mathbf{B} \simeq \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$ *where* $i^2 = -D$, $j^2 = p$ *and* $ij = -ji$.

(2) *The order* $\mathcal{O} \simeq \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$ *where* $e_1 = 1, e_2 = (1+j)/2, e_3 = (aDj + ij)/p$ *and* $e_4 = (i + ij)/2$

(3) *There is a skew-symmetric pairing on* **B** *which is* $\mathbb{Z}$ *valued on* $\mathcal{O}$ *given by* $E(x, y) = tr(x\bar{y}i^{-1})$ *where* $\bar{y}$ *denotes conjugation. Further, the elements* $\eta_1 = e_4 - (p - 1/2), \eta_2 = -aDe_1 - e_3, \eta_3 = e_1$ *and* $\eta_4 = e_2$ *are a symplectic basis for* $\mathcal{O}$

Since **B** is indefinite, any two Eichler orders of level D are conjugate [B-D], so there is no loss of generality in working with this model and we can fix the isomorphism of $\mathbf{B} \otimes \mathbb{R} \simeq M_2(\mathbb{R})$ by defining

$$\Phi_\infty(i) = \begin{pmatrix} 0 & -1 \\ D & 0 \end{pmatrix}, \quad \Phi_\infty(j) = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix}$$

We will call this description of the quaternion algebra *Hashimoto's model*.

Orientations

The above description is not rigid enough. So, following [Ro], [B-D] we rigidify the definition further by defining an *Oriented Eichler Order of level* D_0N as follows. For each prime ℓ dividing D_0 there are two algebra homomorphisms,

$$\mathfrak{o}_\ell : \mathcal{O} \otimes \mathbb{F}_\ell \to \mathbb{F}_\ell$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to a \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to d$$

and similarly for a prime ℓ dividing N there are two distinct algebra homomorphisms

$$\mathfrak{o}_\ell : \mathcal{O} \otimes \mathbb{F}_\ell \to \mathbb{F}_{\ell^2}$$

$$\begin{pmatrix} a & b \\ \ell b^\sigma & a^\sigma \end{pmatrix} \to a \begin{pmatrix} a & b \\ \ell b^\sigma & a^\sigma \end{pmatrix} \to a^\sigma$$

where σ is the non-trivial automorphism of $\mathbb{F}_{\ell^2}$. An oriented Eichler Order is an Eichler order along with a choice of one of these homomorphism $\mathfrak{o}_\ell$, called a ℓ-*orientation*, for each ℓ dividing $D = D_0N$.

2.2. SHIMURA CURVES

With the model of $\mathbf{B}$,$\mathcal{O}$, the isomorphism Φ_∞ and the orientation we can form the family of surfaces we are interested in. Let $\Gamma(1) = \Gamma_{\mathcal{O}}(1)$ be the group

$$\Gamma_{\mathcal{O}}(1) = \{x \in \mathcal{O} | Nm(x) = 1\}$$

where Nm denotes the reduced norm. $\Gamma(1)$ acts on the upper half plane $\mathfrak{H}$ through the embedding Φ_∞. The quotient is an algebraic curve whose complex points represents a component of the moduli of Abelian surfaces whose endomorphism rings contain $\mathcal{O}$. In this case the generic Abelian surface has endomorphism ring actually equal to $\mathcal{O}$.

We further assume that the Abelian surfaces have full level $2k$ structure for some $k > 1$. This is because we will use the level 2 structure and we need further level structure to ensure that there is a universal family over the curve. The k-level structure will not play any other part in our calculations, so will be suppressed but not forgotten in the remaining part of this paper.

The universal family can be described as follows. Let

$$\Gamma(2k) = \Gamma_{\mathcal{O}}(2k) = \{\gamma \in \Gamma(1) | \gamma \in Ker(\Gamma(1) \to Aut(\mathcal{O}/2k\mathcal{O}))\}$$

The *Jacobi Group* $\Gamma(2k) \ltimes \mathcal{O}$ with multiplication defined by $(\gamma, x) \star (\mu, y) = (\gamma\mu, x\mu + y)$ acts on $\mathfrak{H} \times \mathbb{C}^2$ via

$$\left(\begin{pmatrix} a & b \\ c & b \end{pmatrix}, x\right) \cdot (\tau, z_1, z_2) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-1}\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \Phi_\infty(x)\begin{pmatrix} \tau \\ 1 \end{pmatrix}\right)\right)$$

and the quotient is a complex threefold which is a family of Abelian surfaces over the Shimura curve $\mathfrak{H}/\Gamma(2k)$.

The fibre over a point τ is

$$\mathcal{A}_\tau = \mathbb{C}^2/\Phi_\infty(\mathcal{O})\begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

and $\mathcal{A}_\tau \simeq \mathcal{A}_{\tau'}$ if and only if $\tau = \gamma(\tau')$ for some $\gamma \in \Gamma(1)$, with their level structures coinciding if and only if $\gamma \in \Gamma(2k)$.

Let $\mathbf{Y} = \mathbf{Y}(\mathbf{D_0}, \mathbf{N})$ denote the curve $\mathfrak{H}/\Gamma(2k)$. If $D_0 = 1$ the curve is not compact but it can be compactified by adding finitely many cusps. Let $\mathbf{X} = \mathbf{X}(\mathbf{D_0}, \mathbf{N})$ denote the compactified curve. Let $\mathbf{W} = \mathbf{W}(\mathbf{D_0}, \mathbf{N})$ denote the desigularisation of the self product of the universal family over $\mathbf{X}(\mathbf{D_0}, \mathbf{N})$ as described in [Sc1]. Let $\mathcal{W} = \mathcal{W}(D_0, N) = \mathcal{A}_\eta$ denote the generic fibre of this family. As remarked earlier, it is an Abelian surface with endomorphism ring $\mathcal{O}$.

Polarizations

There is an involution on $\mathcal{O}$ defined by $d^\dagger = i^{-1}\bar{d}i$ where i is as in Hashimoto's theorem. The skew symmetric pairing on $\mathbb{C}^2$ defined by

$$\left\langle \Phi_\infty(x)\begin{pmatrix} \tau \\ 1 \end{pmatrix}, \Phi_\infty(y)\begin{pmatrix} \tau \\ 1 \end{pmatrix} \right\rangle = E(x, y) \tag{1}$$

where E is the form coming from part 3 of theorem 2.1, is a Riemann form for A_τ. This is a principal polarization and $d^\dagger$ is the Rosati involution corresponding to it. A well known theorem [Mu] p. 190, says that the fixed points under this involution correspond to elements of the Néron–Severi group of the Abelian surface. It is easy to see that the elements $1, j$ and k are invariant under this involution and are clearly linearly independent, so from that one can see that the rank of the generic

Néron–Severi group is 3. A little more work gives the following, which is a simple corollary to Hashimoto's theorem, but will be useful later on.

COROLLARY 2.2. *The elements* $e_1 = 1$, $e_2 = (1 + j/2)$ *and* $e_3 = (aDj + ij/p)$ *give a basis for the Néron–Severi of the generic fibre. Further, for ample g, the cup product is given by* $< g, g >= 2Nm(\phi_g)$, *where* ϕ_g *is the endomorphism corresponding to g. With that, the intersection matrix is*

	e_1	e_2	e_3
e_1	2	1	0
e_2	1	$\frac{1-p}{2}$	aD
e_3	0	aD	$2bD$

where $a^2D + 1 = bp$

Proof. From the Riemann–Roch theorem, one sees that, for an ample divisor g,

$$< g, g >= 2\chi(\mathcal{L}(g)) \quad \text{and} \quad \chi(\mathcal{L}(g))^2 = \deg(\phi_g)$$

where $\mathcal{L}(g)$ is the corresponding line bundle and ϕ_g is the corresponding endomorphism.

This, coupled with the formula for the degree in terms of the norm, $\deg(\phi) = Nm(\phi)^2$ where Nm is the reduced norm, shows that $< g, g >= \pm 2Nm(\phi_g)$. The sign can be determined from the fact that $< 1, 1 >= 2$ where 1 represents the class of the principal polarization. □

From now on we pick curves in the generic fibre which represent these classes e_i and we denote them by e_i as well. This will not affect anything as all of our results will be modulo divisors homologous to zero in the fibres. We also fix the polarization to be the one in equation (1) and denote it by Θ.

2.3. HUMBERT INVARIANTS

A lemma in [Be2], Lemma 4.1 shows that there is an equivalence of categories between the category of even lattices of rank n with a element ζ of norm 2 with the category of lattices of rank (n−1) such that the quadratic form represents only numbers $\equiv 0$ or 1 mod 4. The correspondence is given by

$$(M, \zeta) \to N(M, \zeta) \quad N \to (M_N, \zeta)$$

where

- $N(M, \zeta)$ is the lattice such that $2N(-2)$ is the orthogonal complement to ζ in $2M + \mathbb{Z}\zeta$
- M_N is the sublattice of $(N(-2) \oplus \mathbb{Z}\zeta) \otimes \mathbb{Q}$ containing $N \oplus \mathbb{Z}\zeta$ and all elements of the form $\frac{1}{2}(x + (x, x)\zeta)$ with x in N.

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Here $N(k)$ denotes the lattice which has the same underlying $\mathbb{Z}$ module as N but with the pairing multiplied by k.

The Néron–Severi lattice of an Abelian surface is an even lattice. A principal polarization is an element of norm 2. So a choice of principal polarization allows us to work with a smaller lattice, namely the primitive Néron–Severi lattice.

$$NS(\mathcal{A}, \Theta) = \{v \in NS(\mathcal{A}) | < v, \Theta >= 0\}$$

with the pairing given by $(v, v)_\Theta = -2 < v, v >$. More generally, one can work with the entire $H^2(\mathcal{A}.\mathbb{Z})$ and the primitive cohomology, $H^2(\mathcal{A}, \Theta)$.

Classically, this was first studied by Humbert [Hu], and so we call the pairing $(,)_\Theta$ the *Humbert norm* and if $v \in NS(\mathcal{A}, \Theta)$ then $(v, v)_\Theta = (v, v)$ is the *Humbert Invariant of v, H(v)*.

From now on unless otherwise mentioned, we will always work with the Humbert norm on a lattice. From the corollary above and the description of the correspondence above we see that the Humbert lattice for the generic Néron–Severi is generated by

$$\bar{e}_2 := \tfrac{1}{2}(2e_2 - < e_2, e_1 > e_1) \quad \text{and} \quad \bar{e}_3 := \tfrac{1}{2}(2e_3 - < e_3, e_1 > e_1)$$

with intersection matrix

	$\bar{e}_2$	$\bar{e}_3$
$\bar{e}_2$	p	$2aD$
$\bar{e}_3$	$2aD$	$4bD$

This matrix has determinant 4D.

Humbert studied the moduli of Abelian surfaces whose Néron–Severi contains an element with a non-zero Humbert invariant, which are now called *Humbert surfaces*. This amounts to saying that the rank of the Néron–Severi is at least 2 and he realised that this was equivalent to saying that the endomorphism ring contains an order in a real quadratic field. One can easily see that, in general, the endomorphism algebra is simply the Clifford algebra of the Humbert lattice.

Humbert's definitions were more analytic and quite different from the above defintions, which are due to Kani. For some computations, however, it is more useful to use the analytic definition, so for that reason we will give it here. The equivalence of the two can be found in Kani [Ka].

Let V' be the set of integral, skew symmetric 4×4 matrices, considered as a subgroup of $M_4(\mathbb{Z})$. Let J be the subgroup generated by

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

and let $V = V'/J$.

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There is an action of $Sp_4(\mathbb{Z})$ on V' given by $v \to M^t v M$ for M in $Sp_4(\mathbb{Z})$ which leaves J invariant and hence descends to an action on V. A v in V' looks like

$$v = \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & b_3 & b_4 \\ -b_1 & -b_3 & 0 & d \\ -b_2 & -b_4 & -d & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ -B^t & D \end{pmatrix}$$

and one defines the *Humbert Invariant* to be

$$(v, v) = H(v) = (b_1 + b_4)^2 - 4(b_1 b_4 - b_3 b_2 - ad) = Tr(B)^2 - 4det(B) + ad$$

$H(v)$ depends only on v mod J and v is said to be *primitive* if it is not divisable by a natural number $\geqslant 2$. A lemma due to Humbert says that two primitive elements v_1 and v_2 are equivalent if and only if $H(v_1) = H(v_2)$.

Let $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$ be a point in the Siegel upper half space $\mathfrak{H}_2$. Every primitive v in V determines a subvariety given by $\tau \in \mathfrak{H}_2$ such that

$$\begin{pmatrix} \tau & I_2 \end{pmatrix} v \begin{pmatrix} \tau \\ I_2 \end{pmatrix} = 0$$
$$\tau A \tau + \tau B - B^t \tau + D = 0$$

and if τ satisfies such an equation it is said to satisfy a *singular relation of invariant H(v)* and the subvariety is called the Humbert surface of invariant $H(v)$. V is called the space of singular relations. Equivalently, a singular relation can be described as a relation between the coefficients of τ of the form

$$v = \alpha\tau_1 + \beta\tau_2 + \gamma\tau_3 + \delta(\tau_2^2 - \tau_1\tau_3) + \varepsilon = 0$$

with invariant $H(v) = \beta^2 - 4\alpha\gamma - 4\delta\varepsilon$. Sometimes it is more convenient to work with this description. We will denote v by $(\alpha, \beta, \gamma, \delta, \varepsilon)$. In fact [Ka], Prop. 5.4, there is a basis for the primitive part of $H^2(\mathcal{A}, \mathbb{Z})$ for which this vector gives the corresponding 2-form in the other description of the Humbert surface.

2.4. CM POINTS AND CM CYCLES

The cycles we are interested are codimension 2 cycles on these families of Abelian surfaces. There are two basic types, horizontal cycles and vertical cycles. Horizontal cycles are roughly images of the Shimura curve under some section. Vertical cycles are those which are supported in fibres over some points. In most fibres, all such cycles will come by restriction of a divisor on universal family to the special fibre. However, there are some points in whose fibres there are extra elements in the Néron–Severi of the fibre, hence extra codimension 2 cycles in the family.

CM Points
As we have seen before, the rank of the generic Néron–Severi is 3, so when there is an extra cycle, the rank goes up to 4. If τ is a point in the moduli where the rank of the Néron–Severi of the fibre $\mathcal{A}_\tau$ is 4, then the Abelian surface $\mathcal{A}_\tau$ is necessarily isogenous to a product of elliptic curves with complex multiplication by an the imaginary quadratic field $K = \mathbb{Q}(\tau)$ and $End_0(\mathcal{A}_\tau) \simeq M_2(K)$. Such a point τ is called a CM point.

Determining a CM point is equivalent to determining an embedding of an imaginary quadratic field $q\colon K \hookrightarrow \mathbf{B}$ such that $\mathbf{B} \otimes K \simeq M_2(K)$ as from Prop. 9.4 of [Sh] there is a unique point τ in $\mathfrak{H}$ such that

$$\Phi_\infty(q(K^\times)) = \{\gamma \in \Phi_\infty(B^\times \cap GL_2^+(\mathbb{Q})) \mid \gamma\tau = \tau\}$$

We can normalize these embeddings as in [Sh] (4.4.3). Once having normalized the embedding of $K \hookrightarrow \mathbf{B}$ one has also normalized the embedding of $K \hookrightarrow M_2(K) = \mathbf{B} \otimes K$. $K \cap End(\mathcal{A}_\tau)$ is an order of discriminant Δ and hence will be denoted by $\mathcal{O}_\Delta$. Through the embedding q and the maps $\mathfrak{o}_\ell$ for ℓ dividing D, we have also chosen an orientation of the order $\mathcal{O}_\Delta$.

CM cycles
Let α be a traceless element of $\mathcal{O}_\Delta$ with $\alpha^2 = \Delta$, so α is purely imaginary. The Rosati involution extends to an involution of $\mathbf{B} \otimes K$ and acts by $(d \otimes \alpha)^\dagger = d^\dagger \otimes \bar{\alpha} = -d^\dagger \otimes \alpha$. Observe that the element $i \otimes \alpha$ is fixed by the Rosati involution. We define a *CM cycle class* to be $\mathcal{Z}_\tau = i \otimes \alpha$, the class of this element in the Néron–Severi of $\mathcal{A}_\tau$. By construction $\mathcal{Z}_\tau$ is *not* a generic class. Further, in this case one can see that the intersection form on the Néron–Severi lattice being thought of as Rosati fixed elements is given by $< v, v >= 2det(\phi_v)$. From that one can see that the CM cycle is orthogonal to the generic Néron–Severi and the Humbert norm is $(\mathcal{Z}_\tau, \mathcal{Z}_\tau) = -4D\Delta = -4Ddisc(\mathcal{O}_\Delta)$.

Remark 2.3. This definition of the CM cycle almost agrees with the definitions in [Sc1] and [Be], where it is defined to be the minimal generator of the orthogonal complement of the generic Néron–Severi. The definitions are the same when Δ is odd, but when Δ is even, our cycle is twice the generator.

The CM cycles are the analogues of CM points on modular and Shimura curves and have many similar properties. They are also defined over number fields. Schoen, in the modular case and Besser, in the Shimura curve case, show that they are homologous to zero in $CH^2(\mathbf{W})$ and so give rise to interesting elements in $CH^2_{hom}(\mathbf{W})$. In fact he shows that in general they are non-trivial in the Griffiths group.

3. Higher Chow Groups and the Localization Sequence

In this section we introduce the higher Chow groups that we use and the Localization theorem. No proofs are given. The proofs of these results may be found in [Bl1].

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3.1. THE GROUP $CH^2(\mathcal{A}, 1)$

The group we use is $CH^2(\mathcal{X}, 1)$ where $\mathcal{X}$ is surface. It is defined as a a certain subquotient of the group of codimension 2 cycles on $\mathcal{X} \times \mathbb{A}^1$. A theorem of Bloch's [Bl1] identifies it with the $\mathcal{K}$-cohomology group $H^1(\mathcal{X}, \mathcal{K}_2)$, where $\mathcal{K}_2$ is the *sheaf* coming from the presheaf given by $U \longrightarrow K_2(U)$. With this description and the Bloch–Gersten–Quillen resolution of the sheaf $\mathcal{K}_2$ one can see that an element of $CH^2(\mathcal{X}, 1)$ is represented by a formal sum

$$\sum(\mathcal{C}_i, f_i)$$

where $\mathcal{C}_i$ are curves on $\mathcal{X}$ and f_i are functions on the C_i such that they satisfy the cocycle condition

$$\sum div(f_i) = 0$$

In $CH^2(\mathcal{X}, 1)$ there are certain elements coming from the product structure on Chow groups,

$$CH^1(\mathcal{X}, 1) \otimes CH^1(\mathcal{X}, 0) \to CH^2(\mathcal{X}, 1) \tag{1}$$

A theorem of Bloch's [Bl1] shows that

$$CH^1(\mathcal{X}, 1) = \Gamma(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) \quad CH^1(\mathcal{X}, 0) = CH^1(\mathcal{X}) = Pic(\mathcal{X})$$

These elements are of the form

$$\sum(\mathcal{C}_i, f_i)$$

where $\mathcal{C}_i$ are divisors on $\mathcal{X}$ and f_i are constant functions hence automatically satisfy the cocycle condition as $div(f_i) = 0$. Such elements are called *decomposable*. More generally one can construct more such elements by looking at norms of elements of extensions of the base field. Let $CH^2(\mathcal{X}, 1)_{dec}$ denote the subgroup of $CH^2(\mathcal{X}, 1)$ generated by such cycles. The group

$$CH^2(\mathcal{X}, 1)_{indec} = CH^2(\mathcal{X}, 1)/CH^2(\mathcal{X}, 1)_{dec}$$

is called the subgroup of *indecomposable* cycles and is sometimes non-trivial.

3.2. THE LOCALIZATION SEQUENCE

The localization sequence for higher Chow groups is an extension of the usual localization sequence for Chow groups to the left. Suppose $\mathcal{X}$ is a smooth projective variety and $\mathcal{Y} \hookrightarrow \mathcal{X}$ is a closed divisor, then we have an exact sequence

$$\ldots \to CH^2(\mathcal{X}, 1) \to CH^2(\mathcal{X}\backslash\mathcal{Y}, 1) \xrightarrow{\partial} CH^1(\mathcal{Y}, 0) \to CH^2(\mathcal{X}, 0) \to \ldots$$

In particular, if $\mathcal{X}$ is a family of surfaces over a curve $\mathcal{Z}$ and $\mathcal{Y}_z$ is the fibre over a point

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$z \in \mathcal{Z}$, then we have a sequence

$$\ldots \to CH^2(\mathcal{X}, 1) \to CH^2(\mathcal{Y}_\eta, 1) \xrightarrow{\partial} \bigoplus_{z \in \mathcal{Z}} CH^1(\mathcal{Y}_z) \to CH^2(\mathcal{X}) \to \ldots$$

where $\mathcal{Y}_\eta$ is the generic fibre of the family. So, from this our strategy becomes clear: to construct relations of rational equivalence between codimension 2 cycles supported in fibres of a family of surfaces it suffices to construct elements of the higher Chow group of the generic fibre and compute the boundary under the map ∂. If $\sum(\mathcal{C}_i, f_i)$ is an element of $CH^2(\mathcal{Y}_\eta, 1)$, so now the $\mathcal{C}_i$ are curves in the generic fibre, the boundary of the element is

$$\partial\Big(\sum(\mathcal{C}_i, f_i)\Big) = \sum div(f_i)$$

where $div(f_i)$ denotes the divisor of f_i being though of as a function on the closure $\bar{\mathcal{C}}_i$ of $\mathcal{C}_i$. This divisor may have vertical componenents. For example, if $(\mathcal{C}, f)$ is a decomposable element in $CH^2(\mathcal{Y}_\eta, 1)$, where f is now a function on the base,

$$\partial((\mathcal{C}, f)) = \sum a_i \mathcal{C}_{x_i}$$

where $div(f) = \sum a_i x_i$ and $\mathcal{C}_{x_i}$ denotes the restriction of $\mathcal{C}$ to the fibre over the point x_i. Note that in this manner one can only get relations between cycles obtained by the restriction of generic cycles to the various fibres. Since the cycles we are interested in are not of this type, one cannot hope to use decomposable elements for our purposes. However, on the generic Abelian surface there exist indecomposable elements In the next section we describe these elements constructed by Collino [Co].

4. Collino's Elements of $CH^2(\mathcal{A}, 1)$

4.1. CONSTRUCTION OF THE ELEMENT

In [Co], Collino constructs certain elements of $CH^g(\mathcal{A}, 1)$ where $\mathcal{A}$ is the Jacobian of a hyperelliptic genus g curve. In particular, if $\mathcal{A}$ is a simple, principally polarized Abelian surface, it is the Jacobian of a smooth genus 2 curve, so one can use his construction. It is as follows:

Let $\mathcal{A} = Jac(\mathcal{C})$ where $\mathcal{C}$ is a genus 2 curve which represents the principal polarization. Since $\mathcal{C}$ is hyperelliptic there is a function $f: \mathcal{C} \to \mathbb{P}^1$ such that $div f = 2(P) - 2(Q)$ where P and Q are two ramification points.

There are maps $i_P(x) = x - (P)$ and $i_Q(x) = x - (Q)$ from $\mathcal{C} \hookrightarrow Jac(\mathcal{C})$. Let $\mathcal{C}_P$ and $\mathcal{C}_Q$ denote the images of $\mathcal{C}$ under i_P and i_Q respectively and let f_P and f_Q denote the function f being thought of as functions on $\mathcal{C}_P$ and $\mathcal{C}_Q$ respectively. Then $(\mathcal{C}_P, f_P) + (\mathcal{C}_Q, f_Q)$ is an element of $CH^2(\mathcal{A}, 1)$. This is because

$$div(f_P) = 2(O) - 2(Q - P) \text{ and } div(f_Q) = 2(P - Q) - 2(O)$$

but $(P - Q) = (Q - P)$ in $Jac(\mathcal{C})$, so

$$div(f_P) + div(f_Q) = 2(O) - 2(Q - P) + 2(P - Q) - 2(O) = 0.$$

Collino proves that this element is indecomposable on the generic Abelian surface. Note that the exact choice of the function f is not that important as any other choice of function with the same divisor would give the same class in the group of indecomposable cycles.

4.2. COMPUTATION OF THE BOUNDARY

Let $\mathbf{X}$ as before be the Shimura curve with full level $2k$ structure and $\mathbf{W}$ and $\mathcal{W}$ be the universal family and the generic fibre respectively. Since we have full level 2 structure, by making a choice of two 2-torsion sections P and Q, we can apply the construction in Section 4.1 to the generic fibre to get an element of $CH^2(\mathcal{W}, 1) = CH^2(\mathcal{A}_\eta, 1)$. In this section we will compute the boundary of this element.

The boundary of this element of $CH^2(\mathcal{W}, 1)$ is of the form

$$\sum_x a_x \mathcal{D}_x$$

where $\mathcal{D}_x$ are codimension 1 cycles in the fibre over the point x. In many fibres, $\mathcal{D}_x$ is simply the restriction of the generic curves $\mathcal{C}_P$ and $\mathcal{C}_Q$ to the fibre, but in some cases it is more interesting. There are two such cases [We]. The first is when the point is a cusp, so the fibre is a degenerate Abelian surface. In this case a theorem of Schoen [Sc1], essentially the Manin–Drinfeld principle, shows that these cycles are torsion in $CH^2(\mathbf{W})$ so one can neglect them as we are only interested in the rational Chow group. The second case is when the curve breaks up into a sum of two elliptic curves intersecting at a point, such that the two ramification points lie on different components. Later on, in Sections 4.3 and 5, we will describe precisely when this happens. In this case the cycle is not simply the restriction and involves the components in a non-trivial manner. To compute the boundary we do the following local computation.

THEOREM 4.1. *Let τ be a point on $\mathbf{X}$ in whose fibre the genus 2 curve $\mathcal{C}_P$ degenerates into a sum of two elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ and such that O lies on $\mathcal{E}_1$ and $Q - P$ lies on $\mathcal{E}_2$. Then*

$$\partial((\mathcal{C}_P, f_P) + (\mathcal{C}_Q, f_Q)) = 2(\mathcal{E}_2 - \mathcal{E}_1)$$

up to the boundaries of decomposable elements and images of cycles homologous to 0 in the fibre.

Proof. In a neighborhood of the point τ the boundary of $(\mathcal{C}_P, f_P)$ is of the form

$$\partial((\mathcal{C}_P, f_P)) = a\mathcal{E}_1 + b\mathcal{E}_2 + \mathcal{H}$$

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where a and b are in $\mathbb{Z}$ and $\mathcal{H}$ denotes the closure of the horizontal sections of $div(f_P)$. Locally, we can always find a decomposable element whose boundary is of the form $b(\mathcal{E}_1 + \mathcal{E}_2)$ so subtracting this element from Collino's element allows us to assume that the boundary is of the form

$$\partial((\mathcal{C}_P, f_P)) = a\mathcal{E}_1 + \mathcal{H}$$

To show that the boundary is not simply the restriction of the closure of a divisor in the generic fibre amounts to showing that $a \neq 0$. To do this we intersect with $\mathcal{E}_2$ and use the fact that a function restricts to a divisor of degree 0 in a cycle not contained in its divisor. Intersecting with $\mathcal{E}_2$ gives

$$0 =< (a\mathcal{E}_1 + \mathcal{H}).\mathcal{E}_2 >= a - 2$$

as by assumption, only $Q - P$ lies on $\mathcal{E}_2$ so $(\mathcal{H}.\mathcal{E}_2) = -2$

A similar computation for $(\mathcal{C}_Q, f_Q)$ shows that

$$\partial((\mathcal{C}_Q, f_Q)) = 2\mathcal{E}_2' + \mathcal{H}'$$

where here $C_Q = \mathcal{E}_1' + \mathcal{E}_2'$ are translates of the $\mathcal{E}_1$ and $\mathcal{E}_2$ which split C_P and $\mathcal{H}'$ is the closure of $divf_Q$.

The cocycle condition $div(f_p) + div(f_Q) = 0$, gives $\mathcal{H} + \mathcal{H}' = 0$, as $\mathcal{H}$ and $\mathcal{H}'$ are the closures of these divisors. Adding the two and observing that $(\mathcal{E}_1 - \mathcal{E}_2) - (\mathcal{E}_1' - \mathcal{E}_2')$ is homologous to zero in the fibre (in fact it is torsion) we get our result. □

Remark 4.2. This calculation works only locally as, a priori, it is not clear that one can find a global decomposable element which has the required boundary. However, we will later give an argument that this is indeed the case.

4.3. COMPUTING SIGNS

In this section we give a recipe for computing the signs of the boundary. As shown in the previous section, the sign depends on the position of the 2-torsion points O and $P - Q$ when the curve splits into a product of two elliptic curves. The best way to describe this is to use genus 2 theta functions with characteristics. A good reference for all the facts used here is [Fr-Kr].

Theta Functions

The *degree 2 Theta function with characteristic* $[\varepsilon, \varepsilon']$ is defined to be

$$\Theta[\varepsilon, \varepsilon'](z, \tau) = \sum_{N \in \mathbb{Z}^2} \exp\left(2\pi i \left(\frac{1}{2}\left(N + \frac{\varepsilon}{2}\right)\tau\left(N + \frac{\varepsilon}{2}\right)^{tr}\right) + \left(N + \frac{\varepsilon}{2}\right)\left(z + \frac{\varepsilon'}{2}\right)^{tr}\right)$$

where

$$\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$$

is a point on $\mathfrak{H}_2$, $z = [(z_1, z_2)]$ is a point on $\mathbb{C}^2$ being thought of as a row matrix and $[\varepsilon, \varepsilon'] = [(\varepsilon_1, \varepsilon_2), (\varepsilon'_1, \varepsilon'_2)]$ is a pair of points in $(\mathbb{Z}/2\mathbb{Z})^2$.

To a characteristic $[\varepsilon, \varepsilon']$ we associate the 2-torsion point $I(\varepsilon'/2)^{tr} + \tau(\varepsilon/2)^{tr}$ on the Abelian surface $\mathcal{A}_\tau$ and call this the *associated 2-torsion point*.

A theta function with characteristic is called *odd* or *even* depending on whether the corresponding function is an odd or even function. This is equivalent to $\varepsilon_1\varepsilon'_1 + \varepsilon_2\varepsilon'_2$ being odd or even.

Up to a nowhere-vanishing factor, the theta function with characteristic is the same as the function that one would get by translating the theta function $\Theta[(0,0)(0,0)](z,\tau)$ by the associated 2-torsion point. This shows that there are six odd theta functions and ten even theta functions.

The functions on the moduli space given by

$$\Theta[(\varepsilon_1, \varepsilon_2), (\varepsilon'_1, \varepsilon'_2)](\tau) = \Theta[(\varepsilon_1, \varepsilon_2), (\varepsilon'_1, \varepsilon'_2)](0, \tau)$$

are called the *thetanullwerte*. Since odd theta functions, being odd functions, vanish identically at 0 there are no odd thetanullwerte.

Let $\Gamma_2 = Sp_4(\mathbb{Z})$ and $\Gamma_2(2)$ be the principal congruence subgroup of level 2. From the transformation formula for the theta functions, one can see that the thetanullwerte descend to give modular forms of weight one-half on $\mathfrak{H}_2/\Gamma_2(2)$. A well known theorem, see for example [Ge], says that each thetanullwert vanishes precisely on one component of the moduli of products of elliptic curves.

The group $\Gamma_2/\Gamma_2(2) \simeq Sp_4(\mathbb{Z}/2\mathbb{Z})$ acts on these theta functions or theta characteristics. It acts transitively on the set of odd theta characteristics and can be used [Ig] to get an isomorphism of $Sp_4(\mathbb{Z}/2\mathbb{Z})$ with $\mathfrak{S}_6$. Let $\Gamma_2(o)$ be the inverse image of $\mathfrak{A}_6$ under this isomorphism and let $\omega \in Sp_4(\mathbb{Z})$ be such that $\omega \notin \Gamma_2(o)$ and $\omega^2 = I$.

At a point τ where the polarization splits into a sum of two elliptic curves, $\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2$, the existence of the curve $\mathcal{E}_1$ implies that the period τ satisfies a certain singular relation, $v = (\alpha, \beta, \gamma, \delta, \varepsilon)$. The curve $\mathcal{E}_2$ corresponds to the relation $-v$. We can make a choice of $\mathcal{E}_1$ or $\mathcal{E}_2$ and look at the $\Gamma_2(o)$ translates, namely $g\mathcal{E}_1 g^{tr}$for all g in $\Gamma_2(o)$.

This allows us to make a uniform choice of elliptic curve $\mathcal{E}_1$, with respect to which we can compute the sign. The sign does not change within components.

To compute the sign, we first need to make a choice of an embedding of the genus 2 curve. In Farkas and Kra [Fr-Kr] chapter VII, they make a choice and with respect to their choice, the genus 2 curve $\mathcal{C}_{P_1}$ is given by the divisor odd theta function $\Theta[(0,1),(0,1)](z,\tau)$ and the six ramification points are the two torsion points associated to the characteristics

$$P_1 = [(0,0),(0,0)] \quad P_2 = [(1,0),(0,0)] \quad P_3 = [(1,0),(1,1)]$$
$$P_4 = [(1,1),(1,1)] \quad P_5 = [(1,1),(1,0)] \quad P_6 = [(0,0),(1,0)]$$

In the component of the moduli of products of elliptic curves given by $\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}$, the odd theta function splits as

$$\Theta[(0,1),(0,1)]((z_1,z_2),\tau) = \Theta[(0,0)](z_1,\tau_1)\,\Theta[(1,1)](z_2,\tau_3)$$

Since the zeroes of the elliptic theta functions are $[(1,1)] = \frac{1}{2}(1+\tau_1)$ and $[(0,0)] = 0$ respectively, the elliptic curves meet at the 2-torsion point $R = [(1,0),(1,0)]$. From this one can see that the points P_1, P_2 and P_6 lie one the elliptic curve $z_1 \times [(0,0)]$ and the points P_3, P_4 and P_5 lie on the elliptic curve $[(1,1)] \times z_2$.

So, for example, if we choose $P = P_1$ and $Q = P_3$ and consider the element of $CH^2(\mathcal{W}, 1)$ given by $(\mathcal{C}_P, f_P) + (\mathcal{C}_Q, f_Q)$, the boundary of this element would have a sign of -1 on the component of the moduli where the curve $\mathcal{C}_P$ splits in to a sum of two elliptic curves meeting at the 2-torsion point $[(1,0),(1,0)]$.

In general, to a component (which corresponds to an even theta characteristic) we can associate a $(3,3)$ configuration of the six ramification points, namely the first triple corresponding to the points lying on our choice of $\mathcal{E}_1$ and the second to the points lying on $\mathcal{E}_2$. To any pair of these ramification points we have a Collino element of $CH^2(\mathcal{W}, 1)$ and the sign of the boundary on this component would depend on the which triple 0 and $P - Q$ belong to. If they belong to the same triple, then the sign is 0 and if they belong to different triples, the sign is $+1$ or -1 depending on whether 0 or $P - Q$ lies on $\mathcal{E}_1$.

The sign at a different component will depend on the action of $\Gamma_2(o)$ on the $(3,3)$ configuration of the six ramification points. To understand this action, we observe that this can be translated in to the well known [Ig],[G-H] situation of the action of $\mathfrak{A}_6$ on the $(3,3)$ configuration of the six odd theta characteristics. This is because the points associated to the six odd theta charateristics are nothing but the ramifications points of the genus 2 curve given by divisor of $\Theta[(0,0),(0,0)](z,\tau)$. As we used $\Theta[(0,1),(0,1)](z,\tau)$ to compute the action on the six ramification points we have, we simply translate our points by the point $[(0,1),(0,1)]$ which gives us the six odd theta characteristics, and then use the description of the action on them.

The table describes the correspondence between pairs of odd theta characteristics and even theta characteristics. We use the convention, as in [Ig] that the six odd theta chacteristics are

$$P_1 = [(0,1),(0,1)] \quad P_2 = [(0,1),(1,1)] \quad P_3 = [(1,0),(1,0)]$$
$$P_4 = [(1,0),(1,1)] \quad P_5 = [(1,1),(0,0)] \quad P_6 = [(1,1),(1,0)]$$

With that convention, the class $(346, 125)$ corresponds to $[(1,1),(1,1)]$ The g in the leftmost column represents the element of $\mathfrak{A}_6$ which has to be applied to $(346, 125)$ to obtain the pair on the right. In [Ig] he makes the isomophism of $\mathfrak{S}_6$ with $Sp_4(\mathbb{Z}/2\mathbb{Z})$ explicit, which along with some help from MAPLE, allows one to do the computation and make the following table.

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Table 1. Correspondence between (3,3) configurations and even Theta Characteristics

	g	Pair	Even Theta Characteristic
I	e	(346,125)	1111
II	(14)(26)	(123,456)	1000
III	(56)(12)	(345,126)	1100
IV	(26)(34)	(124,356)	0110
V	(13)(56)	(145,236)	0011
VI	(13)(45)	(156,234)	1001
VII	(24)(56)	(235,146)	0000
VIII	(23)(34)	(246,135)	0010
IX	(23)(45)	(256,134)	0100
X	(14)(46)	(136,245)	0001

Define the function

$$\varepsilon_{P,Q}(\tau) = \begin{cases} -1 & \text{if } (O) \text{ lies on } (\mathcal{E}_1)_\tau \text{ and } (P-Q) \text{ lies on } (\mathcal{E}_2)_\tau \\ 0 & \text{if } (O) \text{ lies on} (\mathcal{E}_i)_\tau \text{ and } (P-Q) \text{ lies on } (\mathcal{E}_i)_\tau \text{ for } i = 1, 2 \\ 1 & \text{if } (O) \text{ lies on } (\mathcal{E}_2)_\tau \text{ and } (P-Q) \text{ lies on } (\mathcal{E}_1)_\tau \end{cases}$$

From the above discussion, given a point τ on the component corresponding to $[(1,1),(1,1)]$ we can determine $\varepsilon_{P,Q}(g\tau)$ for g in $\Gamma_2(o)$.

4.4. COMPUTING THE RELATION

In this section, we restrict the computations done in the previous sections to Shimura curves.

From Hashimoto's [Ha], Thm 3.5, explicit description of the Eichler order and Shimura curve one can explicitly describe the embedding $\Psi\colon \mathfrak{H} \hookrightarrow \mathfrak{H}_2$ which is compatible with the embedding of the group of units Γ into $\Gamma_2(o)$:

THEOREM 4.3. *The curve* $\mathbf{X}(\mathbf{D_0}, \mathbf{N})$ *satisfies singular relations of the form*

$$\tau_1 + \tau_2 + \frac{p-1}{4}\tau_3 = 0$$
$$2aD\tau_2 + (\tau_2^2 - \tau_1\tau_3) + (a^2D - b)D = 0$$

giving rise to an embedding

$$\Psi(\tau) = \frac{1}{p\tau}\begin{pmatrix} -\bar{\kappa}^2 + \frac{(p-1)aD}{2}\tau + D\kappa^2\tau^2 & \bar{\kappa} - (p-1)aD\tau - D\kappa\tau^2 \\ \bar{\kappa} - (p-1)aD\tau - D\kappa\tau^2 & -1 - 2aD\tau + D\tau^2 \end{pmatrix}$$

where $\kappa = \left(1 + \sqrt{p}/2\right)$

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From this one can see [Ha] that the τ in the image satisfy singular relations with invariants of the form $pX^2 + 4aDXY + 4bDY^2$ for (X, Y) coprime.

Let v be the singular relation given by the existence of an elliptic curve of degree 1, and let $\Psi(\tau)$ be a point on $\Psi(\mathfrak{H})$ which satisfies this relation. Then τ satisfies an equation given by

$$\begin{pmatrix}\Psi(\tau) & I_2\end{pmatrix} v \begin{pmatrix}\Psi(\tau)\\ I_2\end{pmatrix} = 0$$

Since as functions of τ, τ_1,τ_2, τ_3 and $\tau_2^2 - \tau_1\tau_3$ are at worst quadratic when multiplied by $p\tau$ and since the singular relation between the τ_i's is linear, the equation satisfied by τ is quadratic. Hence the point τ is imaginary quadratic so is a CM point on $\mathfrak{H}$.

We can look at the action of $\Gamma_2(o)$ on v, namely $g \cdot v = g^{tr} v g$ for g in $\Gamma_2(\mathrm{o})$. The point $g\Psi(\tau)$, which is given by the action of $\Gamma_2(o)$ on $\mathfrak{H}_2$ need not lie on $\Psi(\mathfrak{H})$. However, for some g there exists a point $\Psi(\tau')$ which satisfies

$$\begin{pmatrix}\Psi(\tau') & I_2\end{pmatrix} g^{tr} v g \begin{pmatrix}\Psi(\tau')\\ I_2\end{pmatrix} = 0$$

If such a point exists it will be unique, and we will denote it by $g(\tau)$. This has the same level 2 structure as $g\tau$, so $\varepsilon_{P,Q}(g(\tau)) = \varepsilon_{P,Q}(g\tau)$. Let $\widetilde{g(\tau)}$ be the point $\omega(g(\tau))$. Then $\varepsilon_{P,Q}(\widetilde{g(\tau)}) = -\varepsilon_{P,Q}(g(\tau))$

The relation we get is then a signed sum over the $g(\tau)$, and putting it all together gives us the following proposition.

PROPOSITION 4.4. *Let $(C_P, f_P) + (C_Q, f_Q)$ be Collino's element in the group $CH^2(\mathcal{A}_\eta, 1)$, where η is the generic point of a Shimura curve* $\mathbf{X}$ *as before. Let τ be a point on the moduli where the curve C_P splits as a sum of elliptic curves $\mathcal{E}_1 + \mathcal{E}_2$, and make a choice of $\mathcal{E}_1$ as before. Assume further that O lies on $\mathcal{E}_1$ and $Q - P$ lies on $\mathcal{E}_2$. Then one has a relation in $CH^2(\mathbf{W}) \otimes \mathbb{Q}$ of the form*

$$\sum_{g \in \Gamma_2(o)} \varepsilon_{P,Q}(g(\tau))(2(\mathcal{E}_2 - \mathcal{E}_1)|_{g(\tau)} - 2(\mathcal{E}_2 - \mathcal{E}_1)|_{\widetilde{g(\tau)}}) \equiv 0$$

up to the relations coming from decomposable elements and cycles homologous to zero in the special fibres, where $\varepsilon_{P,Q}(g(\tau))$ can be computed from the table in Section 4.3.

4.5. ISOGENIES

In this section we will explain how we can use generic isogenies to get more relations between some cycles in $CH^2(\mathbf{W})$.

Let g be an element of $Sp_4(\mathbb{Q})$. Let τ be a point. Then one has an induced isogeny

$$\Phi_g : \mathcal{A}_{g\tau} \to \mathcal{A}_\tau$$

of some degree n^2. For a fixed degree, there are only finitely many classes of g under

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the relation

$$g_1 \sim g_2 \Longleftrightarrow \exists \gamma \in Stab(\mathfrak{H} \times \mathfrak{H})| \; g_1 = \gamma g_2$$

We can use this information to compute the relations one would get by applying such isogenies to the original relation in Proposition 4.4. Equivalently, we could apply the isogenies to Collino's element to get new elements and compute their boundaries.

In the previous section we have used the embedding $\Psi\colon \mathfrak{H} \hookrightarrow \mathfrak{H}_2$. Let g be in $Sp_4(\mathbb{Q})$ of degree n^2 prime to $2k$. $g\Psi$ gives another embedding of embedding of $\mathfrak{H} \hookrightarrow \mathfrak{H}_2$. Let Γ_g denote the group $g\Gamma g^{-1} \cap Sp_4(\mathbb{Z})$, where the action of g is on the image of of Γ in $Sp_4(\mathbb{Z})$. This acts on the new embedding of $\mathfrak{H}$.

Let $\mathbf{X}_g = \mathfrak{H}/\Gamma_g$ (or its compactification, if necessary). There is a map $\mathbf{X}_g \to \mathbf{X}$ given by $\tau \to g^{-1}\tau$ and as above, a corresponding isogeny Φ_g of the universal Abelian surfaces over these curves. Let ζ denote Collino's element in $CH^2(\mathcal{A}_g, 1)$. We can compute relations resulting from this element in the universal family over $\mathbf{X}_g$, $\mathbf{W}_g$ and then push the relation down using the isogeny. This will give us some new relations in $CH^2(\mathbf{W})$. A remark is necessary when the curve $\mathbf{X}_g$ is not compact as then it is not immediately clear that the isogeny extends to a morphism over the cusps. However, as the cycles supported in the cuspidal fibres are torsion as all these curves are quotients of the upper half plane by congruence subgroups [Sl2], one can always multiply the relation by a suitable number to kill those cycles, and then apply the isogeny. As the isogeny preserves the non-cuspidal points, there is no problem.

The relation that one obtains in $CH^2(\mathbf{W}_g)$ are supported in fibres where there is an elliptic curve of degree 1. Let v be the singular relation representing the elliptic curve of degree 1, $\mathcal{E}_1$. By translating by $\Gamma_2(o)$, we can make a uniform choice of the elliptic curve $\mathcal{E}_1$ over the entire moduli of products of elliptic curves. Let τ be a point on $\mathbf{X}_g$ satisfying that singular relation. Then, as described above, the point $g^{-1}\tau$ on $\mathbf{X}$ satisfies a relation corresponding to the existance of elliptic curve of degree n in the fibre.

It is possible for $g^{-1}\tau$ to lie on the image of some other point τ' as

$$g^{-1}\tau = \gamma g^{-1}\tau' \text{ for some } \gamma \text{ in } \Gamma(2k) \Rightarrow g\gamma g^{-1}\tau = \tau'$$

However, we are looking at equivalence by the subgroup of finite index in $g\Gamma g^{-1}$ given by Γ_g, and these points need not be Γ_g equivalent. But since g gives rise to an isogeny of degree coprime with $2k$ the signs of $\mathcal{E}_1 - \mathcal{E}_2$ in all the points in the fibre over a point on $\mathbf{X}$ are the same.

Conversely, a theorem of Kani [Ka] asserts that if τ_0 is a point on $\mathbf{X}$ where there is an elliptic curve of degree n in its fibre, then $\tau_0 = g^{-1}\tau$ for some point τ on $\mathbf{X}_g$ where there is an elliptic curve of degree 1 and g a primitive isogeny of degree n^2.

Let $g_1, \ldots., g_M$ be representatives for the finitely many primitive isogeny classes of degree n^2. Then, since we have made a uniform choice of $\mathcal{E}_1$ over the entire moduli, we have a uniform choice on all the points τ on all the $\mathbf{X}_{g_i}$ which lie on the moduli of products. Since the isogenies are of degree prime to $2k$, they do not affect the level structure, and so one can define $\varepsilon_{P,Q}(g^{-1}\tau) = \varepsilon_{P,Q}(\tau)$ for g of degree n^2.

Applying the corresponding isogenies to the relations on $\mathbf{X}_{g_i}$, $i = 1, \ldots M$ one has the following proposition, which is a generalization of the proposion in the previous section.

PROPOSITION 4.5. *Let τ_1 be a point on $\mathbf{X}_{g_1}$, where g_1 is one of the representatives for the isogenies of degree n^2, and using τ make a uniform choice of elliptic curve $\mathcal{E}_1$ for the whole moduli space. Choose a point τ_i on each of the $\mathbf{X}_{g_i}$*

Then, for each i, there are relations in $CH^2(\mathbf{W}) \otimes \mathbb{Q}$ of the form

$$\sum_{g \in \Gamma_2(o)} \varepsilon_{P,Q}(g_i^{-1} g(\tau_i))(2(\mathcal{E}_2 - \mathcal{E}_1)|_{g_i^{-1}g(\tau_i)} - 2(\mathcal{E}_2 - \mathcal{E}_1)|_{\widetilde{g_i^{-1}g(\tau_i)}}) \equiv 0$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are now elliptic curves of degree n in the fibre over $g_i^{-1}g(\tau)$.

Remark 4.6. While it may happen that $g_i\tau = g_j\tau'$ for some points τ and τ', this can never happen on an embedding of a curve where there is no generic elliptic curve of degree n^2 as otherwise one would have too many elliptic curves in the fibre. This is because the classes of elliptic curves are linearly independent in the Néron–Severi. Both g_i and g_j would contribute 2 each. If generically there were no elliptic curves this would lead to a contradiction as there would be too many linearly independent elements.

Hence we get infinitely many relations between some cycles supported in points where there is an elliptic curve of degree prime to $2k$. Since generically, the Néron–Severi is of rank three, when this happens the Néron–Severi jumps to rank 4, and the Abelian surface is necessarily isogenous to a product of isogenous CM elliptic curves, and there are CM cycles in those fibres.

In the next section we will use the relations above to get relations between the CM cycles.

5. Relations Between CM Cycles

5.1. REWRITING THE RELATION

In this section we first describe the cycles $\mathcal{E}_1 - \mathcal{E}_2$ in terms of the CM cycles and show that we can modify Collino's element by suitably chosen decomposable elements to get a relation only between them. Then we get a more explicit description of the points $g(\tau)$.

In all our computations we will work modulo cycles homologous to 0 in the fibre. From weight considerations, one can see that such cycles map to 0 in the second intermediate Jacobian, though they need not be 0 in the Chow group. Since the CM cycles themselves are defined modulo such cycles, this is not really a restriction. For simplicity, we will only work with the case of Collino's original element.

If we fix the functions f_P and f_Q, then the boundary of Collino's element looks like

$$\sum_\tau \mathcal{D}_\tau$$

where $\mathcal{D}_\tau$ is some element of the Néron–Severi group of the fibre over τ. D_τ can be written in terms of the basis for the rational Néron–Severi given by e_1,e_2,e_3 and $\mathcal{Z}_\tau$,

$$\mathcal{D}_\tau = b_\tau^1 e_1 + b_\tau^2 e_2 + b_\tau^3 e_3 + c_\tau \mathcal{Z}_\tau$$

where $\mathcal{Z}_\tau$ denotes the CM cycle if the point τ is a CM point and is 0 otherwise

LEMMA 5.1. *Let* **W** *be (the compactification of) the universal family of Abelian surfaces as before and let* $\mathcal{W}$ *be the generic fibre. Recall that* e_1,e_2 *and* e_3 *generate the generic Néron–Severi. Then there are decomposable elements* (e_i, f_{e_i}) *for* $i = 1, 2, 3$ *such that*

$$\partial\left(\sum_i (e_i, f_{e_i})\right) = \sum_\tau \left(\sum_i b_\tau^i e_i\right).$$

Hence there is a relation of the form

$$\sum_\tau c_\tau \mathcal{Z}_\tau + \{hom\} \equiv 0$$

in $CH^2(\mathbf{W}) \otimes \mathbb{Q}$, *where* $\{hom\}$ *denotes the image of cycles homologous to zero in the fibre.*

Proof. Let $\pi_i : \overline{e_i} \to \mathbf{X}$ be the maps from the closure of the e_i to the base $\mathbf{X}$. Let M_i be divisors in $CH^1(\mathbf{W}) \otimes \mathbb{Q}$ be such that $(M_i.e_j) = \delta_{ij}$. Such M_i exist as one can always find an orthonormal basis for the rational Néron–Severi. Intersecting M_i with $\sum_\tau \mathcal{D}_\tau$ gives a relation in $CH^2(\overline{e_i})$ of the form

$$\sum_\tau b_\tau^i (\overline{e_i} \cap M_i)|_\tau$$

The direct image

$$(\pi_i)_*\left(\sum_\tau (\overline{e_i} \cap M_i)|_\tau\right) = \sum b_\tau^i \tau$$

is a rational equivalence of points on X so is the divisor of a function, f_{e_i}.

These functions f_{e_i} combined with the elements e_i give the required decomposable elements. Subtracting these elements from Collino's element gives a relation of the form

$$\sum_\tau c_\tau \mathcal{Z}_\tau + \{hom\} \equiv 0$$ □

Remark 5.2. At the cuspidal points, the cycles that remain after subtracting off the boundary of these decomposable elements are orthogonal to the closure of the generic Néron–Severi and as mentioned before are torsion in the Chow group of the special fibre itself, so do not make a contribution after tensoring with the rationals.

One could also say things more intrinsically by using the fact that there is a projector in the ring of correspondences with rational coefficients on $\mathcal{A}$ which takes the varation of Hodge structure $R^2\pi_*(\mathbb{Q})$ onto $Sym^2(R^1p_*(\mathbb{Q}))$ and the fact that the CM cycles lie in this part in the fibres over the CM points. We could then apply this correspondence to the element of $CH^2(\mathcal{A}, 1)$ itself to get a relation involving only CM cycles.

As a result of this lemma one sees that since one gets a relation between CM cycles which all lie in $CH^2_{hom}(\mathbf{W})$, we get a relation there and not merely in the Chow group.

We would now like to determine the points τ and the coefficients c_τ. To determine c_τ we have to write the CM cycle in terms of the basis for the primitive Néron–Severi given by $\bar{e}_i, i = 2, 3$ and $\bar{e}_4 = \frac{1}{2}(2e_4 - < e_4.e_1 > e_1)$, where $e_4 = \mathcal{E}_1$. Since $< e_4, e_4 >= 0$, one has $(\bar{e}_4.\bar{e}_4) = 1$. Define r and s by $(\bar{e}_4.\bar{e}_2) = r$, $(\bar{e}_4.\bar{e}_3) = s$ so the intersection matrix looks like

	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_4$
$\bar{e}_2$	p	$2aD$	r
$\bar{e}_3$	$2aD$	$4bD$	s
$\bar{e}_4$	r	s	1

The idea now is to compute the CM cycle up to a multiple using the property that it is orthogonal to the generic Néron–Severi. Define d_τ to be the smallest multiple of the CM cycle $\mathcal{Z}_\tau$ which lies in the $\mathbb{Z}$-span of $\bar{e}_i, i = 2, 3, 4$. Then one has

PROPOSITION 5.3. *Let d_τ be as above, then*

$$d_\tau \mathcal{Z}_\tau = \frac{aDs - 2bDr}{2}\bar{e}_2 + \frac{2aDr - ps}{2}\bar{e}_3 + 2D\bar{e}_4$$

up to a sign.

Proof. Let

$$\mathcal{Z}_\tau = x_2\bar{e}_2 + x_3\bar{e}_3 + x_4\bar{e}_4$$

Then $(\mathcal{Z}_\tau, \bar{e}_i) = 0$ for $i = 2, 3$, so we get 2 equations

$$0 = px_2 + 2aDx_3 + rx_4$$
$$0 = 2aDx_2 + 4bDx_3 + sx_4$$

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Eliminating the variable x_2 gives

$$x_3 = \frac{2aDr - ps}{4D} x_4$$

and similarly

$$x_2 = \frac{2aDs - 4bDr}{4D} x_4$$

so the vector

$$\left(\frac{2aDs - 4bDr}{4D}, \frac{2aDr - ps}{4D}, 1 \right)$$

is orthogonal to the generic Néron–Severi hence a rational multiple of the CM cycle.

The smallest multiple of the vector such that it is an integral linear combination will be the d_τ times the CM cycle or half the CM cycle depending on the discriminant, at least up to a sign. Define $b_{r,s}$ as the multiple which gives it, hence it is either the smallest multiple or twice the smallest multiple.

From the description above, and the fact that s is even, which we will see presently, it is clear that $b_{r,s}|4D$ and $c_\tau = c_{r,s} = 4D/b_{r,s}$. The smallest multiple is determined by $s_1 = p(s/2) - aDr$, if s_1 is even, then it is D otherwise it is $2D$.

The proposition will then follow from the following lemma

LEMMA 5.4. $|b_{r,s}| = 2D$ *always.*

Proof. Let $v = (\alpha, \beta, \gamma, \delta, \varepsilon)$ be the singular relation corresponding to the elliptic curve $\mathcal{E}_1$. From Hashimoto's explicit description of the embedding, one can compute r and s to be

$$r = \beta - 2\gamma - \frac{1-p}{2}\alpha$$

$$s = 2aD\beta - 2\varepsilon - 2D\delta(a^2D - b)$$

Let

$$s_1 = p(s/2) - aDr = \left(\frac{(p-1)}{2} aD\alpha - (p-1)aD\beta - 2aD\gamma + ((p-1)a^2D)\delta + p\varepsilon \right)$$

It turns out that τ satisfies an equation of discriminant $(1/p)(s_1^2 - 4D(p - r^2/4))$. This is even if and only if s_1 is even. Hence $b_{r,s} = 2D$ always, as if s_1 is even, then the smallest multiple is D hence $b_{r,s} = 2D$ is twice the smallest multiple. If s_1 is odd,then the smallest multiple is $2D$ and $b_{r,s}$ is the smallest multiple. This also says that the discriminant of τ is $(1/p)(s_1^2 - 4D(p - r^2/4))$ up to a square factor. □

This concludes the proof of the proposition. □

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Armed with this information we can compute discriminants of the orders that appear in the relation. They are the points on the moduli where the Néron–Severi is generated by the generic elements along with an elliptic curve of degree 1.

PROPOSITION 5.5. *The discriminants of the points τ where there is a elliptic curve of degree* 1 *are*

$$\Delta/d^2 = \frac{P(s,r) - 4D}{4d^2} = (p(s/2)^2 - aDrs + bDr^2 - D)/d^2$$

where r and s, where r and s run through all possible numbers satisfying $p - r^2/4 \in \mathbb{Z}_{\geqslant 0}$, $4bD - s^2/4 \in \mathbb{Z}_{\geqslant 0}$ and d is such that $d^2|P(s,r) - 4D/4$

Proof. The idea is to compare the determinants of the intersection matrices of the two bases of the rational Néron–Severi. Using the basis coming from $\bar{e}_1, \bar{e}_2$ and the CM cycle $d_\tau \mathcal{Z}_\tau$, at a point of discriminant Δ_0 one gets the determinant of the intersection matrix to be $-16D^2 d_\tau^2 \Delta_0$.

On the other hand, from the computation using the other basis given by $\bar{e}_1, \bar{e}_2, \bar{e}_3$ and the fact that the change of basis matrix has determinant $b_{r,s}^2 = 4D^2$, a simple calculation shows that the determinant is $(4D - P(s,r))4D^2$ where $P(s,r)$ is the quadratic form $ps^2 - 4aDrs + 4bDr^2$.

Comparing the two shows that

$$\Delta_0 = \frac{P(s,r) - 4D}{4d_\tau^2} = \Delta/d_\tau^2$$

where $\Delta = P(s,r) - 4D/4$.

If v_1 and v_2 are two elements of the primitive Néron–Severi, then the lattice generated by them is the Néron–Severi lattice of an Abelian surface with multiplication by an Eichler order of discriminant $v_1^2 v_2^2 - (v_1.v_2)^2/4$. This explains why $p - r^2/4$ and $4bD - s^2/4$ are in $\mathbb{Z}$.

Conversely, if r and s satisfy the conditions that $p - r^2/4$ and $4bD - s^2/4$ are in $\mathbb{Z}$ and d^2 divides $P(s,r) - 4Dn^2/4$ then one can see that the element

$$\mathcal{E} = \frac{1}{2D}\left(d\mathcal{Z}_\tau - \frac{aDs - 2bDr}{2}\bar{e}_2 - \frac{2aDr - ps}{2}\bar{e}_3\right)$$

where τ satisfies an equation of discriminant $P(s,r) - 4D/4d^2$, is a primitive integral element of invariant n^2, hence, from [Ka], one sees that it is an elliptic curve of degree 1. □

Since $P(s,r)$ is positive definite, the discriminant is negative for only finitely many values of s and r.

To generalize Proposition 5.5 for the relations we get from applying isogenies, we have to carry out the same computation using an elliptic curve of odd degree n

instead of 1, and one gets that the discriminants that appear are

$$\Delta = \frac{P(s,r) - 4Dn^2}{4d^2} = \frac{p(s/2)^2 - aDrs + bDr^2 - Dn^2}{d^2}$$

5.2. HEEGNER CYCLES

In this section we define the *Heegner cycles* which are sums of certain CM cycles with the same discriminant. As it turns out, our final result can be expressed in terms of these cycles. A reference for the facts used here is [B-D] or [G-Z].

Let $\mathcal{O}$ be, as before, the Eichler order in the quaternionic division algebra (or $M_2(\mathbb{Q})$. Let K be an imaginary quadratic field and let $\mathcal{O}_\Delta$ be an order of discriminant Δ. If τ is a CM point, then τ determines an embedding of $K \hookrightarrow \mathbf{B}$ and τ is called a *Heegner point of discriminant* Δ if

$$K \cap \mathcal{O} = \mathcal{O}_\Delta$$

Composing with the orientation on the Eichler order $\mathfrak{o}_\ell$ gives an *orientation* on the Heegner point, namely a surjective linear map

$$\kappa_\ell \colon \mathcal{O}_\Delta \to \mathbb{F}_\ell$$

if $\ell | N$ or

$$\kappa_\ell \colon \mathcal{O}_\Delta \to \mathbb{F}_{\ell^2}$$

if $\ell | D_0$.

For a given orientation and level structure, it is well known [G-Z] that there are h_Δ Heegner points of discriminant Δ, where h_Δ is the class number of the order. Further, the Class group $Pic(\mathcal{O}_\Delta)$ acts on the points preserving the orientation. The Atkin–Lehner operator w_ℓ, for $\ell | D$ also acts on the Heegner points by flipping the orientation at ℓ and also changing the ideal class. The action of the group $\Omega_D \ltimes Pic(\mathcal{O}_\Delta)$ is transitive showing that for a fixed level there are $2^t h_\Delta$ Heegner points.

Let m be a number such that

$$m \equiv \begin{cases} a\ square\ mod\ \ell\ \ if\ \ \ell | N \\ a\ nonsquare\ \ mod\ \ell\ \ if\ \ \ell | D_0 \end{cases}$$

and let $\mathbb{Q}_{\ell,m}$ be the extension of $\mathbb{Q}_\ell$ obtained by adding the roots of the equation $x^2 - m = 0$. For $\ell | N$ this is simply $\mathbb{Q}_\ell$ and for $\ell | D_0$ this is a quadratic extension. Let $\mathbb{Z}_{l,m}$ be the ring of integers and $\mathcal{O}_{\Delta,\ell,m} = \mathcal{O}_\Delta \otimes \mathbb{Z}_{l,m}$. Then

$$\mathcal{O}_{\Delta,\ell,m}/\ell\mathcal{O}_{\Delta,\ell,m} = \begin{cases} \mathbb{F}_\ell \oplus \mathbb{F}_\ell & \text{if } \ell|\mathrm{N} \\ \mathbb{F}_{\ell^2} \oplus \mathbb{F}_{\ell^2} & \text{if } \ell|\mathrm{D}_0 \end{cases}$$

and the different orientations correspond to the different canonical maps or equivalently the different primes lying over the primes dividing D.

From that one can see that the orientation classes are in bijection with solutions μ mod $2D$ of the equation $\mu^2 \equiv m\Delta$ *mod* $4D$ as μ will give a solution of $\mu^2 \equiv m\Delta$ *mod* 4ℓ.

Since m^{-1} is a square in $\mathbb{Z}_{\ell,m}$, $m^{-1} = t_\ell^2$ for some $t_\ell \in \mathbb{Z}_{\ell,m}$, so the ideal

$$\mathfrak{L} = \left(\ell, \frac{\mu t_\ell + \sqrt{\Delta}}{2}\right)$$

is a prime lying over ℓ in $\mathbb{Z}_{\ell,m}$. Conversely, given an oriented Heegner point, the kernels of the various orientation maps give rise to the different primes and hence a solution of the equation.

The choice of m is not that important. Any number satisfying the conditions satisfied by m will give the same result.

A Heegner point on $\mathbf{X}$ also comes with the data of a level $2k$ structure, and this has a projection onto the level 2 structure $\mathcal{L}$, and we can further consider the class of $\mathcal{L}$, $[\mathcal{L}]$ which is the even theta characteristic corresponding to it. We can make a uniform choice of an elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ of degree 1, or more generally of degree n, for n odd, over each of these classes. A *Heegner Cycle* is the CM cycle in the fibre over such a point. For a given level 2 structure and orientation there are two Heegner cycles corresponding to which of the two pairs of three 2 torsion points lies on $\mathcal{E}_1$. We define the *sign* of the Heegner cycle to be positive if the component of $\mathcal{E}_2$ in the direction of the CM cycle is positive. This depends on the configuration of the 2-torsion points.

From class field theory, one knows that $Pic(\mathcal{O}_\Delta) \simeq Gal(H_\Delta/K)$, where H_Δ is the ring class field of $\mathcal{O}_\Delta$. From this one can realise the action of *Pic* as a Galois action, and one can see that the action on the Heegner cycles does not change the sign. The action of the Fricke involution $w_D = \prod_\ell w_\ell$ is, up to an element of *Pic*, given by complex conjugation and it flips the orientation and the configuration of the pair of three 2-torsion points and hence the sign.

We then can define a *Heegner cycle of discriminant* Δ*, level* $[\mathcal{L}]$ *and orientation* μ to be a sum

$$\mathcal{Z}_{\Delta,[\mathcal{L}],\mu,-\mu} = \sum_{\mathcal{L}\in[\mathcal{L}]} \sum_{\mathfrak{a}\in Pic(\mathcal{O}_\Delta)} \mathcal{Z}_{\mathfrak{a},\mathcal{L},\mu,I} - \sum_{\mathfrak{a}\in Pic(\mathcal{O}_\Delta)} \mathcal{Z}_{\mathfrak{a},\mathcal{L},-\mu,II}$$

where the I and II denote the possible configurations. This cycle is invariant under the action of *Pic* and changes sign under the action of the Fricke involution so

$$\mathcal{Z}_{\Delta,[\mathcal{L}],\mu,-\mu} = \sum_{\sigma\in Gal(H/\mathbb{Q})} \mathcal{Z}^{\sigma}_{\mathfrak{a}_0,\mathcal{L},\mu_0,I_0}$$

The cycle $\mathcal{Z}_{\Delta,[\mathcal{L}],-\mu,\mu}$ is a different cycle which corresponds to the same configuration but opposite orientation. To reduce notation, and since both the configurations occur, we will suppress the I and II.

For full level $2k$ structure we define the Heegner cycle to be the sum over all the points over the point determined by $(\mathfrak{a}, \mathcal{L}, \mu)$ of the Heegner cycles, and we use the same notation to denote it.

In our situation, if r and s denote the intersection numbers of $\mathcal{E}_1$ with $\bar{e}_2$ and $\bar{e}_3$, then they determine the orientation of the Heegner cycle corresponding to it, which is of discriminant $\Delta = ps^2 - 4adrs + 4bDr^2 - 4D/4d^2$ for some d.

The prime p is a number like m above, namely a square mod ℓ for $\ell|N$ and a non-square mod ℓ for $\ell|D_0$. By 'completing the square' one can then see that $s_1 = p(s/2) - aDr$ satisfies an equation of the form $x^2 = p\Delta d^2 \ mod\ 4D$ and hence determines the orientation!

6. Main Result

In this section we state our main result. We will assume that we always have level $2k$ structure for some odd integer k coprime to N.

THEOREM 6.1. *Let* $\mathbf{X(D_0, N)}$ *be a Shimura curve parametrising Abelian surfaces with endomorphism ring an Eichler order of level* N *in a division algebra of discriminant* D_0. *Let* $\mathbf{W(D_0, N)}$ *denote (the non-singular compactification of) the universal Abelian surface over* $\mathbf{X(D_0, N)}$. *Let* p, a, b *be the invariants which determine Hashimoto's model and let* P *and* Q *denote two* 2*-torsion points.*

Then there are relations in $CH^2_{hom}(\mathbf{W(D_0, N)}) \otimes \mathbb{Q}$ *of the form*

$$\sum_{[\mathcal{L}]} \varepsilon_{P,Q}([\mathcal{L}]) \sum_{r,s} \sum_{d^2|\Delta} d\mathcal{Z}_{\Delta/d^2,[\mathcal{L}],s_1/d,-s_1/d} \equiv 0$$

where $\mathcal{L}$ *runs through all even theta characteristics and* $\varepsilon_{P,Q}([\mathcal{L}])$ *is the sign function as in Section* 4.3, $r, s \in \mathbb{Z}$ *such that* $n^2p - r^2/4 \in \mathbb{Z}_{\geqslant 0}$, $4bDn^2 - s^2/4 \in \mathbb{Z}_{\geqslant 0}$, n *is coprime with* $2k$, r, s, n *mutually coprime,* $s_1 = p(s/2) - aDr)$ *and*

$$\Delta = \frac{ps^2 - 4aDrs + 4bDr^2 - 4Dn^2}{4}$$

with d *running through all* $d^2|\Delta$ *such that* Δ/d^2 *is still a discriminant.*

Proof. This is just a consequence of putting together the statements in the previous sections. □

Remark 6.2. For different choices of the prime p and numbers a and b one could get possibly different relations. One way of interpreting the Hashimoto model is by observing that it gives an embedding of the Shimura curve in to the Siegel upper half space. Different choices of p,a and b correspond to possibly different embeddings and could result in different relations.

6.1. EXAMPLES

I. Suppose $D_0 = 2.3$, $N = 1$ so $D = D_0N = 6$ and $n = 1$. The triple $(p, a, b) = (5, 2, 5)$ determines an embedding. The possible values for (r, s) are

$$\{(1, 0), (1, 2), (1, 4), (1, 6), (1, 8), (1, 10)\}$$

Out of these only $(1, 4)$ and $(1, 6)$ give rise to negative values for Δ, which are -4 and -3 respectively.

Suppose the two torsion points are the ones associated to

$$P = [(0, 0), (0, 0)] \text{ and } Q = [(1, 0), (0, 0)]$$

so they correspond to the odd characteristics $[(0, 1), (0, 1)]$ and $[(1, 1), (1, 0)]$ respectively. Then from the table one can read off the signs at the various components of the moduli of products corrsponding to the even theta characteristics. In this case, for example, the signs are -1 at $[(1, 1), (1, 1)], [(0, 1)(0, 0)]$ and $[(0, 0), (1, 0)]$, 1 at $[(1, 0), (0, 0)], [(0, 1), (1, 0)]$ and $[(0, 0), (1, 1)]$ and 0 elsewhere.

So one has a relation in $CH^2(\mathbf{W}(\mathbf{2}, \mathbf{3}) \otimes \mathbb{Q})$ of the form

$$\begin{aligned}
&\mathcal{Z}_{3,[1000],3,-3} + \mathcal{Z}_{4,[1000],2,-2} + \mathcal{Z}_{3,[0110],3,-3} + \mathcal{Z}_{4,[0110],2,-2} \\
&\quad + \mathcal{Z}_{3,[0011],3,-3} + \mathcal{Z}_{4,[0011],2,-2} - \mathcal{Z}_{3,[1111],3,-3} - \mathcal{Z}_{4,[1111],2,-2} \\
&\quad - \mathcal{Z}_{3,[0100],3,-3} - \mathcal{Z}_{4,[0100],2,-2} - \mathcal{Z}_{3,[0010],3,-3} - \mathcal{Z}_{4,[0010],2,-2} \\
&\quad + \mathcal{Z}_{3,[1000],-3,3} + \mathcal{Z}_{4,[1000],-2,2} + \mathcal{Z}_{3,[0110],-3,3} + \mathcal{Z}_{4,[0110],-2,2} \\
&\quad + \mathcal{Z}_{3,[0011],-3,3} + \mathcal{Z}_{4,[0011],-2,2} - \mathcal{Z}_{3,[1111],-3,3} - \mathcal{Z}_{4,[1111],-2,2} \\
&\quad - \mathcal{Z}_{3,[0100],-3,3} - \mathcal{Z}_{4,[0100],-2,2} - \mathcal{Z}_{3,[0010],-3,3} - \mathcal{Z}_{4,[0010],-2,2} = 0
\end{aligned}$$

where for $[abcd]$ denotes the characteristic $[(a, b), (c, d)]$.

II. Different choices of (p, a, b) leads to different embeddings of the modular curves and hence new relations. For example, if $D = 26$, $(p, a, b) = (5, 2, 21)$, then there are relations between the $\Delta's$ given by $(r, s, \Delta) = (1, 18, -11)$, $(1, 20, -20)$ and $(1, 22, -19)$. However, if (p, a, b) is $(149, 19, 63)$ then one has relations between (r, s, Δ) of the form $(1, 6, -11)$, $(3, 20, -24)$, $(7, 46, -11)$ and $(9, 60, -8)$. In particular, one may get a relation involving more than one Heegner point of the same discriminant.

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References

[B-D] Bertolini, M. and Darmon, H.: Heegner points on Mumford-Tate curves, *Invent. Math.* **126**(3) (1996), 413–456.

[Be] Besser, A.: CM cycles over Shimura curves, *J. Algebraic Geom.* **4**(4) (1995), 659–691.

[Be2] Besser, A.: Elliptic fibrations of $K3$ surfaces and QM Kummer surfaces, *Math. Z.* **228**(2) (1998), 283–308.

[Bo] Borcherds, R. E.: The Gross–Kohnen–Zagier theorem in higher dimensions, *Duke Math. J.* **97**(2) (1999), 219–233.

[Bl1] Bloch, S.: Algebraic cycles and higher K-theory, *Adv. in Math.* **61**(3) (1986), 267–304.

[Co] Collino, A.: Griffiths' infinitesimal invariant and higher K-theory on hyperelliptic Jacobians, *J. Algebraic Geom.* **6**(3) (1997), 393–415.

[Fr-Kr] Farkas, H. M. and Kra, I.: Riemann surfaces, Grad. Texts in Math., 71, Springer-Verlag, New York, 1980, xi+337 pp.

[Ge] Van der Geer, G.: On the geometry of a Siegel modular threefold, *Math. Ann.* **260**(3) (1982), 317–350.

[G-H] Hirzebruch, F. and Van der Geer, G.: *Lectures on Hilbert Modular Surfaces*, Based on notes taken by W. Hausmann and F. J. Koll. Seminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], 77. Presses de l'Université de Montréal, Montréal, Que., 1981. 193 pp. ISBN: 2-7606-0562-0.

[Gr] Gross, B. H.: Heegner points on $X_0(N)$. In *Modular Forms* (*Durham*, *1983*), Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., Horwood, Chichester, 1984, pp. 87–105.

[G-Z] Gross, B. H. and Zagier, D. B.: Heegner points and derivatives of L-series, *Invent. Math.* **84**(2) (1986), 225–320.

[Ha] Hashimoto, K.: Explicit form of quaternion modular embeddings, *Osaka J. Math.* **32**(3) (1995), 533–546.

[Hu] Humbert, G.: *Collected Works*, Gauthier-Villars, Paris, 1929.

[Ig] Igusa, J-I.: On Siegel modular forms of genus 2(II), *Amer. J. Math.* **86** (1964), 392–412.

[Ka] Kani, E.: Elliptic curves on Abelian surfaces, *Manuscripta Math.* **84**(2) (1994), 199–223.

[Ne] Nekovář, J.: Kolyvagin's method for Chow groups of Kuga-Sato varieties, *Invent. Math.* **107**(1) (1992), 99–125.

[Mu] Mumford, D.: *Abelian Varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970.

[Ro] Roberts, D. P.: Shimura Curves Analogous to $X_0(N)$, Thesis, Harvard University, May 1989.

[Sc2] Schoen, C.: Complex multiplication cycles and a conjecture of Beilinson and Bloch, *Trans. Amer. Math. Soc.* **339**(1) (1993), 87–115.

[Sc1] Schoen, C.: Complex multiplication cycles on elliptic modular threefolds, *Duke Math. J.* **53**(3) (1986), 771–794.

[Sl2] Scholl, A. J.: Vanishing cycles and non-classical parabolic cohomology, *Invent. Math.* **124**(1–3) (1996), 503–524.

[Sh] Shimura, G.: *Introduction to the Arithmetic Theory of Automorphic Functions*, Reprint of the 1971 original. Publications of the Mathematical Society of Japan, 11. Kano Memorial Lectures, 1. Princeton University Press, Princeton, NJ, 1994.

[Vi] Vignéras, M-F.: *Arithmétique des algèbres de quaternions* (French) [Arithmetic of quaternion algebras] Lecture Notes in Mathematics, 800, Springer, Berlin, 1980.

[We] Weil, A.: Zum Beweis des Torellischen Satzes, *Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa.* 1957 (1957), 33–53 (Collected Works, vol II).

[Za] Zagier, D.: *Modular Points, Modular Curves, Modular Surfaces and Modular Forms*. Workshop, Bonn 1984 (Bonn, 1984), 225–248, Lecture Notes in Math. 1111, Springer, Berlin-New York, 1985.

[Zh] Zhang, S.: Heights of Heegner cycles and derivatives of *L*-series. Preprint.